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Research Article

Rayleigh Mixture Distribution

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This paper presents Rayleigh mixtures of distributions in which the weight functions are assumed to be chi-square, t and F sampling distributions. The exact probability density functions of the mixture of two correlated Rayleigh random variables have been derived. Different moments, characteristic functions, shape characteristics, and the estimates of the parameters of the proposed mixture distributions using method of moments have also been provided.

1. Introduction

In statistics, a mixture distribution is expressed as a convex combination of other probability distributions. It can be used to model a statistical population with subpopulations, where components of mixture probability densities are the densities of the subpopulations, and the weights are the proportion of each subpopulation in the overall population. Mixture distribution may suitably be used for certain data set where different subsets of the whole data set possess different properties that can best be modeled separately. They can be more mathematically manageable, because the individual mixture components are dealt with more nicely than the overall mixture density. The families of mixture distributions have a wider range of applications in different fields such as fisheries, agriculture, botany, economics, medicine, genetics, psychology, paleontoogy, electrophoresis, finance, communication theory, sedimentology/geology, and zoology.

Pearson [1] is considered as the torch bearer in the field of mixtures distributions. He studied the estimation of the parameters of the mixture of two normal distributions. After a long period of time, some basic properties of mixture distributions were studied by Robins (1948). Some of other researchers [2–5] have studied in greater detail the finite mixture of distributions. Roy et al. [6–12] defined and studied poisson, binomial, negative binomial, gamma, chi-square and Erlang mixtures of some standard distributions. In the light of the above-mentioned distributions, here we have studied Rayleigh mixtures of distributions in which the weight functions are assumed to be chi-square, t- and F-distribution, and the moments, characteristic function, and shape characteristics of these mixtures distributions have also been studied.

2. Preliminaries

Suppose the random variable X has a probability density function (pdf) $f(x \mid \theta)$ and if the parameter space Θ is a discrete random variable containing parameter values $\theta_1, \theta_2, \ldots, \theta_k$ such that the distribution of Θ is $P(\Theta = \theta_i) = p_i$, then the unconditional distribution of X is

$$m(x) = \sum_{i=1}^{k} p_i f(x \mid \theta_i). \tag{2.1}$$

This is called a mixture of the distributions $f(x \mid \theta_i)$ with weight p_i, $i = 1, 2, \ldots, k$. The above definition may be extended to the case for large k.

It can be generalized to the case when the parameter space Θ is absolutely continuous random variable having pdf $\tau(\theta)$. We will have, then, a continuous mixture of densities $f(x \mid \theta)$ with weight function $\tau(\theta)$. In this case, the unconditional distribution of X is

$$m(x) = \int_{\Theta} f(x \mid \theta) \tau(\theta) d\theta. \tag{2.2}$$

3. Main Results

In this paper we first define the general form of Rayleigh mixture distribution. Then we furnished the Rayleigh mixture of some well-known sampling distributions such as chi-square, t- and F-distributions. The exact distribution of the mixture of two correlated Rayleigh distributions has been studied.

The main results of this study have been presented in the form of some definitions and theorems.

Definition 1. A random variable X is said to have Rayleigh mixture distribution if its probability density function is defined by

$$f\left(x; \sigma^2, n\right) = \int_0^{\infty} \frac{r e^{-r^2/2\sigma^2}}{\sigma^2} \tau(x, r; n) dr, \tag{3.1}$$

where $\tau(x, r; \tau, n)$ is a probability density function or any sampling distribution such as chi-square, t- and F-distribution.

The name Rayleigh mixture distributions is given due to the fact that the derived distribution (3.1) is the weighted sum of $\tau(x, r; \tau, n)$ with weight factor equal to the probabilities of Rayleigh distribution.

3.1. Formulation of Rayleigh Mixture Distribution

The Rayleigh mixtures of distributions in which the weight functions are assumed to be chi-square, t- and F-distribution. In a statistical theory, we will use chi-square distribution as a weight function if sampling statistic follows chi-square distribution. For example, sampling variance is followed by chi-square distribution and we can use chi-square distribution as a weight function. Similarly, we will use t-distribution and F-distribution if and only if sampling statistic follows t-distribution and F-distribution, respectively. For example, if population variance is unknown and sample size is very small, then the sampling mean follows t-distribution and the ratio of sampling variances follows F-distribution. Now we define Rayleigh mixtures of distributions for different weight functions as follows.

3.1.1. Rayleigh Mixtures of Chi-Square Distribution

Definition 2. A random variable χ^2 is said to have a Rayleigh mixture of chi-square distribution with parameter σ^2 with degrees of freedom n if its probability density function is defined by

$$f\left(\chi^2;\sigma^2,n\right)=\int_0^\infty \frac{re^{-r^2/2\sigma^2}}{\sigma^2}\frac{e^{-\chi^2/2}\left(\chi^2\right)^{(n/2)+r-1}}{2^{(n/2)+r}\,\Gamma\left(n/2+r\right)}dr;\quad 0<\chi^2<\infty, \tag{3.2}$$

where the weight function $\tau(x,r;\tau,n)$ in (3.1) is the chi-square sampling distribution. Here the notation "Γ" in (3.2) is a gamma function such that $\Gamma a=(a-1)!=(a-1)(a-2)\cdots 3\cdot 2\cdot 1$.

3.1.2. Rayleigh Mixtures of t-Distribution

Definition 3. A random variable t is defined to have a Rayleigh mixture of t-distributions with parameter σ^2 and degrees of freedom n if its probability density function is defined as

$$f\left(t;\sigma^2,n\right)=\int_0^\infty \frac{re^{-r^2/2\sigma^2}}{\sigma^2}\frac{t^{2r}}{n^{(1/2)+r}B(1/2+r,n/2)\left(1+t^2/n\right)^{(n+1)/2+r}}dr,\qquad -\infty<t<\infty, \tag{3.3}$$

where the weight function $\tau(x,r;\tau,n)$ in (3.1) is the student t-distribution.

