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Research Article

Common Fixed Point Theorems of the Asymptotic Sequences in Ordered Cone Metric Spaces

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We introduce the notions of the asymptotic $\mathcal{SMK}$-sequence with respect to the stronger Meir-Keeler cone-type mapping $\xi : \operatorname{int}(P) \cup \{\theta\} \to [0,1)$ and the asymptotic $\mathcal{WMK}$-sequence with respect to the weaker Meir-Keeler cone-type mapping $\phi : \operatorname{int}(P) \cup \{\theta\} \to \operatorname{int}(P) \cup \{\theta\}$ and prove some common fixed point theorems for these two asymptotic sequences in cone metric spaces with regular cone P. Our results generalize some recent results.

1. Introduction and Preliminaries

Let (X, d) be a metric space, D a subset of X, and $f : D \to X$ a map. We say f is contractive if there exists $\alpha \in [0,1)$ such that for all $x, y \in D$,

$$d(fx, fy) \leq \alpha \cdot d(x, y). \tag{1.1}$$

The well-known Banach's fixed point theorem asserts that if $D = X$, f is contractive and (X, d) is complete, then f has a unique fixed point in X. It is well known that the Banach contraction principle [1] is a very useful and classical tool in nonlinear analysis. Also, this principle has many generalizations. For instance, Kannan [2] and Chatterjea [3] introduced two conditions that can replace (1.1) in Banach's theorem.

(Kannan [2]) There exists $\alpha \in [0,1)$ such that for all $x, y \in X$,

$$d(fx, fy) \leq \frac{\alpha}{2}\left[d(x, fx) + d(y, fy)\right]. \tag{1.2}$$

(Chatterjea [3]) There exists $\alpha \in [0,1)$ such that for all $x, y \in X$,

$$d(fx, fy) \leq \frac{\alpha}{2}[d(x, fy) + d(y, fx)]. \tag{1.3}$$

After these three conditions, many papers have been written generalizing some of the conditions (1.1), (1.2), and (1.3). In 1969, Boyd and Wong [4] showed the following fixed point theorem.

Theorem 1.1 (see [4]). *Let (X, d) be a complete metric space and $f : X \to X$ a map. Suppose there exists a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\phi(0) = 0$, $\phi(t) < t$ for all $t > 0$ and ϕ is right upper semicontinuous such that*

$$d(fx, fy) \leq \phi(d(x, y)) \quad \forall x, y \in X. \tag{1.4}$$

Then, f has a unique fixed point in X.

Later, Meir-Keeler [5], using a result of Chu and Diaz [6], extended Boyd-Wong's result to mappings satisfying the following more general condition:

$$\forall \eta > 0\ \exists \delta > 0 \text{ such that } \eta \leq d(x, y) < \eta + \delta \Longrightarrow d(fx, fy) < \eta, \tag{1.5}$$

and Meir-Keeler proved the following very interesting fixed point theorem which is a generalization of the Banach contraction principle.

Theorem 1.2 (Meir-Keeler [5]). *Let (X, d) be a complete metric space and let f be a Meir-Keeler contraction, that is, for every $\eta > 0$, there exists $\delta > 0$ such that $d(x, y) < \eta + \delta$ implies $d(fx, fy) < \eta$ for all $x, y \in X$. Then, f has a unique fixed point.*

Subsequently, some authors worked on this notion of Meir-Keeler contraction (e.g., [7–10]).

Huang and Zhang [11] introduced the concept of cone metric space by replacing the set of real numbers by an ordered Banach space, and they showed some fixed point theorems of contractive type mappings on cone metric spaces. The category of cone metric spaces is larger than metric spaces. Subsequently, many authors like Abbas and Jungck [12] have generalized the results of Huang and Zhang [11] and studied the existence of common fixed points of a pair of self-mappings satisfying a contractive type condition in the framework of normal cone metric spaces. However, authors like Rezapour and Hamlbarani [13] studied the existence of common fixed points of a pair of self and nonself mappings satisfying a contractive type condition in the situation in which the cone does not need to be normal. Many authors studied this subject, and many results on fixed point theory are proved (see, e.g., [13–27]).

