Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2011, Article ID 198469, 18 pages
doi:10.1155/2011/198469

Research Article

Valuing Options in Heston's Stochastic Volatility Model: Another Analytical Approach

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Received 21 April 2011; Accepted 27 July 2011

Academic Editor: Julián López-Gómez

We are concerned with the valuation of European options in the Heston stochastic volatility model with correlation. Based on Mellin transforms, we present new solutions for the price of European options and hedging parameters. In contrast to Fourier-based approaches, where the transformation variable is usually the log-stock price at maturity, our framework focuses on directly transforming the current stock price. Our solution has the nice feature that it requires only a single integration. We make numerical tests to compare our results with Heston's solution based on Fourier inversion and investigate the accuracy of the derived pricing formulae.

1. Introduction

The pricing methodology proposed by Black and Scholes [1] and Merton [2] is maybe the most significant and influential development in option pricing theory. However, the assumptions underlying the original works were questioned ab initio and became the subject of a wide theoretical and empirical study. Soon it became clear that extensions are necessary to fit the empirical data. The main drawback in the original Black/Scholes/Merton (BSM) model is the assumption of a constant volatility.

To reflect the empirical evidence of a nonconstant volatility and to explain the so-called volatility smile, different approaches were developed. Dupire [3] applies a partial differential equation (PDE) method and assumes that volatility dynamics can be modeled as a deterministic function of the stock price and time.

A different approach is proposed by Sircar and Papanicolaou [4]. Based on the PDE framework, they develop a methodology that is independent of a particular volatility process.

The result is an asymptotic approximation consisting of a BSM-like price plus a Gaussian variable capturing the risk from the volatility component.

The majority of the financial community, however, focuses on stochastic volatility models. These models assume that volatility itself is a random process and fluctuates over time. Stochastic volatility models were first studied by Johnson and Shanno [5], Hull and White [6], Scott [7], and Wiggins [8]. Other models for the volatility dynamics were proposed by E. Stein and J. Stein [9], Heston [10], Schöbel and Zhu [11], and Rogers and Veraart [12]. In all these models the stochastic process governing the asset price dynamics is driven by a subordinated stochastic volatility process that may or may not be independent.

While the early models could not produce closed-form formulae, it was E. Stein and J. Stein [9] (S&S) who first succeeded in deriving an analytical solution. Assuming that volatility follows a mean reverting Ornstein-Uhlenbeck process and is uncorrelated with asset returns, they present an analytic expression for the density function of asset returns for the purpose of option valuation. Schöbel and Zhu [11] generalize the S&S model to the case of nonzero correlation between instantaneous volatilities and asset returns. They present a closed-form solution for European options and discuss additional features of the volatility dynamics.

The maybe most popular stochastic volatility model was introduced by Heston [10]. In his influential paper he presents a new approach for a closed-form valuation of options specifying the dynamics of the squared volatility (variance) as a square-root process and applying Fourier inversion techniques for the pricing procedure. The characteristic function approach turned out to be a very powerful tool. As a natural consequence it became standard in option pricing theory and was refined and extended in various directions (Bates [13], Carr and Madan [14], Bakshi and Madan [15], Lewis [16], Lee [17], Kahl and Jäckel [18], Kruse and Nögel [19], Fahrner [20], or Lord and Kahl [21] among others). See also the study by Duffie et al. [22, 23] for the mathematical foundations of affine processes.

Beside Fourier and Laplace transforms, there are other interesting integral transforms used in theoretical and applied mathematics. Specifically, the Mellin transform gained great popularity in complex analysis and analytic number theory for its applications to problems related to the Gamma function, the Riemann zeta function, and other Dirichlet series. Its applicability to problems arising in finance theory has not been studied much yet [24, 25]. Panini and Srivastav introduce in [25] Mellin transforms in the theory of option pricing and use the new approach to value European and American plain vanilla and basket options on non-dividend paying stocks. The approach is extended in [24] to power options with a nonlinear payoff and American options written on dividend paying assets. The purpose of this paper is to show how the framework can be extended to the stochastic volatility problem. We derive an equivalent representation of the solution and discuss its interesting features.

The paper is structured as follows. In Section 2 we give a formulation of the pricing problem for European options in the square-root stochastic volatility model. Based on Mellin transforms, the solution for puts is presented in Section 3. Section 4 is devoted to further analysis of our new solution. We provide a direct connection to Heston's pricing formula and give closed-form expressions for hedging parameters. Also, an explicit solution for European calls is presented. Numerical calculations are made in Section 5. We test the accuracy of our closed-form solutions for a variety of parameter combinations. Section 6 concludes this paper.

2. Problem Statement

Let $S(t) = S_t$ be the price of a dividend paying stock at time t and V_t its instantaneous variance. Following Heston we assume that the risk neutral dynamics of the asset price are governed by the system of stochastic differential equations (SDEs):

$$\begin{aligned} dS_t &= (r-q)S_t dt + \sqrt{V_t}\, S_t\, dW_t, \\ dV_t &= \kappa(\theta - V_t)dt + \xi\sqrt{V_t}\, dZ_t, \end{aligned} \tag{2.1}$$

with initial values $S_0, V_0 \in (0, \infty)$ and where $r, q, \kappa, \theta, \xi > 0$. The parameter r is the risk-free interest rate, and q is the dividend yield. Both are assumed to be constant over time. κ is the speed of mean reversion to the mean reversion level θ, and ξ is the so-called volatility of volatility. W_t and Z_t are two correlated Brownian motions with $dW_t dZ_t = \rho dt$, where $\rho \in (-1, 1)$ is the correlation coefficient. The Feller condition $\kappa\theta > (1/2)\xi^2$ guarantees that the variance process never reaches zero and always stays positive. For practical uses it is also worth mentioning that in most cases the correlation coefficient ρ is negative. This means that an up move in the asset is normally accompanied by a down move in volatility.

Let $P^E(S, V, t)$ be the current price of a European put option with strike price X and maturity T. The option guarantees its holder a terminal payoff given by

$$P^E(S, V, T) = \max(X - S(T), 0). \tag{2.2}$$

Using arbitrage arguments it is straightforward to derive a two-dimensional partial differential equation (PDE) that must be satisfied by any derivative F written on S and V:

$$F_t + (r-q)SF_S + \frac{1}{2}VS^2F_{SS} + \left(\kappa(\theta - V) - \lambda\xi\sqrt{V}\right)F_V + \frac{1}{2}\xi^2 VF_{VV} + \rho\xi VSF_{SV} - rF = 0, \tag{2.3}$$

on $0 < S, V < \infty$, $0 < t < T$ (throughout this paper partial derivatives with respect to the underlying variables will be denoted by subscripts) (see [16]). λ is called the market price of volatility risk. Heston provides some reasons for the assumption that λ is proportional to volatility, that is, $\lambda = k\sqrt{V}$ for some constant k. Therefore, $\lambda\xi\sqrt{V} = k\xi V = \lambda^* V$ (say). Hence, without loss of generality, λ can be set to zero as has been done in [26, 27]. For a constant volatility the two-dimensional PDE reduces to the fundamental PDE due to Black/Scholes and Merton and admits a closed-form solution given by the celebrated BSM formula. If F is a European put option, that is, $F(S, V, t) = P^E(S, V, t)$, then we have

$$P_t^E + (r-q)SP_S^E + \frac{1}{2}VS^2P_{SS}^E + \kappa(\theta - V)P_V^E + \frac{1}{2}\xi^2 VP_{VV}^E + \rho\xi VSP_{SV}^E - rP^E = 0, \tag{2.4}$$

where $P^E(S,V,t) : R^+ \times R^+ \times [0,T] \rightarrow R^+$. The boundary conditions are given by

$$P^E(S,V,T) = \max(X - S(T), 0),$$
$$P^E(0,V,t) = Xe^{-r(T-t)},$$
$$P^E(S,0,t) = \max\left(Xe^{-r(T-t)} - S(t)e^{-q(T-t)}, 0\right), \tag{2.5}$$
$$\lim_{S\to\infty} P^E(S,V,t) = 0,$$
$$\lim_{V\to\infty} P^E(S,V,t) = Xe^{-r(T-t)}. \tag{2.6}$$