3.1.3. Rayleigh Mixtures of F-Distribution

Definition 4. A random variable F is defined to have a Rayleigh mixture of F-distributions with parameter σ^2 and degrees of freedom n_1 and n_2, if its probability density function is defined as

$$f\left(F;\sigma^2,n_1,n_2\right)=\int_0^\infty \frac{re^{-r^2/2\sigma^2}}{\sigma^2}\frac{\left(n_1/n_2\right)^{(n_1/2)+r}F^{n_1/2+r-1}}{B(n_1/2+r,n_2/2)\left(1+(n_1/n_2)F\right)^{(n_1+n_2)/2+r}}dr,\qquad 0<F<\infty, \tag{3.4}$$

where the weight function $\tau(x,r;\tau,n)$ in (3.1) is the F-distribution.

3.1.4. Mixture of Two Correlated Rayleigh Distributions

Let X and Y be two independent Rayleigh variables with probability density function (pdf). The joint distribution of X and Y with correlation coefficient $\rho(-1 \leq \rho \leq 1)$ can be constructed by the following formula:

$$f(x,y) = f(x)g(y)\left[1 + \rho(1 - 2F(x)) \times (1 - 2G(y))\right] \tag{3.5}$$

which was developed by Farlie-Gumbl-Morgenstern (1979).

Using the formula, the mixture of two correlated distributions is as follows:

$$\begin{aligned} f(x,y;\sigma_1,\sigma_1;\rho) = {} & \frac{xy}{(\sigma_1\sigma_2)^2} \\ & \times \left\{ e^{-(1/2)((x^2/\sigma_1^2)+(y^2/\sigma_2^2))} + \rho e^{-(1/2)((x^2/\sigma_1^2)+(y^2/\sigma_2^2))} - 2\rho e^{-(1/2)((2x^2/\sigma_1^2)+(y^2/\sigma_2^2))} \right. \\ & \left. -2\rho e^{-(1/2)((x^2/\sigma_1^2)+(2y^2/\sigma_2^2))} + 4\rho e^{-((x^2/\sigma_1^2)+(y^2/\sigma_2^2))} \right\}, \end{aligned} \tag{3.6}$$

where $x > 0, y > 0; \sigma_1, \sigma_2 > 0$ and $-1 \leq \rho \leq 1$.

3.2. Derivation of Characteristics of Rayleigh Mixture Distribution

Moments and different characteristics of the Rayleigh mixture of distributions are presented by the following theorems.

Theorem 3.1. *If χ^2 follows a Rayleigh mixture of chi-square distribution with parameter σ^2 with degrees of freedom n, then the sth raw moment of this mixture distribution about origin is given by*

$$\mu_s' = \int_0^\infty \frac{re^{-r^2/2\sigma^2}}{\sigma^2} \frac{2^s \overline{)n/2 + r + s}}{\overline{)n/2 + r}} dr. \tag{3.7}$$

Hence, the mean and the variance of this mixture distribution are as follows:

$$\begin{aligned} Mean &= n + \sigma\sqrt{2\pi}, \\ Variance &= 2n + 2\sigma\sqrt{2\pi} + 2\sigma^2(4 - \pi). \end{aligned} \tag{3.8}$$

Proof. We know the sth raw moment defined by

$$\begin{aligned}
\mu_s' &= E\left[\left(\chi^2\right)^s\right] \\
&= \iint_0^\infty \frac{re^{-r^2/2\sigma^2}}{\sigma^2} \frac{e^{-\chi^2/2}\left(\chi^2\right)^{n/2+r+s-1}}{2^{n/2+r})\overline{n/2+r}} dr d\chi^2 \\
&= \int_0^\infty \frac{re^{-r^2/2\sigma^2}}{\sigma^2} \frac{2^s)\overline{n/2+r+s}}{)\overline{n/2+r}} dr.
\end{aligned} \tag{3.9}$$

If we put $s = 1$ in (3.9), we get

$$\begin{aligned}
\mu_1' &= \int_0^\infty \frac{re^{-r^2/2\sigma^2}}{\sigma^2} \frac{2)\overline{n/2+r+1}}{)\overline{n/2+r}} dr \\
&= n + \sigma\sqrt{2\pi}.
\end{aligned} \tag{3.10}$$

If we put $s = 2$ in (3.9), we have

$$\begin{aligned}
\mu_2' &= \int_0^\infty \frac{re^{-r^2/2\sigma^2}}{\sigma^2} \frac{2^2)\overline{n/2+r+2}}{)\overline{n/2+r}} dr \\
&= 4\int_0^\infty \frac{re^{-r^2/2\sigma^2}}{\sigma^2} (n/2+r+1)(n/2+r) dr \\
&= n^2 + 4\sigma(n+1)\sqrt{2})\overline{\frac{3}{2}} + 2n + 8\sigma^2 \quad \text{(On simplification)} \\
&= n^2 + \sigma(n+1)2\sqrt{2\pi} + 2n + 8\sigma^2.
\end{aligned} \tag{3.11}$$

Hence, the variance is defined by

$$\begin{aligned}
\mu_2 &= \mu_2' - \left(\mu_1'\right)^2 \\
&= 2n + 2\sigma\sqrt{2\pi} + 2\sigma^2(4-\pi).
\end{aligned} \tag{3.12}$$

This completes the proof. □

Theorem 3.2. *If χ^2 follows a Rayleigh mixture of chi-square distributions with parameter σ^2 and degrees of freedom n, then its characteristic function is given by*

$$\Phi_{\chi^2}(t) = (1-2it)^{-n/2} \int_0^\infty \frac{re^{-r^2/2\sigma^2}}{\sigma^2} (1-2it)^{-r} dr. \tag{3.13}$$