Throughout this paper, by $\mathbb{R}$ we denote the set of all real numbers, while $\mathbb{N}$ is the set of all natural numbers, and we initiate our discussion by introducing some preliminaries and notations.

Definition 1.3 (see [11]). Let E be a real Banach space and P a nonempty subset of E. $P \neq \{\theta\}$, where θ denotes the zero element of E, is called a cone if and only if

(i) P is closed,

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$,

(iii) $x \in P$ and $-x \in P \Rightarrow x = \theta$.

For given a cone $P \subset E$, we can define a partial ordering with respect to P by $x \preccurlyeq y$ or $x \succcurlyeq y$ if and only if $y - x \in P$ for all $x, y \in E$. The real Banach space E equipped with the partial ordered induced by P is denoted by $(E, \preccurlyeq)$. We shall write $x \prec y$ to indicate that $x \preccurlyeq y$ but $x \neq y$, while $x \preccurlyeq y$ will stand for $y - x \in \operatorname{int}(P)$, where $\operatorname{int}(P)$ denotes the interior of P.

Proposition 1.4 (see [28]). *Suppose P is a cone in a real Bancah space E. Then,*

(i) *If* $e \preccurlyeq f$ *and* $f \ll g$, *then* $e \ll g$.

(ii) *If* $e \ll f$ *and* $f \preccurlyeq g$, *then* $e \ll g$.

(iii) *If* $e \ll f$ *and* $f \ll g$, *then* $e \ll g$.

(iv) *If* $a \in P$ *and* $a \preccurlyeq e$ *for each* $e \in \operatorname{int}(P)$, *then* $a = \theta$.

Proposition 1.5 (see [29]). *Suppose* $e \in \operatorname{int}(P)$, $\theta \preccurlyeq a_n$, *and* $a_n \to \theta$. *Then, there exists* $n_0 \in \mathbb{N}$ *such that* $a_n \ll e$ *for all* $n \geq n_0$.

The cone P is called normal if there exists a real number $K > 0$ such that for all $x, y \in E$,

$$\theta \preccurlyeq x \preccurlyeq y \implies \|x\| \leq K\|y\|. \tag{1.6}$$

The least positive number K satisfying above is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent, that is, if $\{x_n\}$ is a sequence such that

$$x_1 \preccurlyeq x_2 \preccurlyeq \cdots \preccurlyeq x_n \preccurlyeq \cdots \preccurlyeq y, \tag{1.7}$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

Definition 1.6 (see [11]). Let X be a nonempty set, E a real Banach space, and P a cone in E. Suppose the mapping $d : X \times X \to (E, \preccurlyeq)$ satisfies

(i) $\theta \preccurlyeq d(x, y)$, for all $x, y \in X$,

(ii) $d(x, y) = \theta$ if and only if $x = y$,

(iii) $d(x, y) = d(y, x)$, for all $x, y \in X$,

(iv) $d(x, y) + d(y, z) \succcurlyeq d(x, z)$, for all $x, y, z \in X$.

Then, d is called a cone metric on X, and (X, d) is called a cone metric space.

Definition 1.7 (see [11]). Let (X, d) be a cone metric space, and let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $\theta \ll c$ there is $n_0 \in \mathbb{N}$ such that

$$d(x_n, x) \ll c, \quad \forall n > n_0, \tag{1.8}$$

then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x.

Definition 1.8 (see [11]). Let (X, d) be a cone metric space, and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is a Cauchy sequence if for any $c \in E$ with $\theta \ll c$, there is $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_m) \ll c, \quad \forall n,\ m > n_0. \tag{1.9}$$

Definition 1.9 (see [11]). Let (X, d) be a cone metric space. If every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

Remark 1.10 (see [11]). If P is a normal cone, then $\{x_n\}$ converges to x if and only if $d(x_n, x) \to \theta$ as $n \to \infty$. Further, in the case $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta$ as $m, n \to \infty$.