The first condition is the terminal condition. It specifies the final payoff of the option. The second condition states that for a stock price of zero the put price must equal the discounted strike price. The third condition specifies the payoff for a variance (volatility) of zero. In this case the underlying asset evolves completely deterministically and the put price equals its lower bound derived by arbitrage considerations. The next condition describes the option's price for ever-increasing asset prices. Obviously, since a put option gives its holder the right to sell the asset the price will tend to zero if S tends to infinity. Finally, notice that if variance (volatility) becomes infinite the current asset price contains no information about the terminal payoff of the derivative security, except that the put entitles its holder to sell the asset for X. In this case the put price must equal the discounted strike price, that is, its upper bound, again derived by arbitrage arguments.

In a similar manner the European call option pricing problem with solution $C^E(S,V,t)$ is characterized as the unique solution of (2.4) subject to

$$C^E(S,V,T) = \max(S(T) - X, 0),$$
$$C^E(0,V,t) = 0,$$
$$C^E(S,0,t) = \max\left(S(t)e^{-q(T-t)} - Xe^{-r(T-t)}, 0\right), \tag{2.7}$$
$$\lim_{S\to\infty} C^E(S,V,t) = \infty,$$
$$\lim_{V\to\infty} C^E(S,V,t) = S(t)e^{-q(T-t)}.$$

3. Analytic Solution Using Mellin Transforms

The objective of this section is to solve (2.4) subject to (2.5)–(2.6) in (semi) closed form. The derivation of a solution is based on Mellin transforms. For a locally Lebesgue integrable function $f(x)$, $x \in R^+$, the Mellin transform $M(f(x),\omega)$, $\omega \in C$, is defined by

$$M(f(x),\omega) := \tilde{f}(\omega) = \int_0^\infty f(x)\ x^{\omega-1}\, dx. \tag{3.1}$$

As a complex function the Mellin transform is defined on a vertical strip in the ω-plane, whose boundaries are specified by the asymptotic behavior of the function $f(x)$ as $x \rightarrow 0^+$

and $x \to \infty$ (Fourier transforms (at least those which are typical in option pricing) usually exist in horizontal strips of the complex plane. This is the key conceptual difference between the two frameworks). For conditions that guarantee the existence and the connection to Fourier and Laplace transforms, see [28] or [29]. Conversely, if $\tilde{f}(\omega)$ is a continuous, integrable function with fundamental strip (a, b), then, if c is such that $a < c < b$ and $\tilde{f}(c + it)$ is integrable, the inverse of the Mellin transform is given by

$$f(x) = M^{-1}\left(\tilde{f}(\omega)\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(\omega)x^{-\omega}d\omega. \tag{3.2}$$

Let $\tilde{P}^E = \tilde{P}^E(\omega, V, t)$ denote the Mellin transform of $P^E(S, V, t)$. It is easily verified that $\tilde{P}^E$ exists in the entire half plane with $\mathrm{Re}(\omega) > 0$, where $\mathrm{Re}(\omega)$ denotes the real part of ω. A straightforward application to (2.4) gives

$$\tilde{P}^E_t + (a_1V + b_1)\tilde{P}^E_V + (a_2V + b_2)\tilde{P}^E_{VV} + (a_0V + b_0)\tilde{P}^E = 0, \tag{3.3}$$

where

$$\begin{aligned} a_1 &= -(\omega\rho\xi + \kappa), \qquad b_1 = \kappa\theta, \\ a_2 &= \frac{1}{2}\xi^2, \qquad b_2 = 0, \\ a_0 &= \frac{1}{2}\omega(\omega + 1), \qquad b_0 = q\omega - r(\omega + 1). \end{aligned} \tag{3.4}$$

This is a one-dimensional PDE in the complex plane with nonconstant coefficients. To provide a unique solution for $0 < V < \infty,\ 0 < t < T$, we need to incorporate the boundary conditions from the previous section. The transformed terminal and boundary conditions are given by, respectively,

$$\tilde{P}^E(\omega, V, T) = X^{\omega+1}\left(\frac{1}{\omega} - \frac{1}{\omega + 1}\right), \tag{3.5}$$

$$\tilde{P}^E(\omega, 0, t) = e^{(q\omega - r(\omega+1))(T-t)} \cdot X^{\omega+1}\left(\frac{1}{\omega} - \frac{1}{\omega + 1}\right), \tag{3.6}$$

and condition (2.6) becomes

$$\lim_{V\to\infty}\left|\tilde{P}^E(\omega, V, t)\right| = \infty. \tag{3.7}$$

Now, we change the time variable from t to $\tau = T - t$ and convert the backward in time PDE into a forward in time PDE with solution domain $0 < V,\ \tau < \infty$. With $\tilde{P}^E(\omega, V, t) = \tilde{P}^E(\omega, V, \tau)$, the resulting equation is

$$-\tilde{P}^E_\tau + (a_1V + b_1)\tilde{P}^E_V + (a_2V + b_2)\tilde{P}^E_{VV} + (a_0V + b_0)\tilde{P}^E = 0, \tag{3.8}$$

where the coefficients a_0, a_1, a_2, b_0, b_1, and b_2 are given in (3.4) and the terminal condition (3.5) becomes an initial condition

$$\widetilde{P}^E(\omega, V, 0) = X^{\omega+1}\left(\frac{1}{\omega} - \frac{1}{\omega+1}\right). \tag{3.9}$$

Additionally we have

$$\begin{gathered} \widetilde{P}^E(\omega, 0, \tau) = e^{(q\omega - r(\omega+1))\tau} \cdot X^{\omega+1}\left(\frac{1}{\omega} - \frac{1}{\omega+1}\right), \\ \lim_{V\to\infty}\left|\widetilde{P}^E(\omega, V, \tau)\right| = \infty. \end{gathered} \tag{3.10}$$

To simplify the PDE (3.8) further, we assume that the solution $\widetilde{P}^E(\omega, V, \tau)$ can be written in the form

$$\widetilde{P}^E(\omega, V, \tau) = e^{(q\omega - r(\omega+1))\tau} \cdot h(\omega, V, \tau) \tag{3.11}$$

with an appropriate function $h(\omega, V, \tau)$. It follows that h must satisfy

$$-h_\tau + (a_1 V + b_1)h_V + a_2 V h_{VV} + a_0 V h = 0, \tag{3.12}$$

with initial and boundary conditions

$$\begin{gathered} h(\omega, V, 0) = X^{\omega+1}\left(\frac{1}{\omega} - \frac{1}{\omega+1}\right), \\ h(\omega, 0, \tau) = X^{\omega+1}\left(\frac{1}{\omega} - \frac{1}{\omega+1}\right), \\ \lim_{V\to\infty}|h(\omega, V, \tau)| = \infty. \end{gathered} \tag{3.13}$$