Proof. The characteristic function is defined as

$$
\begin{aligned}
\Phi_{\chi^2}(t) &= E\left[e^{it\chi^2}\right] \\
&= \iint_0^{\infty} \frac{re^{-r^2/2\sigma^2}}{\sigma^2} \frac{e^{-(\chi^2/2)(1-2it)}\left(\chi^2\right)^{n/2+r-1}}{2^{n/2+r}\Gamma\left(n/2+r\right)} dr\, d\chi^2 \\
&= \int_0^{\infty} \frac{re^{-r^2/2\sigma^2}}{\sigma^2} \frac{1}{(1-2it)^{n/2+r}} dr \\
&= (1-2it)^{-n/2} \int_0^{\infty} \frac{re^{-r^2/2\sigma^2}}{\sigma^2} (1-2it)^{-r} dr
\end{aligned}
\tag{3.14}
$$

and hence proved □

Theorem 3.3. *If t follows a Rayleigh mixture of t-distributions with parameter σ^2 and degrees of freedom n, then the sth raw moment about origin is*

$$
\begin{aligned}
\mu'_{2s+1} &= \mu_{2s+1} = 0, \\
\mu'_{2s} &= \mu_{2s} = n^s \int_0^{\infty} \frac{re^{-r^2/2\sigma^2}}{\sigma^2} \frac{\Gamma\left(r+s+1/2\right)\Gamma\left(n/2-s\right)}{\Gamma\left(r+1/2\right)\Gamma\left(n/2\right)} dr.
\end{aligned}
\tag{3.15}
$$

And hence

$$
\textit{Mean} = 0, \qquad \textit{Variance} = \frac{n}{(n-2)}\left[1+\sigma\sqrt{2\pi}\right] \quad \textit{for } n > 2.
\tag{3.16}
$$

Therefore,

$$
\textit{Skewness: } \beta_1 = 0, \qquad \textit{Kurtosis: } \beta_2 = \frac{(n-2)\left[2\sigma^2 + 2\sigma\sqrt{2\pi} + 3\right]}{(n-4)\left[1+\sigma\sqrt{2\pi}\right]^2}, \quad n > 4.
\tag{3.17}
$$

Proof. The $(2s+1)$th raw moment (odd order moments) about origin is given by

$$
\begin{aligned}
\mu'_{2s+1} &= E\left[t^{2s+1}\right] \\
&= \int_{-\infty}^{\infty}\int_0^{\infty} \frac{re^{-r^2/2\sigma^2}}{\sigma^2} \frac{t^{2r+2s+1}}{n^{1/2+r}B(1/2+r, n/2)\left(1+t^2/n\right)^{(n+1)/2+r}} dr\, dt \\
&= \int_0^{\infty} \frac{re^{-r^2/2\sigma^2}}{\sigma^2 n^{1/2+r} B(1/2+r, n/2)} \int_{-\infty}^{\infty} \frac{t^{2r+2s+1}}{\left(1+t^2/n\right)^{(n+1)/2+r}} dt\, dr
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{\infty} \frac{re^{-r^2/2\sigma^2}}{\sigma^2 n^{1/2+r} B(1/2+r, n/2)} \int_{-\infty}^{\infty} \psi(t)\,dt\,dr \\
&= 0 \quad \left[\text{Since, } \psi(t) = \frac{t^{2r+2s+1}}{(1+t^2/n)^{(n+1)/2+r}} \text{ is an odd function of } t\right].
\end{aligned}
\tag{3.18}
$$

If $s = 0$, then $\mu_1' = \text{Mean} = 0$. If $s = 1$ then $\mu_3' = \mu_3 = 0$. So, $\mu_{2s+1}' = \mu_{2s+1} = 0$.

Now, the $(2s)$th raw moment about origin is given by

$$
\begin{aligned}
\mu_{2s} = \mu_{2s}' &= E\left[t^{2s}\right] \\
&= \int_{-\infty}^{\infty}\int_0^{\infty} \frac{re^{-r^2/2\sigma^2}}{\sigma^2} \frac{t^{2r+2s}}{n^{1/2+r} B(1/2+r, n/2)(1+t^2/n)^{(n+1)/2+r}}\,dr\,dt \\
&= \int_0^{\infty} \frac{re^{-r^2/2\sigma^2}\ n^{s+r+1/2}}{\sigma^2 n^{1/2+r} B(1/2+r, n/2)} \int_0^{\infty} \frac{(t^2/n)^{s+r+1/2-1}}{(1+t^2/n)^{(n+1)/2+r}}\,d\left(\frac{t^2}{n}\right)dr \\
&= \int_0^{\infty} \frac{re^{-r^2/2\sigma^2}\ n^{s+r+1/2}}{\sigma^2 n^{1/2+r} B(1/2+r, n/2)} \int_0^{\infty} \frac{u^{s+r+1/2-1}}{(1+u)^{(n+1)/2+r}}\,du\,dr; \qquad \left[\text{Putting } u = \frac{t^2}{n}\right] \\
&= n^s \int_0^{\infty} \frac{re^{-r^2/2\sigma^2}}{\sigma^2} \frac{\Gamma(r+s+1/2)\,\Gamma(n/2-s)}{\Gamma(r+1/2)\,\Gamma(n/2)}\,dr.
\end{aligned}
\tag{3.19}
$$

If $s = 1$ then,

$$
\begin{aligned}
\mu_2' &= \mu_2 \\
&= n\int_0^{\infty} \frac{re^{-r^2/2\sigma^2}}{\sigma^2} \frac{\Gamma(r+3/2)\,\Gamma(n/2-1)}{\Gamma(r+1/2)\,\Gamma(n/2)}\,dr \\
&= \frac{n}{n-2} + \left(\frac{2n}{n-2}\right)\sqrt{2}\sigma\,\Gamma\left(\frac{3}{2}\right); \quad n > 2 \\
&= \frac{n}{n-2}\left[1 + \sigma\sqrt{2\pi}\right]; \qquad \left[\because\ \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}\right].
\end{aligned}
\tag{3.20}
$$