In this paper, we introduce the notions of the asymptotic $\mathcal{SMK}$-sequence with respect to the stronger Meir-Keeler cone-type mapping $\xi : \operatorname{int}(P) \cup \{\theta\} \to [0, 1)$ and the asymptotic $\mathcal{WMK}$-sequence with respect to the weaker Meir-Keeler cone-type mapping $\phi : \operatorname{int}(P) \cup \{\theta\} \to \operatorname{int}(P) \cup \{\theta\}$ and prove some common fixed point theorems for these two asymptotic sequences in cone metric spaces with regular cone P.

2. Common Fixed Point Theorems for the Asymptotic $\mathcal{SMK}$-Sequences

In 1973, Geraghty [30] introduced the following generalization of Banach's contraction principle.

Theorem 2.1 (see [30]). *Let (X, d) be a complete metric space, and let S denote the class of the functions $\beta : [0, \infty) \to [0, 1)$ which satisfy the condition*

$$\beta(t_n) \longrightarrow 1 \Longrightarrow t_n \longrightarrow 0. \tag{2.1}$$

Let $f : X \to X$ be a mapping satisfying

$$d(fx, fy) \le \beta(d(x, y)) \cdot d(x, y), \quad \text{for } x, y \in X, \tag{2.2}$$

where $\beta \in S$. Then, f has a unique fixed point $z \in X$.

In this section, we first introduce the notions of the stronger Meir-Keeler cone-type mapping $\xi : \operatorname{int}(P) \cup \{\theta\} \to [0, 1)$ and the asymptotic $\mathcal{SMK}$-sequence with respect to this stronger Meir-Keeler cone-type mapping ξ, and we next prove some common fixed point theorems for the asymptotic $\mathcal{SMK}$-sequence in cone metric spaces.

Definition 2.2. Let (X, d) be a cone metric space with cone P, and let

$$\xi : \mathrm{int}(P) \cup \{\theta\} \longrightarrow [0,1). \tag{2.3}$$

Then, the function ξ is called a stronger Meir-Keeler cone-type mapping, if for each $\eta \in \mathrm{int}(P)$ with $\eta \gg \theta$ there exists $\delta \gg \theta$ such that for $x, y \in X$ with $\eta \preccurlyeq d(x,y) \ll \delta + \eta$ there exists $\gamma_\eta \in [0,1)$ such that $\xi(d(x,y)) < \gamma_\eta$.

Example 2.3. Let $E = \mathbb{R}$, $P = \{x \in E : x \succcurlyeq \theta\}$ a normal cone, $X = [0,\infty)$, and let $d : X \times X \to E$ be the Euclidean metric. Define $\xi : \mathrm{int}(P) \cup \{\theta\} \to [0,1)$ by $\xi(d(x,y)) = \gamma$ where $\gamma \in [0,1)$, $x, y \in X$, then ξ is a stronger Meir-Keeler cone-type mapping.

Example 2.4. Let $E = \mathbb{R}$, $P = \{x \in E : x \succcurlyeq \theta\}$ a normal cone, $X = [0,\infty)$, and let $d : X \times X \to E$ be the Euclidean metric. Define $\xi : \mathrm{int}(P) \cup \{\theta\} \to [0,1)$ by $\xi(d(x,y)) = \|d(x,y)\|/(\|d(x,y)\| + 1)$ for $x, y \in X$, then ξ is a stronger Meir-Keeler cone-type mapping.