Observe that, for $\kappa = \theta = \xi = 0$, that is, if the stock price dynamics are given by the standard BSM model with constant volatility, the PDE for h is solved as

$$h(\omega, V, \tau) = X^{\omega+1}\left(\frac{1}{\omega} - \frac{1}{\omega+1}\right)e^{(1/2)\omega(\omega+1)V\tau}. \tag{3.14}$$

In this case the equation for $\widetilde{P}^E(\omega, V, \tau)$ becomes

$$\widetilde{P}^E(\omega, V, \tau) = X^{\omega+1}\left(\frac{1}{\omega} - \frac{1}{\omega+1}\right)e^{((1/2)\omega(\omega+1)V + q\omega - r(\omega+1))\tau}, \tag{3.15}$$

and the price of a European put option can be expressed as

$$P^E(S,V,\tau) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{P}^E(\omega,V,\tau)S^{-\omega}\,d\omega, \tag{3.16}$$

with $0 < c < \infty$. In [24] it is shown that the last equation is equivalent to the BSM formula for European put options.

The final step in deriving a general solution for h or equivalently for $\widetilde{P}^E$ for a nonconstant volatility is to assume the following functional form of the solution:

$$h(\omega,V,\tau) = \widetilde{c}\cdot H(\omega,\tau)\cdot e^{G(\omega,\tau)\cdot a_0\cdot V}, \tag{3.17}$$

with $H(\omega,0) = 1$, $G(\omega,0) = 0$ and where we have set

$$\widetilde{c} = X^{\omega+1}\left(\frac{1}{\omega} - \frac{1}{\omega+1}\right). \tag{3.18}$$

Inserting the functional form for h in (3.12), determining the partial derivatives, and simplifying yield two ordinary differential equations (ODEs). We have

$$G_\tau(\omega,\tau) = A\cdot G^2(\omega,\tau) + B\cdot G(\omega,\tau) + C, \tag{3.19}$$

$$H_\tau(\omega,\tau) = a_0\cdot b_1\cdot G(\omega,\tau)\cdot H(\omega,\tau), \tag{3.20}$$

where $A = a_0 a_2$, $B = a_1$, and $C = 1$. The ODE for $G(\omega,\tau)$ is identified as a Riccati equation with constant coefficients. These types of equations also appear in frameworks based on Fourier transforms (see [10, 11, 13], among others). Having solved for G, a straightforward calculation shows that $H(\omega,\tau)$ equals

$$H(\omega,\tau) = e^{a_0 b_1 \int_0^\tau G(\omega,x)dx}. \tag{3.21}$$

Thus, we first present the solution for G. The transformation

$$G(\omega,\tau) = \frac{1}{A}u(\omega,\tau) - \frac{B}{2A} \tag{3.22}$$

gives

$$u_\tau(\omega,\tau) = u^2(\omega,\tau) + \frac{4AC - B^2}{4}. \tag{3.23}$$

Note that this is a special case of the more general class of ODEs given by

$$u_\tau(\omega,\tau) = au^2(\omega,\tau) + b\tau^n, \tag{3.24}$$

with $n \in N$ and a and b constants. This class of ODEs has solutions of the form

$$u(\omega,\tau) = -\frac{1}{a}\frac{F_\tau(\omega,\tau)}{F(\omega,\tau)}, \tag{3.25}$$

where

$$F(\omega,\tau) = \sqrt{\tau}\left(c_1 J_{1/2m}\left(\frac{1}{m}\sqrt{ab}\,\tau^m\right) + c_2 Y_{1/2m}\left(\frac{1}{m}\sqrt{ab}\,\tau^m\right)\right). \tag{3.26}$$

The parameters c_1, c_2 are again constants depending on the underlying boundary conditions, $m = (1/2)(n+2)$, and J and Y are Bessel functions of the first and second kind, respectively. See [30] for a reference. Setting $m = 1$ and incorporating the boundary conditions, $u(\omega,\tau)$ is solved as

$$u(\omega,\tau) = \frac{k}{2}\frac{\tan((1/2)k\tau) + B/k}{1-(B/k)\tan((1/2)k\tau)}, \tag{3.27}$$

where we have set

$$k = k(\omega) = \sqrt{4AC - B^2} = \sqrt{\xi^2\omega(\omega+1) - (\omega\rho\xi+\kappa)^2}. \tag{3.28}$$

Thus, we immediately get

$$\begin{aligned} G(\omega,\tau) &= -\frac{B}{2A} + \frac{k}{2A}\frac{\tan((1/2)k\tau) + B/k}{1-(B/k)\tan((1/2)k\tau)} \\ &= -\frac{B}{2A} + \frac{k}{2A}\,\frac{k\sin((1/2)k\tau) + B\cos((1/2)k\tau)}{k\cos((1/2)k\tau) - B\sin((1/2)k\tau)}. \end{aligned} \tag{3.29}$$

Using $k^2 + B^2 = 4A$, it is easily verified that an equivalent expression for $G(\omega,\tau)$ equals

$$G(\omega,\tau) = \frac{2\sin((1/2)k\tau)}{k\cos((1/2)k\tau) + (\omega\rho\xi+\kappa)\sin((1/2)k\tau)} \tag{3.30}$$

with $k = k(\omega)$ from above. To solve for $H(\omega,\tau)$ we first mention that (see [31])

$$\int \frac{B\cos x + C\sin x}{b\cos + c\sin x}dx = \frac{Bc - Cb}{b^2+c^2}\ln(b\cos x + c\sin x) + \frac{Bb+Cc}{b^2+c^2}x. \tag{3.31}$$

Therefore,

$$\int_0^\tau G(\omega,x)dx = -\frac{B\tau}{2A} + \frac{1}{A}\ln\left(\frac{k}{k\cos((1/2)k\tau) - B\sin((1/2)k\tau)}\right), \tag{3.32}$$

$$H(\omega,\tau) = e^{(\kappa\theta/\xi^2)[(\omega\rho\xi+\kappa)\tau + 2\ln(k/k\cos((1/2)k\tau) + (\omega\rho\xi+\kappa)\sin((1/2)k\tau))]}. \tag{3.33}$$

Finally, we have arrived at the following result.

Theorem 3.1. *A new Mellin-type pricing formula for European put options in Heston's [10] mean reverting stochastic volatility model is given by*

$$P^E(S,V,\tau) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{P}^E(\omega,V,\tau)S^{-\omega}d\omega, \tag{3.34}$$

with $0 < c < c^*$ *and where*

$$\widetilde{P}^E(\omega,V,\tau) = \widetilde{c}\cdot e^{(q\omega - r(\omega+1))\tau}\cdot H(\omega,\tau)\cdot e^{G(\omega,\tau)a_0 V} \tag{3.35}$$

with $G(\omega,\tau)$ *and* $H(\omega,\tau)$ *from above. The parameters* $\widetilde{c}$ *and* k *are given in* (3.18) *and* (3.28)*, respectively. The choice of* c^* *will be commented on below.*

Remark 3.2. Note that similar to Carr and Madan [14] the final pricing formula only requires a single integration.