If $s = 2$, then,

$$
\begin{aligned}
\mu_4' = \mu_4 &= n^2 \int_0^{\infty} \frac{re^{-r^2/2\sigma^2}}{\sigma^2} \frac{\Gamma(r+5/2)\,\Gamma(n/2-2)}{\Gamma(r+(1/2))\,\Gamma(n/2)}\,dr \\
&= \frac{n^2}{(n-2)(n-4)} \int_0^{\infty} \frac{re^{-r^2/2\sigma^2}}{\sigma^2}\left(r^2+4r+3\right)dr \\
&\left[\text{Since, } \frac{\Gamma(r+5/2)\,\Gamma(n/2-2)}{\Gamma(r+1/2)\,\Gamma(n/2)} = \frac{(r+3/2)(r+1/2)}{(n/2-1)(n/2-2)} = \frac{r^2+4r+3}{(n-2)(n-4)}\right].
\end{aligned}
\tag{3.21}
$$

Hence,

$$\begin{aligned}
\mu_4' &= \mu_4 \\
&= \frac{n^2}{(n-2)(n-4)}\left[2\sigma^2\Gamma 2 + 4\sigma\sqrt{2}\,\Gamma\frac{3}{2} + 3\right] \\
&= \frac{n^2}{(n-2)(n-4)}\left[2\sigma^2 + 2\sigma\sqrt{2\pi} + 3\right], \quad n > 4.
\end{aligned} \tag{3.22}$$

To find the Skewness and Kurtosis of this mixture distribution

$$\begin{aligned}
&\text{Skewness: } \beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0, \\
&\text{Kurtosis: } \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{(n-2)\left[2\sigma^2 + 2\sigma\sqrt{2\pi} + 3\right]}{(n-4)\left[1 + \sigma\sqrt{2\pi}\right]^2}.
\end{aligned} \tag{3.23}$$

This completes the proof. □

Theorem 3.4. *If F follows a Rayleigh mixture of F-distributions having parameter σ^2 with degrees of freedom n_1 and n_2, respectively, then the sth raw moment about origin is given by*

$$\mu_s' = \left(\frac{n_2}{n_1}\right)^s \int_0^\infty \frac{re^{-r^2/2\sigma^2}}{\sigma^2} \frac{\Gamma(n_1/2 + r + s)\,\Gamma(n_2/2 - s)}{\Gamma(n_1/2 + r)\,\Gamma(n_2/2)} dr. \tag{3.24}$$

Hence, the mean and variance of this distribution are

$$\begin{aligned}
&\textit{Mean} = \frac{n_2}{n_1(n_2-2)}\left[n_1 + \sigma\sqrt{2\pi}\right], \\
&\textit{Variance} = \left(\frac{n_2}{n_1}\right)^2\left[\frac{n_1^2 + 2n_1 + 2(n_1+1)\ \sigma\sqrt{2\pi} + 8\sigma^2}{(n_2-2)\ (n_2-4)} - \frac{\left(n_1 + \sigma\sqrt{2\pi}\right)^2}{(n_2-2)^2}\right],
\end{aligned} \tag{3.25}$$

respectively.

Proof. The sth raw moment about origin is given by

$$\begin{aligned}
\mu_s' = E(F^s) &= \iint_0^\infty \frac{re^{-r^2/2\sigma^2}}{\sigma^2} \frac{(n_1/n_2)^{n_1/2+r} F^{n_1/2+r+s-1}}{B(n_1/2+r, n_2/2)(1 + (n_1/n_2)F)^{(n_1+n_2)/2+r}} dr dF \\
&= \int_0^\infty \frac{re^{-r^2/2\sigma^2}}{\sigma^2} \frac{(n_1/n_2)^{n_1/2+r}}{B(n_1/2+r, n_2/2)} \int_0^\infty \frac{F^{n_1/2+r+s-1}}{(1 + (n_1/n_2)F)^{(n_1+n_2)/2\,+r}}\ dF dr \\
&= \left(\frac{n_2}{n_1}\right)^s \int_0^\infty \frac{re^{-r^2/2\sigma^2}}{\sigma^2} \frac{\Gamma(n_1/2 + r + s)\,\Gamma(n_2/2 - s)}{\Gamma(n_1/2 + r)\,\Gamma(n_2/2)} dr.
\end{aligned} \tag{3.26}$$

If we put $s = 1$, we get

$$
\begin{aligned}
\text{Mean} &= \mu_1' \\
&= \left(\frac{n_2}{n_1}\right) \int_0^\infty \frac{re^{-r^2/2\sigma^2}}{\sigma^2} \frac{\Gamma(n_1/2 + r + 1)\,\Gamma(n_2/2 - 1)}{\Gamma(n_1/2 + r)\,\Gamma(n_2/2)} dr \\
&= \frac{n_2}{n_1(n_2 - 2)} \left[n_1 + \sigma\sqrt{2\pi}\right].
\end{aligned}
\tag{3.27}
$$

Putting $s = 2$, we get

$$
\begin{aligned}
\mu_2' &= \left(\frac{n_2}{n_1}\right)^2 \int_0^\infty \frac{re^{-r^2/2\sigma^2}}{\sigma^2} \frac{\Gamma(n_1/2 + r + 2)\,\Gamma(n_2/2 - 2)}{\Gamma(n_1/2 + r)\,\Gamma(n_2/2)} dr \\
&= \left(\frac{n_2}{n_1}\right)^2 \frac{1}{(n_2 - 2)(n_2 - 4)} \left[n_1^2 + 2n_1 + 2(n_1 + 1)\ \sigma\sqrt{2\pi} + 8\sigma^2\right].
\end{aligned}
\tag{3.28}
$$