Definition 2.5. Let (X, d) be a cone metric space with a cone P, $\xi : \mathrm{int}(P) \cup \{\theta\} \to [0,1)$ a stronger Meir-Keeler cone-type mapping, and let

$$\{f_n\}_{n\in\mathbb{N}}, \; f_n : X \longrightarrow X \tag{2.4}$$

be a sequence of mappings. Suppose that there exists $\alpha \in \mathbb{N}$ such that the sequence $\{f_n\}_{n\in\mathbb{N}}$ satisfy that

$$d\left(f_i^\alpha x, f_j^\alpha y\right) \preccurlyeq \xi(d(x,y)) \cdot d(x,y), \quad \forall x, y \in X, \text{ and } i, j \in \mathbb{N}. \tag{2.5}$$

Then, we call $\{f_n\}_{n\in\mathbb{N}}$ an asymptotic $\mathcal{SMK}$-sequence with respect to this stronger Meir-Keeler cone-type mapping ξ.

Example 2.6. Let $E = \mathbb{R}^2$ and $P = \{(x,y) \in \mathbb{R}^2 | x, y \succcurlyeq \theta\}$ a normal cone in E. Let

$$X = \left\{(x,0) \in \mathbb{R}^2 \mid x \geq 0\right\} \cup \left\{(0,y) \in \mathbb{R}^2 \mid y \geq 0\right\}, \tag{2.6}$$

and we define the mapping $d : X \times X \to E$ by

$$\begin{gathered} d((x,0),(y,0)) = \left(\frac{9}{5}|x-y|, |x-y|\right), \\ d((0,x),(0,y)) = \left(|x-y|, \frac{3}{5}|x-y|\right), \\ d((x,0),(0,y)) = d((0,y),(x,0)) = \left(\frac{9}{5}x + y, x + \frac{3}{5}y\right). \end{gathered} \tag{2.7}$$

Let the asymptotic $\mathcal{SMK}$-sequence of mappings, $\{f_n\}_{n\in\mathbb{N}}$, $f_n : X \to X$ be

$$f_n(x,0) = (0,3^n x),$$
$$f_n(0,y) = \left(\frac{1}{3^{n+1}}y,0\right), \tag{2.8}$$

and let $\xi : \operatorname{int}(P) \to [0,1)$ be

$$\xi((x,y)) = \begin{cases} \frac{1}{3}\|d(x,y)\|, & \text{if } d(x,y) \preccurlyeq \frac{1}{2}, \\ \frac{\|d(x,y)\|}{\|d(x,y)\|+1}, & \text{if } d(x,y) \gg \frac{1}{2}. \end{cases} \tag{2.9}$$

Then, ξ is a stronger Meir-Keeler cone-type mapping and for $\alpha = 2$, and let $\{f_n\}_{n\in\mathbb{N}}$ be an asymptotic $\mathcal{SMK}$-sequence with respect to this stronger Meir-Keeler cone-type mapping ξ.

Now, we will prove the following common fixed point theorem of the asymptotic $\mathcal{SMK}$-sequence with respect to this stronger Meir-Keeler cone-type mapping for cone metric spaces with regular cone.

Theorem 2.7. *Let (X,d) be a complete cone metric space, P a regular cone in E, and let $\xi : \operatorname{int}(P) \cup \{\theta\} \to [0,1)$ be a stronger Meir-Keeler cone-type mapping. Suppose*

$$\{f_n\}_{n\in\mathbb{N}},\ f_n : X \longrightarrow X \tag{2.10}$$

is an asymptotic $\mathcal{SMK}$-sequence with respect to this stronger Meir-Keeler cone-type mapping ξ. Then, $\{f_n\}_{n\in\mathbb{N}}$ has a unique common fixed point in X.

Proof. Since $\{f_n\}_{n\in\mathbb{N}}$ is an asymptotic $\mathcal{SMK}$-sequence with respect to this stronger Meir-Keeler cone-type mapping ξ, there exists $\alpha \in \mathbb{N}$ such that

$$d\left(f_i^\alpha x, f_j^\alpha y\right) \preccurlyeq \xi(d(x,y)) \cdot d(x,y), \quad \forall x,y \in X, \text{ and } i,j \in \mathbb{N}. \tag{2.11}$$