We now consider the issue of specifying c^*. Recall that, to guarantee the existence of the inverse Mellin transform of $\widetilde{P}^E(\omega,V,\tau)$ in a vertical strip of the ω-plane, we need $\widetilde{P}^E(c + iy, V,\tau)$ to be integrable, and hence analytic. From (3.30) and (3.33) we have that $G(\omega,\tau)$ and $H(\omega,\tau)$ have the same points of singularity with

$$\begin{aligned}
\lim_{\omega\to 0} G(\omega,\tau) &= \frac{2\sin((1/2)i\kappa\tau)}{i\kappa\cos((1/2)i\kappa\tau) + \kappa\sin((1/2)i\kappa\tau)} \\
&= \frac{2}{i\kappa}\sin\left(\frac{1}{2}i\kappa\tau\right)e^{(1/2)\kappa\tau} \\
&= \frac{1-e^{-\kappa\tau}}{\kappa}, \\
&\lim_{\omega\to 0} H(\omega,\tau) = 1.
\end{aligned} \tag{3.36}$$

Furthermore, since

$$k(\omega) = \sqrt{\xi^2\omega^2(1-\rho^2) + \omega(\xi^2 - 2\rho\xi\kappa) - \kappa^2}, \tag{3.37}$$

it follows that $k(\omega)$ has two real roots given by

$$\omega_{1/2} = \frac{-(\xi - 2\rho\kappa) \pm \sqrt{(\xi-2\rho\kappa)^2 + 4\kappa^2(1-\rho^2)}}{2\xi(1-\rho^2)}, \tag{3.38}$$

where $\rho \neq \pm 1$ and where only the positive root ω_1 is of relevance. For $\rho = \pm 1$ we have a single root

$$\omega_1 = \frac{\kappa^2}{\xi^2 \mp 2\xi\kappa}. \tag{3.39}$$

We deduce that all singular points of G and H are real, starting with ω_1 being a removable singularity. We therefore define c^* as the first nonremovable singularity of G and H in $\{\omega \in C \mid 0 < \text{Re}(\omega) < \infty,\ \text{Im}(\omega) = 0\}$, that is, the first real root of $f(\omega)$ except ω_1, where $f(\omega)$ is defined by

$$f(\omega) = k(\omega)\cos\left(\frac{1}{2}k(\omega)\tau\right) + (\omega\rho\xi + \kappa)\sin\left(\frac{1}{2}k(\omega)\tau\right). \tag{3.40}$$

If $f(\omega)$ has no roots or no other roots except ω_1 in $\{\omega \in C \mid 0 < \text{Re}(\omega) < \infty,\ \text{Im}(\omega) = 0\}$, then we set $c^* = \infty$. By definition it follows that $\omega_1 \leq c^*$, with the special cases $\lim_{\tau\to 0} c^* = \infty$ and $\lim_{\tau\to\infty} c^* = \omega_1$.

4. Further Analysis

In the previous section a Mellin transform approach was used to solve the European put option pricing problem in Heston's mean reverting stochastic volatility model. The outcome is a new characterization of European put prices using an integration along a vertical line segment in a strip of the positive complex half plane. Our solution has a clear and well-defined structure. The numerical treatment of the solution is simple and requires a single integration procedure. However, the final expression for the option's price can still be modified to provide further insights on the analytical solution. First we have the following proposition.

Proposition 4.1. *An equivalent (and more convenient) way of expressing the solution in Theorem 3.1 is*

$$P^E(S,V,\tau) = Xe^{-r\tau}P_1 - Se^{-q\tau}P_2, \tag{4.1}$$

with $S = S(t)$ being the current stock price,

$$\begin{aligned} P_1 &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{Xe^{-r\tau}}{Se^{-q\tau}}\right)^{\omega}\frac{1}{\omega}H(\omega,\tau)e^{G(\omega,\tau)a_0V}d\omega, \\ P_2 &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{Xe^{-r\tau}}{Se^{-q\tau}}\right)^{\omega+1}\frac{1}{\omega+1}H(\omega,\tau)e^{G(\omega,\tau)a_0V}d\omega, \end{aligned} \tag{4.2}$$

where $0 < c < c^*$.

Proof. The statement follows directly from Theorem 3.1 by simple rearrangement. □

Remark 4.2. Equation (4.1) together with (4.2) provides a direct connection to Heston's original pricing formula given by

$$P^E(S,V,\tau) = Xe^{-r\tau}\Pi_1 - Se^{-q\tau}\Pi_2, \tag{4.3}$$

with

$$\Pi_1 = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-i\omega \ln X}\varphi(\omega)}{i\omega}\right)d\omega,$$
$$\Pi_2 = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-i\omega \ln X}\varphi(\omega - i)}{i\omega\varphi(-i)}\right)d\omega, \tag{4.4}$$

where the function $\varphi(\omega)$ is the log-characteristic function of the stock at maturity $S(T)$:

$$\varphi(\omega) = E\left[e^{i\omega \ln S(T)}\right]. \tag{4.5}$$

Remark 4.3. By the fundamental concept of a risk-neutral valuation, we have

$$\begin{aligned} P^E(S, V, \tau) &= E_t^Q\left[e^{-r\tau}(X - S(T)) \cdot 1_{\{S(T)<X\}}\right] \\ &= Xe^{-r\tau}E_t^Q\left[1_{\{S(T)<X\}}\right] - Se^{-q\tau}E_t^{Q^*}\left[1_{\{S(T)<X\}}\right], \end{aligned} \tag{4.6}$$

with $E_t^{\cdot}$ being the time t expectation under the corresponding risk-neutral probability measure, while Q^* denotes the equivalent martingale measure given by the Radon-Nikodym derivative

$$\frac{dQ^*}{dQ} = \frac{S(T)e^{-r\tau}}{Se^{-q\tau}}. \tag{4.7}$$

So the framework allows an expression of the above probabilities as the inverse of Mellin transforms.

A further advantage of the new framework is that hedging parameters, commonly known as Greeks, are easily determined analytically. The most popular Greek letters widely used for risk management are delta, gamma, vega, rho, and theta. Each of these sensitivities measures a different dimension of risk inherent in the option. The results for Greeks are summarized in the next proposition.