Then the variance,

$$
\begin{aligned}
\mu_2 &= \mu_2' - \left(\mu_1'\right)^2 \\
&= \left(\frac{n_2}{n_1}\right)^2 \left[\frac{n_1^2 + 2n_1 + 2(n_1 + 1)\ \sigma\sqrt{2\pi} + 8\sigma^2}{(n_2 - 2)(n_2 - 4)} - \frac{\left(n_1 + \sigma\sqrt{2\pi}\right)^2}{(n_2 - 2)^2}\right],
\end{aligned}
\tag{3.29}
$$

hence proved. □

Theorem 3.5. *If F follows a Rayleigh mixture of F-distributions having parameter σ^2 with degrees of freedom n_1 and n_2, respectively, then its characteristic function is given by*

$$
\Phi_F(t) = \int_0^\infty \frac{re^{-r^2/2\sigma^2}}{\sigma^2} \sum_{x=0}^{\infty} \frac{1}{x!} \cdot \left(\frac{n_2 it}{n_1}\right)^x \frac{\Gamma(n_1/2 + r + x)\,\Gamma(n_2/2 - x)}{\Gamma(n_1/2 + r)\,\Gamma(n_2/2)} dr. \tag{3.30}
$$

From here we may get the mean and variance of this distribution.

Proof. The characteristic function of the random variable F is given by

$$
\begin{aligned}
\Phi_F(t) &= E\left[e^{itF}\right] \\
&= \int_0^\infty e^{itF} \int_0^\infty \frac{re^{-r^2/2\sigma^2}}{\sigma^2} \frac{(n_1/n_2)^{n_1/2+r} F^{n_1/2+r-1}}{B(n_1/2 + r, n_2/2)(1 + (n_1/n_2)F)^{(n_1+n_2)/2+r}} dr\,dF \\
&= \int_0^\infty \frac{re^{-r^2/2\sigma^2}}{\sigma^2} \sum_{x=0}^{\infty} \frac{1}{x!} \cdot \left(\frac{n_2 it}{n_1}\right)^x \frac{\Gamma(n_1/2 + r + x)\,\Gamma(n_2/2 - x)}{\Gamma(n_1/2 + r)\,\Gamma(n_2/2)} dr.
\end{aligned}
\tag{3.31}
$$

Hence the sth raw moment about origin is

$$\begin{aligned}
\mu'_s &= \text{coefficient of } \frac{(it)^s}{s!} \text{ in } \Phi_F(t) \\
&= \left(\frac{n_2}{n_1}\right)^s \int_0^\infty \frac{re^{-r^2/2\sigma^2}}{\sigma^2} \frac{\overline{)n_1/2+r+s}\ \overline{)n_2/2-s}}{\overline{)n_1/2+r}\ \overline{)n_2/2}} dr.
\end{aligned} \tag{3.32}$$

If $s = 1$, then

$$\begin{aligned}
\mu'_1 &= \left(\frac{n_2}{n_1}\right) \int_0^\infty \frac{re^{-r^2/2\sigma^2}}{\sigma^2} \frac{\overline{)n_1/2+r+1}\ \overline{)n_2/2-1}}{\overline{)n_1/2+r}\ \overline{)n_2/2}} dr \\
&= \frac{n_2}{n_1(n_2-2)} \left[n_1 + \sigma\sqrt{2\pi}\right].
\end{aligned} \tag{3.33}$$

If we put $s = 2$, we get

$$\begin{aligned}
\mu'_2 &= \left(\frac{n_2}{n_1}\right)^2 \int_0^\infty \frac{re^{-r^2/2\sigma^2}}{\sigma^2} \frac{\overline{)n_1/2+r+2}\ \overline{)n_2/2-2}}{\overline{)n_1/2+r}\ \overline{)n_2/2}} dr \\
&= \left(\frac{n_2}{n_1}\right)^2 \frac{1}{(n_2-2)(n_2-4)} \left[n_1^2 + 2n_1 + 2(n_1+1)\ \sigma\sqrt{2\pi} + 8\sigma^2\right].
\end{aligned} \tag{3.34}$$

Therefore,

$$\begin{aligned}
\text{Variance: } \mu_2 &= \mu'_2 - \left(\mu'_1\right)^2 \\
&= \left(\frac{n_2}{n_1}\right)^2 \left[\frac{n_1^2 + 2n_1 + 2(n_1+1)\ \sigma\sqrt{2\pi} + 8\sigma^2}{(n_2-2)(n_2-4)} - \frac{\left(n_1 + \sigma\sqrt{2\pi}\right)^2}{(n_2-2)^2}\right].
\end{aligned} \tag{3.35}$$

Driving coefficient of Skewness = β_1 and coefficient of Kurtosis = β_2 is a tedious job; we have avoided the task here. □

The different moments of the random variable which is the resultant of the product of two correlated Rayleigh random variables are obtained by following theorem.

Theorem 3.6. *For $-1 \le \rho \le 1$, the (a,b)th product moment of the mixture of two correlated Rayleigh random variables is denoted by $\mu'(a,b;\rho)$ and given by*

$$\mu'(a,b;\rho) = \sigma_1^a \sigma_2^b \times \Gamma\left(\frac{a}{2}+1\right)\Gamma\left(\frac{b}{2}+1\right) \times \left[2^{a+b/2} + \rho\left(2^{a/2}-1\right)\left(2^{b/2}-1\right)\right]. \tag{3.36}$$