Given $x_0 \in X$ and we define the sequence $\{x_n\}$ recursively as follows:

$$x_n = f_n^\alpha x_{n-1}, \quad \forall n \in \mathbb{N}. \tag{2.12}$$

Hence, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d\left(f_n^\alpha x_{n-1}, f_{n+1}^\alpha x_n\right) \\ &\preccurlyeq \xi(d(x_{n-1},x_n)) \cdot d(x_{n-1},x_n) \\ &\ll d(x_{n-1},x_n). \end{aligned} \tag{$*$}$$

Thus, the sequence $\{d(x_n, x_{n+1})\}$ is descreasing. Regularity of P guarantees that the mentioned sequence is convergent. Let $\lim_{n\to\infty} d(x_n, x_{n+1}) = \eta \geq 0$. Then, there exists $\kappa_0 \in \mathbb{N}$ such that for all $n \geq \kappa_0$

$$\eta \preccurlyeq d(x_n, x_{n+1}) \ll \eta + \delta. \tag{2.13}$$

For each $n \in \mathbb{N}$, since ξ is a stronger Meir-Keeler type mapping, for these η and $\delta \gg 0$ we have that for $x_{\kappa_0+n}, x_{\kappa_0+n+1} \in X$ with $\eta \preccurlyeq d(x_{\kappa_0+n}, x_{\kappa_0+n+1}) \ll \delta + \eta$, there exists $\gamma_\eta \in [0,1)$ such that $\xi(d(x_{\kappa_0+n}, x_{\kappa_0+n+1})) < \gamma_\eta$. Thus, by $(*)$, we can deduce

$$\begin{aligned} d(x_{\kappa_0+n}, x_{\kappa_0+n+1}) &= \xi(d(x_{\kappa_0+n-1}, x_{\kappa_0+n})) \cdot d(x_{\kappa_0+n-1}, x_{\kappa_0+n}) \\ &\ll \gamma_\eta \cdot d(x_{\kappa_0+n-1}, x_{\kappa_0+n}), \end{aligned} \tag{2.14}$$

and it follows that for each $n \in \mathbb{N}$

$$\begin{aligned} d(x_{\kappa_0+n}, x_{\kappa_0+n+1}) &\ll \gamma_\eta \cdot d(x_{\kappa_0+n-1}, x_{\kappa_0+n}) \\ &\ll \cdots \\ &\ll \gamma_\eta^n \cdot d(x_{\kappa_0+1}, x_{\kappa_0+2}). \end{aligned} \tag{2.15}$$

So,

$$\lim_{n\to\infty} d(x_{\kappa_0+n}, x_{\kappa_0+n+1}) = \theta, \quad \text{since } \gamma_\eta < 1. \tag{2.16}$$

We now claim that $\lim_{n\to\infty} d(x_{\kappa_0+n}, x_{\kappa_0+m}) = \theta$ for $m > n$. For $m, n \in \mathbb{N}$ with $m > n$, we have

$$d(x_{\kappa_0+n}, x_{\kappa_0+m}) \preccurlyeq \sum_{i=n}^{m-1} d(x_{\kappa_0+i}, x_{\kappa_0+i+1}) \prec \frac{\gamma_\eta^{m-1}}{1-\gamma_\eta} d(x_{\kappa_0+1}, x_{\kappa_0+2}), \tag{2.17}$$

and hence $d(x_n, x_m) \to \theta$, since $0 < \gamma_\eta < 1$. So $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete cone metric space, there exists $\nu \in X$ such that $\lim_{n\to\infty} x_n = \nu$.

We next prove that ν is a unique periodic point of f_j, for all $j \in \mathbb{N}$. Since for all $j \in \mathbb{N}$,

$$\begin{aligned} d\left(\nu, f_j^\alpha \nu\right) &= d(\nu, x_n) + d\left(x_n, f_j^\alpha \nu\right) \\ &= d(\nu, x_n) + d\left(f_n^\alpha x_{n-1}, f_j^\alpha \nu\right) \\ &= d(\nu, x_n) + \xi(d(x_{n-1}, \nu)) \cdot d(x_{n-1}, \nu) \\ &\ll d(\nu, x_n) + \gamma_\eta \cdot d(x_{n-1}, \nu), \end{aligned} \tag{2.18}$$

we have $d(\nu, f_j^\alpha \nu) \to \theta$. This implies that $\nu = f_j^\alpha \nu$. So, ν is a periodic point of f_j, for all $j \in \mathbb{N}$.