Proposition 4.4. *Setting*

$$I(\omega, \tau) = H(\omega, \tau)e^{G(\omega,\tau)a_0 V}, \tag{4.8}$$

the analytical expressions for the delta, gamma, vega, rho, and theta in the case of European put options are given by, respectively,

$$P_S^E(S,V,\tau) = \frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{X}{S}\right)^{\omega+1}\frac{1}{\omega+1}e^{(q\omega-r(\omega+1))\tau}I(\omega,\tau)d\omega, \tag{4.9}$$

$$P_{SS}^E(S,V,\tau) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{S}\left(\frac{X}{S}\right)^{\omega+1}e^{(q\omega-r(\omega+1))\tau}I(\omega,\tau)d\omega, \tag{4.10}$$

$$P_V^E(S,V,\tau) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{X}{2}\left(\frac{X}{S}\right)^{\omega}e^{(q\omega-r(\omega+1))\tau}G(\omega,\tau)I(\omega,\tau)d\omega. \tag{4.11}$$

Recall that the rho of a put option is the partial derivative of P^E with respect to the interest rate and equals

$$P_r^E(S,V,\tau) = \frac{-X\tau}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{X}{S}\right)^{\omega}\frac{1}{\omega}e^{(q\omega-r(\omega+1))\tau}I(\omega,\tau)d\omega. \tag{4.12}$$

Finally, the theta of the put, that is, the partial derivative of P^E with respect to τ, is determined as

$$P_\tau^E(S,V,\tau) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{X}{S}\right)^{\omega}\frac{X}{\omega(\omega+1)}e^{(q\omega-r(\omega+1))\tau}I(\omega,\tau)J(\omega,\tau)d\omega, \tag{4.13}$$

with

$$J(\omega,\tau) = q\omega - r(\omega+1) + \frac{1}{2}\omega(\omega+1)(\kappa\theta G(\omega,\tau) + VG_\tau(\omega,\tau)), \tag{4.14}$$

where

$$G_\tau(\omega,\tau) = \left(1 - \frac{(\omega\rho\xi+\kappa)^2}{\xi^2\omega(\omega+1)}\right)\frac{1}{\cos^2\left((1/2)k\tau + \tan^{-1}(-(\omega\rho\xi+\kappa)/k)\right)}. \tag{4.15}$$

Proof. The expressions follow directly from Theorem 3.1 or Proposition 4.1. The final expression for $J(\omega,\tau)$ follows by straightforward differentiation and (3.20). □

We point out that instead of using the put call parity relationship for valuing European call options a direct Mellin transform approach is also possible. However, a slightly modified definition is needed to guarantee the existence of the integral. We therefore propose to define the Mellin transform for calls as

$$M\left(C^E(S,V,t),\omega\right) = \tilde{C}^E(\omega,V,t) = \int_0^\infty C^E(S,V,t)\ S^{-(\omega+1)}\,dS, \tag{4.16}$$

where $1 < \mathrm{Re}(\omega) < \infty$. Conversely, the inverse of this modified Mellin transform is given by

$$C^E(S,V,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{C}^E(\omega,V,t)S^{\omega}\,d\omega, \tag{4.17}$$

where $1 < c$. Using the modification and following the lines of reasoning outlined in Section 3, it is straightforward to derive at the following theorem.

Theorem 4.5. *The Mellin-type closed-form valuation formula for European call options in the square-root stochastic volatility model of Heston [10] equals*

$$C^E(S,V,\tau) = Se^{-q\tau}P_2^* - Xe^{-r\tau}P_1^*, \tag{4.18}$$

where

$$\begin{aligned} P_2^* &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{Se^{-q\tau}}{Xe^{-r\tau}}\right)^{\omega-1}\frac{1}{\omega-1}H^*(\omega,\tau)e^{G^*(\omega,\tau)a_0^*V}\,d\omega, \\ P_1^* &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{Se^{-q\tau}}{Xe^{-r\tau}}\right)^{\omega}\frac{1}{\omega}H^*(\omega,\tau)e^{G^*(\omega,\tau)a_0^*V}\,d\omega, \end{aligned} \tag{4.19}$$

with

$$\begin{aligned} H^*(\omega,\tau) &= e^{(\kappa\theta/\xi^2)[-(\omega\rho\xi-\kappa)\tau+2\ln(k^*/k^*\cos((1/2)k^*\tau)-(\omega\rho\xi-\kappa)\sin((1/2)k^*\tau))]}, \\ G^*(\omega,\tau) &= \frac{2\sin((1/2)k^*\tau)}{k^*\cos((1/2)k^*\tau)-(\omega\rho\xi-\kappa)\sin((1/2)k^*\tau)}, \\ k^* &= k^*(\omega) = \sqrt{\xi^2\omega(\omega-1)-(\omega\rho\xi-\kappa)^2}, \end{aligned} \tag{4.20}$$

and $a_0^* = (1/2)\omega(\omega-1)$. *Furthermore, one has that* $1 < c < c^*$ *with* c^* *being characterized equivalently as at the end of the previous section.*

Remark 4.6. Again, a direct analogy to Heston's original pricing formula is provided. Also, the corresponding closed-form expressions for the Greeks follow immediately.

5. Numerical Examples

In this section we evaluate the results of the previous sections for the purpose of computing and comparing option prices for a range of different parameter combinations. Since our numerical calculations are not based on a calibration procedure, we will use notional parameter specifications. As a benchmark we choose the pricing formula due to Heston based on Fourier inversion (H). From the previous analysis it follows that the numerical inversion in both integral transform approaches requires the calculation of logarithms with complex arguments. As pointed out in [11, 18] this calculation may cause problems especially for options with long maturities or high mean reversion levels. We therefore additionally implement

Table 1: European option prices in Heston's stochastic volatility model for different asset prices S and maturities τ. Fixed parameters are $X = 100$, $r = 0.04$, $q = 0.02$, $V = 0.09$, $\kappa = 3$, $\theta = 0.12$, $\xi = 0.2$, $\rho = -0.5$, and $c = 2$.