Proof. We know that

$$\begin{aligned}
\mu'(a,b;\rho) &= E\left(X^a Y^b\right) \\
&= \iint_0^\infty x^a y^b \times \frac{xy}{(\sigma_1\sigma_2)^2} \\
&\quad \times \left\{ e^{-(1/2)(x^2/\sigma_1^2+y^2/\sigma_2^2)} + \rho e^{-(1/2)(x^2/\sigma_1^2+y^2/\sigma_2^2)} - 2\rho e^{-(1/2)(2x^2/\sigma_1^2+y^2/\sigma_2^2)} \right. \\
&\qquad \left. -2\rho e^{-(1/2)(x^2/\sigma_1^2+2y^2/\sigma_2^2)} + 4\rho e^{-(x^2/\sigma_1^2+y^2/\sigma_2^2)} \right\} dx\,dy \\
&= \iint_0^\infty \frac{x^{a+1}y^{b+1}}{(\sigma_1\sigma_2)^2} \times e^{-(1/2)(x^2/\sigma_1^2+y^2/\sigma_2^2)} dx\,dy \\
&\quad + \rho \iint_0^\infty \frac{x^{a+1}y^{b+1}}{(\sigma_1\sigma_2)^2} \times e^{-(1/2)(x^2/\sigma_1^2+y^2/\sigma_2^2)} dx\,dy \\
&\quad - 2\rho \iint_0^\infty \frac{x^{a+1}y^{b+1}}{(\sigma_1\sigma_2)^2} \times e^{-(1/2)(2x^2/\sigma_1^2+y^2/\sigma_2^2)} dx\,dy \\
&\quad - 2\rho \iint_0^\infty \frac{x^{a+1}y^{b+1}}{(\sigma_1\sigma_2)^2} \times e^{-(1/2)(x^2/\sigma_1^2+2y^2/\sigma_2^2)} dx\,dy \\
&\quad + 4\rho \iint_0^\infty \frac{x^{a+1}y^{b+1}}{(\sigma_1\sigma_2)^2} \times e^{-(x^2/\sigma_1^2+y^2/\sigma_2^2)} dx\,dy.
\end{aligned} \tag{3.37}$$

Now taking the first integral from (3.37)

$$\iint_0^\infty \frac{x^{a+1}y^{b+1}}{(\sigma_1\sigma_2)^2} \times e^{-(1/2)(x^2/\sigma_1^2+y^2/\sigma_2^2)} dx\,dy \tag{3.38}$$

and making a transformation $p = (1/2)(x^2/\sigma_1^2)$ and $q = (1/2)(y^2/\sigma_2^2)$ we obtain the following result:

$$\left(\sigma_1\sqrt{2}\right)^a \int_0^\infty p^{a/2} e^{-p} dp \times \left(\sigma_2\sqrt{2}\right)^b \int_0^\infty q^{b/2} e^{-q} dq = 2^{(a+b)/2}\sigma_1^a\sigma_2^b \times \Gamma\left(\frac{a}{2}+1\right)\Gamma\left(\frac{b}{2}+1\right). \tag{3.39}$$

Using the similar mathematical simplification we get the following results for the 2nd integral

$$2^{(a+b)/2}\rho\sigma_1^a\sigma_2^b \times \Gamma\left(\frac{a}{2}+1\right)\Gamma\left(\frac{b}{2}+1\right), \tag{3.40}$$

for the 3rd integral

$$2^{b/2}\rho\sigma_1^a\sigma_2^b \times \Gamma\left(\frac{a}{2}+1\right)\Gamma\left(\frac{b}{2}+1\right), \tag{3.41}$$

for the 4th integral

$$2^{a/2}\rho\sigma_1^a\sigma_2^b \times \Gamma\left(\frac{a}{2}+1\right)\Gamma\left(\frac{b}{2}+1\right), \tag{3.42}$$

for the 5th integral

$$\rho\sigma_1^a\sigma_2^b \times \Gamma\left(\frac{a}{2}+1\right)\Gamma\left(\frac{b}{2}+1\right). \tag{3.43}$$

Now putting all of these values in (3.37) and simplifying the result we get our desired result stated in the theorem.

Special findings of the above theorem if $\rho = 0$ then the product moment of the two correlated Rayleigh variables is nothing but the product of ath and bth moments of two independent Rayleigh variables. In such case the product moment is as follows:

$$\begin{aligned} E\left(X^aY^b\right) &= 2^{(a+b)/2} \times \sigma_1^a\sigma_2^b \times \Gamma\left(\frac{a}{2}+1\right)\Gamma\left(\frac{b}{2}+1\right) \\ &= 2^{a/2}\sigma_1^a\Gamma\left(\frac{a}{2}+1\right) \times 2^{b/2}\sigma_2^b\Gamma\left(\frac{b}{2}+1\right) \\ &= E(X^a) \cdot E\left(Y^b\right). \end{aligned} \tag{3.44}$$

□

Theorem 3.7. *If X and Y are two correlated Rayleigh variates having joint density given in* (3.6)*, then probability density function of $W = X/Y$ is given by*

$$\begin{aligned} f(w;\sigma_1,\sigma_2;\rho) = &\sqrt{\frac{\pi}{2}} \times w^2\sigma_1\sigma_2 \\ &\times \left[\left\{1+\rho\left(1+\sqrt{2}\right)\right\}\left(w^2\sigma_2^2+\sigma_1^2\right)^{-(3/2)}\right. \\ &\left. -2\rho\left\{\left(2w^2\sigma_2^2+\sigma_1^2\right)^{-(3/2)}+\left(w^2\sigma_2^2+2\sigma_1^2\right)^{-(3/2)}\right\}\right], \end{aligned} \tag{3.45}$$

where, $w > 0$; $\sigma_1 > 0$, $\sigma_2 > 0$ and $-1 \le \rho < 1$.