Let μ be another periodic point of f_i, for all $i \in \mathbb{N}$. Then,

$$d(\mu,\nu) = d\left(f_i^{\alpha}\mu, f_j^{\alpha}\nu\right) \preccurlyeq \xi(d(\mu,\nu)) \cdot d(\mu,\nu) \ll \gamma_\eta d(\mu,\nu). \tag{2.19}$$

Then, $\mu = \nu$.

Since $f_i\nu = f_i(f_i^{\alpha}\nu) = f_j^{\alpha}(f_i\nu)$, we have that $f_i\nu$ is also a periodic point of f_i, for all $j \in \mathbb{N}$. Therefore, $\nu = f_i\nu$, for all $j \in \mathbb{N}$, that is, ν is a unique common fixed point of $\{f_n\}_{n\in\mathbb{N}}$. □

Example 2.8. It is easy to get that $(0,0)$ is a unique common fixed point of the asymptotic $\mathcal{SMK}$-sequence $\{f_n\}_{n\in\mathbb{N}}$ of Example 2.6.

If the stronger Meir-Keeler cone-type mapping $\xi(t) = c$ for some $c \in [0,1)$, then we are easy to get the following corollaries.

Corollary 2.9. *Let (X,d) be a complete cone metric space, P a regular cone of a real Banach space E, and let $c \in [0,1)$. Suppose the sequence of mappings*

$$\{f_n\}_{n\in\mathbb{N}}, \ f_n : X \longrightarrow X \tag{2.20}$$

satisfy that for some $\alpha \in \mathbb{N}$,

$$d\left(f_i^{\alpha}x, f_j^{\alpha}y\right) \preccurlyeq c \cdot d(x,y), \quad \forall x,y \in X, \text{ and } i,j \in \mathbb{N}. \tag{2.21}$$

Then, $\{f_n\}_{n\in\mathbb{N}}$ has a unique common fixed point in X.

Corollary 2.10 (see [11]). *Let (X,d) be a complete cone metric space, P a regular cone of a real Banach space E, and let $c \in [0,1)$. Suppose the mapping $f : X \to X$ satisfies that for some $\alpha \in \mathbb{N}$,*

$$d(f^{\alpha}x, f^{\alpha}y) \preccurlyeq c \cdot d(x,y), \quad \forall x,y \in X. \tag{2.22}$$

Then, f has a unique fixed point in X.

Definition 2.11. Let (X,d) be a cone metric space with a cone P, and let

$$\xi, \xi_{i,j} : \operatorname{int}(P) \cup \{\theta\} \to [0,1), \quad \forall i,j \in \mathbb{N} \tag{2.23}$$

be stronger Meir-Keeler cone-type mappings with

$$\sup_{i,j\in\mathbb{N}} \xi_{i,j}(t) \leq \xi(t) \quad \forall t \in P. \tag{2.24}$$

Suppose the sequence $\{f_n\}_{n\in\mathbb{N}}$, $f_n : X \to X$ satisfy that for some $\alpha \in \mathbb{N}$,

$$d\left(f_i^{\alpha}x, f_j^{\alpha}y\right) \preccurlyeq \xi_{i,j}(d(x,y)) \cdot d(x,y), \quad \forall x,y \in X, \ i,j \in \mathbb{N}. \tag{2.25}$$

Then, we call $\{f_n\}_{n\in\mathbb{N}}$ a generalized asymptotic $\mathcal{SMK}$-sequence with respect to the stronger Meir-Keeler cone-type mappings $\{\xi_{i,j}\}_{i,j\in\mathbb{N}}$.