	Puts				Calls			
(S, τ)	H	H(RCA)	MT	Diff	H	H(RCA)	MT	Diff
(80; 0.25)	19.8379	19.8379	19.8379	$1.7 \cdot 10^{-6}$	0.4339	0.4339	0.4339	$1.7 \cdot 10^{-6}$
(90; 0.25)	11.6806	11.6806	11.6806	$1.1 \cdot 10^{-6}$	2.2267	2.2268	2.2268	$1.1 \cdot 10^{-6}$
(100; 0.25)	5.9508	5.9508	5.9508	$4.9 \cdot 10^{-7}$	6.4471	6.4471	6.4471	$4.9 \cdot 10^{-7}$
(110; 0.25)	2.6708	2.6708	2.6708	$6.4 \cdot 10^{-6}$	13.1172	13.1173	13.1173	$6.4 \cdot 10^{-5}$
(120; 0.25)	1.0870	1.0870	1.0870	$7.5 \cdot 10^{-6}$	21.4835	21.4835	21.4835	$7.4 \cdot 10^{-6}$
(80; 0.5)	20.5221	20.5221	20.5221	$3.4 \cdot 10^{-6}$	1.7062	1.7062	1.7062	$3.4 \cdot 10^{-6}$
(90; 0.5)	13.5342	13.5342	13.5342	$2.2 \cdot 10^{-6}$	4.6188	4.6188	4.6188	$2.2 \cdot 10^{-6}$
(100; 0.5)	8.4302	8.4302	8.4302	$1.1 \cdot 10^{-6}$	9.4153	9.4153	9.4153	$1.1 \cdot 10^{-6}$
(110; 0.5)	5.0232	5.0232	5.0232	$3.0 \cdot 10^{-7}$	15.9088	15.9088	15.9088	$3.0 \cdot 10^{-7}$
(120; 0.5)	2.8995	2.8995	2.8995	$9.7 \cdot 10^{-7}$	23.6856	23.6856	23.6856	$9.7 \cdot 10^{-7}$
(80; 1.0)	22.1413	22.1413	22.1413	$6.7 \cdot 10^{-6}$	4.4783	4.4782	4.4783	$6.7 \cdot 10^{-6}$
(90; 1.0)	16.2923	16.2923	16.2923	$4.7 \cdot 10^{-6}$	8.4312	8.4312	8.4312	$4.7 \cdot 10^{-6}$
(100; 1.0)	11.7819	11.7819	11.7819	$2.3 \cdot 10^{-6}$	13.7229	13.7229	13.7229	$2.3 \cdot 10^{-6}$
(110; 1.0)	8.4207	8.4207	8.4207	$2.5 \cdot 10^{-7}$	20.1636	20.1636	20.1636	$2.5 \cdot 10^{-7}$
(120; 1.0)	5.9755	5.9755	5.9755	$2.3 \cdot 10^{-6}$	27.5204	27.5204	27.5204	$2.3 \cdot 10^{-6}$
(80; 2.0)	24.5972	24.5972	24.5972	$1.3 \cdot 10^{-6}$	9.1487	9.1487	9.1487	$1.3 \cdot 10^{-5}$
(90; 2.0)	19.8041	19.8041	19.8041	$8.2 \cdot 10^{-6}$	13.9635	13.9635	13.9635	$8.2 \cdot 10^{-6}$
(100; 2.0)	15.9136	15.9136	15.9136	$3.6 \cdot 10^{-6}$	19.6809	19.6809	19.6809	$3.6 \cdot 10^{-6}$
(110; 2.0)	12.7852	12.7852	12.7852	$7.2 \cdot 10^{-7}$	26.1604	26.1604	26.1604	$7.2 \cdot 10^{-7}$
(120; 2.0)	10.2833	10.2833	10.2833	$5.2 \cdot 10^{-6}$	33.2664	33.2664	33.2664	$5.2 \cdot 10^{-6}$
(80; 3.0)	26.1731	26.1731	26.1731	$1.4 \cdot 10^{-6}$	12.8222	12.8222	12.8222	$1.4 \cdot 10^{-6}$
(90; 3.0)	21.9865	21.9865	21.9865	$7.3 \cdot 10^{-6}$	18.0533	18.0533	18.0533	$7.3 \cdot 10^{-7}$
(100; 3.0)	18.5011	18.5011	18.5011	$2.3 \cdot 10^{-8}$	23.9855	23.9855	23.9855	$2.3 \cdot 10^{-8}$
(110; 3.0)	15.6055	15.6055	15.6055	$6.9 \cdot 10^{-6}$	30.5076	30.5076	30.5076	$6.9 \cdot 10^{-6}$
(120; 3.0)	13.2004	13.2004	13.2004	$1.2 \cdot 10^{-6}$	37.5201	37.5201	37.5201	$1.2 \cdot 10^{-6}$

the rotation count algorithm proposed by Kahl and Jäckel in [18] to overcome these possible inconsistencies (H(RCA)). The Mellin transform solution (MT) is based on (3.34) for puts and (4.18) for calls, respectively. The limits of integration $c \pm i\infty$ are truncated at $c \pm i500$. Although any other choice of truncation is possible, this turned out to provide comparable results. To assess the accuracy of the alternative solutions, we determine the absolute difference between H(RCA) and MT (Diff). Table 1 gives a first look at the results for different asset prices and expiration dates. We distinguish between in-the-money (ITM), at-the-money (ATM), and out-of-the-money (OTM) options. Fixed parameters are $X = 100$, $r = 0.04$, $q = 0.02$, $V = 0.09$, $\kappa = 3$, $\theta = 0.12$, $\xi = 0.2$, and $\rho = -0.5$, whereas S and τ vary from 80 to 120 currency units and three months to three years, respectively. Using these values, we have for the European put $\omega_1 = 9.6749$ constant, while c^* varies over time from 54.7066 ($\tau = 0.25$) to 11.7046 ($\tau = 3.0$) and for the European call $\omega_1 = 31.0082$ with c^* changing from 116.7385 ($\tau = 0.25$) to 33.7810 ($\tau = 3.0$). We therefore use $c = 2$ for the calculations (in both cases). Our major finding is that the pricing formulae derived in this paper provide comparable results for all parameter

Table 2: European option prices in Heston's stochastic volatility model for different asset prices S and correlations ρ. Fixed parameters are $X = 100$, $r = 0.05$, $q = 0.02$, $V = 0.04$, $\kappa = 2$, $\theta = 0.05$, $\xi = 0.2$, and $c = 2$.

	Puts				Calls			
(S, ρ)	H	H(RCA)	MT	Diff	H	H(RCA)	MT	Diff
(80; −1.00)	18.4620	18.4620	18.4620	$1.7 \cdot 10^{-6}$	0.1350	0.1350	0.1350	$1.7 \cdot 10^{-6}$
(100; −1.00)	5.0431	5.0431	5.0431	$2.1 \cdot 10^{-6}$	6.5170	6.5170	6.5170	$2.1 \cdot 10^{-6}$
(120; −1.00)	1.0353	1.0353	1.0353	$2.6 \cdot 10^{-5}$	22.3103	22.3103	22.3103	$2.6 \cdot 10^{-5}$
(80; −0.75)	18.5533	18.5533	18.5533	$1.3 \cdot 10^{-6}$	0.2263	0.2263	0.2263	$1.3 \cdot 10^{-6}$
(100; −0.75)	5.0403	5.0403	5.0403	$4.1 \cdot 10^{-6}$	6.5143	6.5143	6.5143	$4.1 \cdot 10^{-6}$
(120; −0.75)	0.9541	0.9541	0.9541	$6.6 \cdot 10^{-6}$	22.2291	22.2291	22.2291	$6.6 \cdot 10^{-6}$
(80; −0.50)	18.6413	18.6413	18.6413	$1.0 \cdot 10^{-6}$	0.3143	0.3143	0.3143	$1.0 \cdot 10^{-6}$
(100; −0.50)	5.0371	5.0371	5.0371	$4.4 \cdot 10^{-6}$	6.5111	6.5111	6.5111	$4.4 \cdot 10^{-6}$
(120; −0.50)	0.8695	0.8695	0.8695	$2.5 \cdot 10^{-6}$	22.1445	22.1445	22.1445	$2.5 \cdot 10^{-6}$
(80; −0.25)	18.7269	18.7269	18.7269	$7.9 \cdot 10^{-6}$	0.3999	0.3999	0.3999	$7.9 \cdot 10^{-6}$
(100; −0.25)	5.0332	5.0332	5.0332	$4.7 \cdot 10^{-6}$	6.5072	6.5072	6.5072	$4.7 \cdot 10^{-6}$
(120; −0.25)	0.7812	0.7812	0.7812	$1.5 \cdot 10^{-6}$	22.0562	22.0562	22.0562	$1.5 \cdot 10^{-6}$
(80; 0.00)	18.8104	18.8104	18.8104	$4.9 \cdot 10^{-5}$	0.4834	0.4834	0.4834	$4.9 \cdot 10^{-5}$
(100; 0.00)	5.0285	5.0285	5.0285	$2.7 \cdot 10^{-5}$	6.5025	6.5025	6.5025	$3.0 \cdot 10^{-5}$
(120; 0.00)	0.6887	0.6887	0.6887	$6.0 \cdot 10^{-5}$	21.9637	21.9637	21.9637	$6.0 \cdot 10^{-5}$
(80; 0.25)	18.8921	18.8921	18.8921	$1.1 \cdot 10^{-6}$	0.5651	0.5651	0.5651	$1.1 \cdot 10^{-6}$
(100; 0.25)	5.0229	5.0229	5.0229	$5.3 \cdot 10^{-6}$	6.4969	6.4969	6.4969	$5.3 \cdot 10^{-6}$
(120; 0.25)	0.5913	0.5913	0.5913	$9.6 \cdot 10^{-6}$	21.8663	21.8663	21.8663	$9.5 \cdot 10^{-6}$
(80; 0.50)	18.9721	18.9721	18.9721	$2.2 \cdot 10^{-6}$	0.6451	0.6451	0.6450	$2.2 \cdot 10^{-6}$
(100; 0.50)	5.0166	5.0166	5.0166	$5.7 \cdot 10^{-6}$	6.4906	6.4906	6.4906	$5.7 \cdot 10^{-6}$
(120; 0.50)	0.4882	0.4881	0.4881	$1.2 \cdot 10^{-6}$	21.7931	21.7630	21.7630	$1.2 \cdot 10^{-6}$
(80; 1.00)	19.1275	19.1275	19.1275	$9.60 \cdot 10^{-6}$	0.8005	0.8005	0.8005	$1.4 \cdot 10^{-5}$
(100; 1.00)	5.0027	5.0027	5.0027	$4.2 \cdot 10^{-6}$	6.4767	6.4767	6.4767	$5.7 \cdot 10^{-6}$
(120; 1.00)	0.2566	0.2566	0.2566	$1.3 \cdot 10^{-6}$	21.5316	21.5316	21.5316	$2.0 \cdot 10^{-6}$