Proof. Under the transformation $x = z$, $y = z/w$ in (3.6) with the Jacobean

$$J((x,y) \longrightarrow (w,z)) = \frac{w^2}{z}, \tag{3.46}$$

the pdf of w and z is given by

$$\begin{aligned} f(w,z) = \frac{z^2}{w(\sigma_1\sigma_2)^2}&\left\{(1+\rho)e^{-z^2/2(1/\sigma_1^2+1/w^2\sigma_2^2)} - 2\rho e^{-z^2/2(2/\sigma_1^2+1/w^2\sigma_2^2)}\right. \\ &\left. -2\rho e^{-z^2/2(1/\sigma_1^2+2/w^2\sigma_2^2)} + 4\rho e^{-z^2(1/\sigma_1^2+1/w^2\sigma_2^2)}\right\}. \end{aligned} \tag{3.47}$$

Now integrating over z we get the marginal distribution of w as

$$f(w) = \frac{(1+\rho)}{w(\sigma_1\sigma_2)^2}\int_0^\infty z^2 e^{-z^2/2(1/\sigma_1^2+1/w^2\sigma_2^2)}dz - \frac{2\rho}{w(\sigma_1\sigma_2)^2}\int_0^\infty z^2 e^{-z^2/2(2/\sigma_1^2+1/w^2\sigma_2^2)}dz - \frac{2\rho}{w(\sigma_1\sigma_2)^2}\int_0^\infty z^2 e^{-z^2/2(1/\sigma_1^2+2/w^2\sigma_2^2)}dz + \frac{4\rho}{w(\sigma_1\sigma_2)^2}\int_0^\infty z^2 e^{-z^2(1/\sigma_1^2+1/w^2\sigma_2^2)}dz. \tag{3.48}$$

Taking each of the integral separately and making transformation

$$\begin{aligned} z &= \sqrt{2p}\left(\frac{1}{\sigma_1^2}+\frac{1}{w^2\sigma_2^2}\right)^{-1/2} \quad \text{for the first integral,} \\ z &= \sqrt{2p}\left(\frac{2}{\sigma_1^2}+\frac{1}{w^2\sigma_2^2}\right)^{-1/2} \quad \text{for the second integral,} \\ z &= \sqrt{2p}\left(\frac{1}{\sigma_1^2}+\frac{2}{w^2\sigma_2^2}\right)^{-1/2} \quad \text{for the third integral,} \\ z &= \sqrt{p}\left(\frac{1}{\sigma_1^2}+\frac{1}{w^2\sigma_2^2}\right)^{-1/2} \quad \text{for the fourth integral.} \end{aligned} \tag{3.49}$$

We have got the following results:

$$\begin{aligned} &\sqrt{\frac{\pi}{2}}\left(\frac{1}{\sigma_1^2}+\frac{1}{w^2\sigma_2^2}\right)^{-3/2}, \qquad \sqrt{\frac{\pi}{2}}\left(\frac{2}{\sigma_1^2}+\frac{1}{w^2\sigma_2^2}\right)^{-3/2}, \\ &\sqrt{\frac{\pi}{2}}\left(\frac{1}{\sigma_1^2}+\frac{2}{w^2\sigma_2^2}\right)^{-3/2}, \qquad \frac{\sqrt{\pi}}{4}\left(\frac{1}{\sigma_1^2}+\frac{1}{w^2\sigma_2^2}\right)^{-3/2} \end{aligned} \tag{3.50}$$

for the first, second, third, and fourth integration, respectively. □

Combining all of the obtained results for the integrals in (3.48) we achieve the result stat-
ed as in the theorem.

Theorem 3.8. *For nonnegative integer a and $-1<\rho<1$ the ath moment for $W = X/Y$ is*

$$E(W^a) = \frac{1}{\sigma_2\sqrt{2}}\left(\frac{\sigma_1}{\sigma_2}\right)^{a+1} \times \Gamma\left(\frac{a+3}{2}\right)\Gamma\left(-\frac{a}{2}\right) \times \left[1+\rho\left\{1-2\sqrt{2}\left(1+2^{a+2}\right)\right\}\right]. \tag{3.51}$$

Proof. According to definition of expectation we can obtain the following result for the ath moment:

$$E(W^a) = \int_0^\infty w^{a+2} \times \sqrt{\frac{\pi}{2}} \left[\left\{ 1 + \rho\left(1 + \sqrt{2}\right) \right\} \left(w^2\sigma_2^2 + \sigma_1^2 \right)^{-3/2} \right.$$
$$\left. -2\rho \left\{ \left(2w^2\sigma_2^2 + \sigma_1^2 \right)^{-3/2} + \left(w^2\sigma_2^2 + 2\sigma_1^2 \right)^{-3/2} \right\} \right] dw. \tag{3.52}$$

By making transformation $w = (\sigma_1/\sigma_2)\sqrt{m}$, $w = (\sigma_1/\sigma_2)\sqrt{m/2}$ and $w = (\sigma_1/\sigma_2)\sqrt{2m}$ for the first, second, and third parts of the component containing the integral we have

$$E(W^a) = \frac{\sqrt{\pi}}{2^{3/2}} \times \frac{\sigma_1^{a+1}}{\sigma_2^{a+2}} \left\{ 1 + \rho\left(1 + \sqrt{2}\right) \right\} \times B\left(\frac{a+3}{2}, -\frac{a}{2} \right)$$
$$- \frac{\rho\sqrt{\pi}}{2^{(a+4)/2}} \times \frac{\sigma_1^{a+1}}{\sigma_2^{a+2}} \left\{ 1 + \rho\left(1 + \sqrt{2}\right) \right\} \times B\left(\frac{a+3}{2}, -\frac{a}{2} \right) \tag{3.53}$$
$$- \frac{\rho\sqrt{\pi}}{2^{-(a+2)/2}} \times \frac{\sigma_1^{a+1}}{\sigma_2^{a+2}} \left\{ 1 + \rho\left(1 + \sqrt{2}\right) \right\} \times B\left(\frac{a+3}{2}, -\frac{a}{2} \right).$$