Example 2.12. Let $E = \mathbb{R}^2$ and $P = \{(x,y) \in \mathbb{R}^2 | x, y \succcurlyeq \theta\}$ a normal cone in E. Let

$$X = \left\{(x,0) \in \mathbb{R}^2 \mid x \geq 0\right\} \cup \left\{(0,y) \in \mathbb{R}^2 \mid y \geq 0\right\}, \tag{2.26}$$

and we define the mapping $d : X \times X \to E$ by

$$\begin{gathered} d((x,0),(y,0)) = \left(\frac{9}{5}|x-y|, |x-y|\right), \\ d((0,x),(0,y)) = \left(|x-y|, \frac{3}{5}|x-y|\right), \\ d((x,0),(0,y)) = d((0,y),(x,0)) = \left(\frac{9}{5}x + y, x + \frac{3}{5}y\right). \end{gathered} \tag{2.27}$$

Let $\{f_n\}_{n\in\mathbb{N}}$, $f_n : X \to X$ be

$$\begin{gathered} f_n(x,0) = (0, 2^n x), \\ f_n(0,y) = \left(\frac{1}{2^{n+1}}y, 0\right), \end{gathered} \tag{2.28}$$

and let $\xi_{i,j}, \xi : P \to [0,1)$ be

$$\begin{gathered} \xi_{i,j}(t) = \begin{cases} \dfrac{1}{2}, & \text{if } t \preccurlyeq 1, \\ \dfrac{1}{2} + \dfrac{1}{4\|t\|^{i+j}}, & \text{if } t \gg 1, \end{cases} \\ \xi(t) = \begin{cases} \dfrac{3}{4}, & \text{if } t \preccurlyeq 3, \\ \dfrac{\|t\|}{\|t\|+1}, & \text{if } t \gg 3. \end{cases} \end{gathered} \tag{2.29}$$

Then, $\{\xi_{i,j}\}_{i,j\in\mathbb{N}}$ be stronger Meir-Keeler cone-type mappings with

$$\sup_{i,j\in\mathbb{N}} \xi_{i,j}(t) \preccurlyeq \xi(t) \quad \forall t \in P, \tag{2.30}$$

and for $\alpha = 2$, let $\{f_n\}_{n\in\mathbb{N}}$ be a generalized asymptotic $\mathcal{SMK}$-sequence with respect to the stronger Meir-Keeler cone-type mappings $\{\xi_{i,j}\}_{i,j\in\mathbb{N}}$.

Follows Theorem 3.4, we are easy to conclude the following results.

Theorem 2.13. *Let (X, d) be a complete cone metric space, P a regular cone of a real Banach space E, let*

$$\xi, \xi_{i,j} : \operatorname{int}(P) \cup \{\theta\} \rightarrow [0,1), \quad \forall i, j \in \mathbb{N} \tag{2.31}$$

be stronger Meir-Keeler cone-type mappings with

$$\sup_{i,j\in\mathbb{N}} \xi_{i,j}(t) \preccurlyeq \xi(t) \quad \forall t \in P, \tag{2.32}$$

and let

$$\{f_n\}_{n\in\mathbb{N}}, \ f_n : X \longrightarrow X \tag{2.33}$$

be a generalized asymptotic $\mathcal{SMK}$-sequence with respect to the stronger Meir-Keeler cone-type mappings $\{\xi_{i,j}\}_{i,j\in\mathbb{N}}$. Then, $\{f_n\}_{n\in\mathbb{N}}$ has a unique common fixed point in X.

Example 2.14. It is easy to get that $(0, 0)$ is a unique common fixed point of the generalized $\mathcal{SMK}$-sequence $\{f_n\}_{n\in\mathbb{N}}$ of Example 2.12.