combinations. The absolute differences are very small (of order 10^{-6} to 10^{-8} for puts and 10^{-5} to 10^{-8} for calls). They can be neglected from a practical point of view. In addition, since the numerical integration is accomplished in both frameworks equivalently efficient, the calculations are done very quickly.

Next, we examine how the option prices vary if the correlation between the underlying asset and its instantaneous variance changes. Although from a practical point of view it may be less realistic to allow for a positive correlation, we do not make any restrictions on ρ and let it range from −1.00 to 1.00. We fix time to maturity to be 6 months. Also, to provide a variety of parameter combinations, we change some of the remaining parameters slightly: $X = 100$, $r = 0.05$, $q = 0.02$, $V = 0.04$, $\kappa = 2$, $\theta = 0.05$, and $\xi = 0.2$. We abstain from presenting the numerical values of ω_1 and c^* in this case and choose again $c = 2$ for the integration. Our findings are reported in Table 2. Again, the Mellin transform approach gives satisfactory results as the absolute differences show. For both puts and calls they are of order 10^{-5} to 10^{-6}. Analyzing the results in detail, one basically observes two different kinds of behavior. For ITM put options we have an increase in value for increasing values of ρ. The maximum difference is 0.6655 or 3.60%. The opposite is true for OTM puts. Here we have a strict decline

Table 3: Delta values of European option prices in Heston's stochastic volatility model for different asset prices S and maturities τ. Fixed parameters are $X = 100$, $r = 0.06$, $q = 0.03$, $V = 0.16$, $\kappa = 3$, $\theta = 0.16$, $\xi = 0.1$, $\rho = -0.75$, and $c = 2$.

	Puts				Calls			
(S, τ)	Δ_H	$\Delta_{H(RCA)}$	Δ_{MT}	Diff	Δ_H	$\Delta_{H(RCA)}$	Δ_{MT}	Diff
(80; 0.25)	−0.8318	−0.8318	−0.8318	$2.4 \cdot 10^{-7}$	0.1607	0.1607	0.1607	$2.4 \cdot 10^{-7}$
(90; 0.25)	−0.6422	−0.6422	−0.6422	$2.4 \cdot 10^{-7}$	0.3503	0.3503	0.3503	$2.4 \cdot 10^{-7}$
(100; 0.25)	−0.4348	−0.4348	−0.4348	$2.4 \cdot 10^{-7}$	0.5578	0.5578	0.5578	$2.4 \cdot 10^{-7}$
(110; 0.25)	−0.2625	−0.2625	−0.2625	$2.4 \cdot 10^{-7}$	0.7300	0.7300	0.7300	$2.4 \cdot 10^{-7}$
(120; 0.25)	−0.1447	−0.1447	−0.1447	$2.5 \cdot 10^{-7}$	0.8479	0.8479	0.8479	$2.5 \cdot 10^{-7}$
(80; 0.5)	−0.7118	−0.7118	−0.7118	$4.8 \cdot 10^{-7}$	0.2734	0.2734	0.2734	$4.8 \cdot 10^{-7}$
(90; 0.5)	−0.5558	−0.5558	−0.5558	$4.8 \cdot 10^{-7}$	0.4294	0.4294	0.4294	$4.8 \cdot 10^{-7}$
(100; 0.5)	−0.4085	−0.4085	−0.4085	$4.8 \cdot 10^{-7}$	0.5766	0.5766	0.5766	$4.8 \cdot 10^{-7}$
(110; 0.5)	−0.2863	−0.2863	−0.2863	$4.8 \cdot 10^{-7}$	0.6988	0.6988	0.6988	$4.7 \cdot 10^{-7}$
(120; 0.5)	−0.1936	−0.1936	−0.1936	$4.8 \cdot 10^{-7}$	0.7915	0.7915	0.7915	$4.8 \cdot 10^{-7}$
(80; 1.0)	−0.5892	−0.5892	−0.5892	$8.6 \cdot 10^{-8}$	0.3812	0.3812	0.3812	$8.6 \cdot 10^{-8}$
(90; 1.0)	−0.4738	−0.4738	−0.4738	$8.7 \cdot 10^{-8}$	0.4966	0.4966	0.4966	$8.7 \cdot 10^{-8}$
(100; 1.0)	−0.3723	−0.3723	−0.3723	$8.6 \cdot 10^{-8}$	0.5981	0.5981	0.5981	$8.6 \cdot 10^{-8}$
(110; 1.0)	−0.2878	−0.2878	−0.2878	$8.0 \cdot 10^{-8}$	0.6827	0.6827	0.6827	$8.0 \cdot 10^{-8}$
(120; 1.0)	−0.2199	−0.2199	−0.2199	$8.6 \cdot 10^{-8}$	0.7505	0.7505	0.7505	$8.6 \cdot 10^{-8}$
(80; 2.0)	−0.4684	−0.4684	−0.4684	$1.7 \cdot 10^{-7}$	0.4733	0.4733	0.4733	$1.7 \cdot 10^{-7}$
(90; 2.0)	−0.3895	−0.3895	−0.3895	$1.7 \cdot 10^{-7}$	0.5523	0.5523	0.5523	$1.7 \cdot 10^{-7}$
(100; 2.0)	−0.3222	−0.3222	−0.3222	$1.7 \cdot 10^{-7}$	0.6196	0.6196	0.6196	$1.7 \cdot 10^{-7}$
(110; 2.0)	−0.2659	−0.2659	−0.2659	$1.7 \cdot 10^{-7}$	0.6758	0.6758	0.6758	$1.7 \cdot 10^{-7}$
(120; 2.0)	−0.2193	−0.2193	−0.2193	$1.7 \cdot 10^{-7}$	0.7224	0.7224	0.7224	$1.7 \cdot 10^{-7}$
(80; 3.0)	−0.3969	−0.3969	−0.3969	$2.4 \cdot 10^{-7}$	0.5170	0.5170	0.5170	$2.4 \cdot 10^{-7}$
(90; 3.0)	−0.3361	−0.3361	−0.3361	$2.4 \cdot 10^{-7}$	0.5779	0.5779	0.5779	$2.4 \cdot 10^{-7}$
(100; 3.0)	−0.2847	−0.2847	−0.2847	$2.4 \cdot 10^{-7}$	0.6292	0.6292	0.6292	$2.4 \cdot 10^{-7}$
(110; 3.0)	−0.2417	−0.2417	−0.2417	$2.4 \cdot 10^{-7}$	0.6723	0.6723	0.6723	$2.4 \cdot 10^{-7}$
(120; 3.0)	−0.2056	−0.2056	−0.2056	$2.4 \cdot 10^{-7}$	0.7083	0.7083	0.7083	$2.4 \cdot 10^{-7}$