Simplifying this we have the stated result of the theorem. □

Theorem 3.9. *The moment generating function of W is*

$$M_W(t) = \sum_{a=o}^{\infty} \frac{t^a}{a!} \times \frac{1}{\sigma_2\sqrt{2}} \left(\frac{\sigma_1}{\sigma_2} \right)^{a+1}$$
$$\times \Gamma\left(\frac{a+3}{2} \right) \Gamma\left(-\frac{a}{2} \right) \times \left[1 + \rho\left\{ 1 - 2\sqrt{2}\left(1 + 2^{a+2} \right) \right\} \right]. \tag{3.54}$$

Proof. The moment generating function of V at t is given by

$$M_W(t) = E\left(e^{tw}\right)$$
$$= \sum_{a=0}^{\infty} \frac{t^a}{a!} E(W^a) \tag{3.55}$$
$$= \sum_{a=o}^{\infty} \frac{t^a}{a!} \times \frac{1}{\sigma_2\sqrt{2}} \left(\frac{\sigma_1}{\sigma_2} \right)^{a+1} \times \Gamma\left(\frac{a+3}{2} \right) \Gamma\left(-\frac{a}{2} \right) \times \left[1 + \rho\left\{ 1 - 2\sqrt{2}\left(1 + 2^{a+2} \right) \right\} \right].$$

□

3.3. *Parameter Estimation of Rayleigh Mixture Distribution*

We know the well-known method of the maximum likelihood estimation is very complicated for the parameter estimation of mixture distribution and method of moment is very suitable in these cases. Hence, we used method of moments (MoMs) estimation of technique for estimation of the parameter of the Rayleigh mixture distribution.

3.3.1. Parameter Estimation of Rayleigh Mixture of Chi-Square Distribution

Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution defined in (3.2) where the parameter σ^2 is unknown.

The first sample raw moment is

$$m_1' = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X} \quad (\text{say}). \tag{3.56}$$

And as we already got

$$\mu_1' = n + \sigma\sqrt{2\pi}. \tag{3.57}$$

Hence

$$n + \sigma\sqrt{2\pi} = \overline{X}. \tag{3.58}$$

Therefore,

$$\hat{\sigma}^2 = \frac{\left(\overline{X} - n\right)^2}{2\pi}. \tag{3.59}$$

3.3.2. Parameter Estimation of Rayleigh Mixture of t-Distribution

Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution defined in (3.3) where the para-
meter σ^2 is unknown. We want to estimate this parameter by method of moment.

The second sample raw moment is obtained as

$$m_2' = \frac{1}{n}\sum_{i=1}^{n} X_i^2 = S^2 \quad (\text{say}). \tag{3.60}$$

We have already found

$$\mu_2' = \mu_2 = \frac{n}{n-2}\left[1 + \sigma\sqrt{2\pi}\right], \quad n > 2. \tag{3.61}$$

Hence, by the method of moments, we get

$$\frac{n}{n-2}\left[1 + \sigma\sqrt{2\pi}\right] = S^2, \quad n > 2. \tag{3.62}$$

Therefore,

$$\hat{\sigma}^2 = \frac{1}{2\pi}\left[S^2\frac{(n-2)}{n} - 1\right]^2. \tag{3.63}$$

3.3.3. Parameter Estimation of Mixture of F-Distribution

Let $X_1,\ X_2,\ \ldots,\ X_n$ be a random sample from the distribution as specified in (3.4) where the parameter σ^2 is unknown. We want to estimate this parameter by method of moments.

The first sample raw moment is

$$m_1' = \frac{1}{n}\sum_{i=1}^{n} \mathrm{X}_i = \overline{\mathrm{X}} \quad (\text{say}). \tag{3.64}$$

And we already get

$$\mu_1' = \frac{n_2}{n_1(n_2-2)}\left[n_1 + \sigma\sqrt{2\pi}\right]. \tag{3.65}$$

Hence, from the method of moments estimator is

$$\frac{n_2}{n_1(n_2-2)}\left[n_1 + \sigma\sqrt{2\pi}\right] = \overline{\mathrm{X}}. \tag{3.66}$$

Therefore,

$$\hat{\sigma}^2 = \left(\frac{1}{2\pi}\right)\left(\frac{n_1}{n_2}\right)^2\left[\overline{\mathrm{X}}(n_2-2) - n_1\right]^2. \tag{3.67}$$

4. Concluding Remarks

In this paper, we have presented the Rayleigh mixtures of distributions in which the weight functions are assumed to be chi-square, t- and F-distributions, and the mixture of two correlated Rayleigh distributions has been presented. The moments, characteristic function and shape characteristics of these mixtures distributions have also been studied. The Rayleigh distribution is frequently used to model wave heights in oceanography and in communication theory to describe hourly median and instantaneous peak power of received radio signals. It could also be used to model the frequency of different wind speeds over a year at wind turbine sites. The Rayleigh mixture of sampling distribution may be used in the similar nature but with some additional informative environment. Suppose we want to know the distribution of the average fish caught by fisherman in the Bay of Bengal of a particular day. Fishing depends on height of the wave and wind speed in that zone. As we know the average amount fish catch by the fisherman depends on the weather of the Sea. If the wave heights are very high the fishermen are prohibited to go to the sea for fishing if it is not so much dangerous but still the sea is unstable they are asked to be very careful during fishing. This means that average amount of fishing and standard deviation of the amount fish catch by the fishermen varies based on heights of the wave. The distribution of wave heights follows Rayleigh distribution and distribution of average catch fish at a normal situation follows t-distribution but at the Bay of Bangle it is seriously affected by height of wave; hence the average number of fish catch at the Bay of Bangle will follow Rayleigh mixture of t-distribution. Similarly, the distribution of the variability of the number of fishes catch by the fishermen at the Bay of Bangle follows Rayleigh mixture of chi-square distribution. We hope the findings of the paper will be useful for the practitioners that have been mentioned above.

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