in price if ρ is increased. The magnitude of price reactions to changes in ρ increases, too. The maximum change in the downward move is 0.7787 or equivalently 75.21%. The same behavior is observed for ATM options. However, the changes are much more moderate with a maximum percentage change of 0.80%. For European calls the situation is different. OTM calls increase significantly in value if ρ is increased, whereas ITM and ATM call prices decrease for an increasing ρ. The maximum percentage changes are 492.96% (OTM), 3.49% (ITM), and 0.62% (ATM), respectively.

Finally, we compare the values of delta for different $(S; \tau)$ combinations. For the calculation of the delta of a European put, we use (4.9). The corresponding delta value for a call is easily determined from the price function presented in the text. S and τ vary from 80 to 120 currency units and three months to three years, respectively. Again, the remaining parameters are slightly altered and equal $X = 100$, $r = 0.06$, $q = 0.03$, $V = 0.16$, $\kappa = 3$, $\theta = 0.16$, $\xi = 0.10$, $\rho = 0.75$, and $c = 2$. The results are summarized in Table 3. Once more, we observe a high consistency with Heston's framework based on Fourier inversion. For all parameter combinations our results agree with Heston's with a great degree of precision.

In summary, our numerical experiments suggest that the new framework is able to compete with Heston's solution based on Fourier inversion. The accuracy of the results is very satisfying, and the framework is flexible enough to account for all the pricing features inherent in the model. The findings justify the assessment of the Mellin transform approach as a very competitive alternative.

6. Conclusion

We have applied a new integral transform approach for the valuation of European options on dividend paying stocks in a mean reverting stochastic volatility model with correlation. Using the new framework our main results are new analytical characterizations of options' prices and hedging parameters. Our equivalent solutions may be of interest for theorists as well as practitioners. On one hand they provide further insights on the analytic solution, on the other hand they are easily and quickly treated numerically by applying efficient numerical integration schemes. We have done extensive numerical tests to demonstrate the flexibility and to assess the accuracy of the alternative pricing formulae. The results are gratifying and convincing. The new method is very competitive and should be regarded as a real alternative to other approaches, basically Fourier inversion methods, existing in the literature. Also, since the transformation variable is the current value of the asset instead of its terminal price, the new framework may turn out to be applicable to path-dependent problems.

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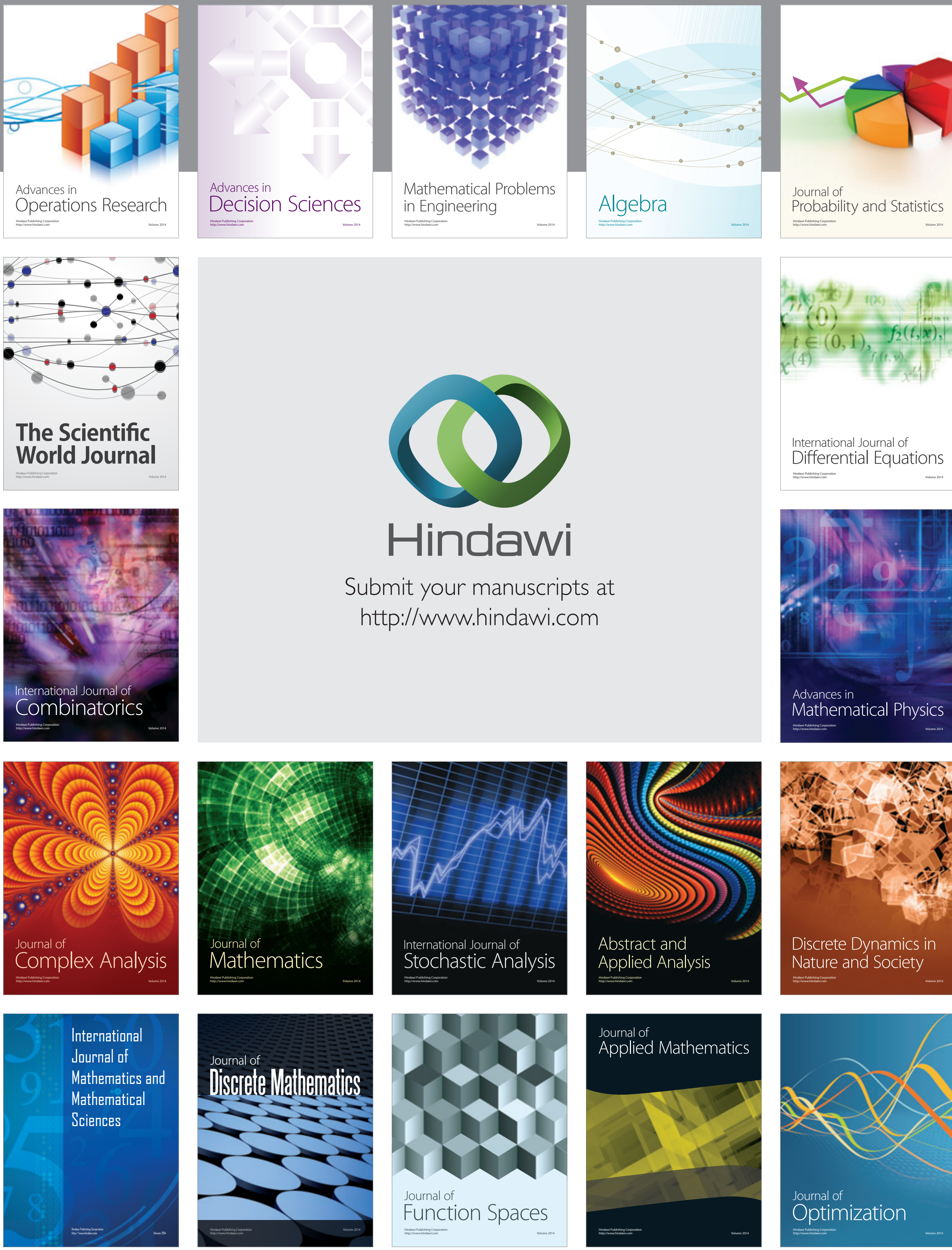

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