3. Common Fixed Point Theorems for the Asymptotic $\mathcal{WMK}$-Sequences

In this section, we first introduce the notions of the weaker Meir-Keeler cone-type mapping $\phi : \operatorname{int}(P) \cup \{\theta\} \rightarrow \operatorname{int}(P) \cup \{\theta\}$ and the asymptotic $\mathcal{WMK}$-sequence with respect to this weaker Meir-Keeler cone-type mapping ϕ, and we next prove some common fixed point theorems for the asymptotic $\mathcal{WMK}$-sequence in cone metric spaces.

Definition 3.1. Let (X, d) be a cone metric space with cone P, and let

$$\phi : \operatorname{int}(P) \cup \{\theta\} \longrightarrow \operatorname{int}(P) \cup \{\theta\}. \tag{3.1}$$

Then, the function ϕ is called a weaker Meir-Keeler cone-type mapping, if for each $\eta \in \operatorname{int}(P)$ with $\eta \gg \theta$ there exists $\delta \gg \theta$ such that for $x, y \in X$ with $\eta \preccurlyeq d(x, y) \ll \delta + \eta$ there exists $n_0 \in \mathbb{N}$ such that $\phi^{n_0}(d(x, y)) \ll \eta$.

Example 3.2. Let $E = \mathbb{R}$, $P = \{x \in E : x \succcurlyeq \theta\}$ a normal cone, $X = [0, \infty)$, and let $d : X \times X \rightarrow E$ be the Euclidean metric. Define $\phi : \operatorname{int}(P) \cup \{\theta\} \rightarrow \operatorname{int}(P) \cup \{\theta\}$ by $\phi(d(x, y)) = (1/3)d(x, y)$ for $x, y \in X$, then ϕ is a weaker Meir-Keeler cone-type mapping.

Definition 3.3. Let (X, d) be a cone metric space with a cone P, $\phi : \operatorname{int}(P) \cup \{\theta\} \rightarrow \operatorname{int}(P) \cup \{\theta\}$ be a weaker Meir-Keeler cone-type mapping, and let

$$\{f_n\}_{n\in\mathbb{N}}, \ f_n : X \rightarrow X \tag{3.2}$$

be a sequence of mappings. Suppose that there exists $\alpha \in \mathbb{N}$ such that the sequence $\{f_n\}_{n\in\mathbb{N}}$ satisfy that

$$d\left(f_i^{\alpha}x, f_j^{\alpha}y\right) \preccurlyeq \phi(d(x,y)), \quad \forall x,y \in X,\ i,j \in \mathbb{N}. \tag{3.3}$$

Then, we call $\{f_n\}_{n\in\mathbb{N}}$ an asymptotic $\mathcal{WMK}$-sequence with respect to this weaker Meir-Keeler cone-type mapping ξ.

Now, we will prove the following common fixed point theorem of the asymptotic $\mathcal{WMK}$-sequence with respect to this weaker Meir-Keeler cone-type mapping for cone metric spaces with regular cone.

Theorem 3.4. *Let (X,d) be a complete cone metric space, P a regular cone in E, and let $\phi : \operatorname{int}(P) \cup \{\theta\} \to \operatorname{int}(P) \cup \{\theta\}$ be a weaker Meir-Keeler cone-type mapping, and ϕ also satisfies the following conditions:*

(i) *$\phi(\theta) = \theta$; $\phi(t) \ll t$ for all $t \gg \theta$,*

(ii) *for $t_n \in \operatorname{int}(P) \cup \{\theta\}$, if $\lim_{n\to\infty} t_n = \gamma \gg \theta$, then $\lim_{n\to\infty}\phi(t_n) \ll \gamma$,*

(iii) *$\{\phi^n(t)\}_{n\in\mathbb{N}}$ is decreasing.*

Suppose that

$$\{f_n\}_{n\in\mathbb{N}},\ f_n : X \longrightarrow X \tag{3.4}$$

is an asymptotic $\mathcal{WMK}$-sequence with respect to this weaker Meir-Keeler cone-type mapping ϕ. Then, $\{f_n\}_{n\in\mathbb{N}}$ has a unique common fixed point in X.

Proof. Since $\{f_n\}_{n\in\mathbb{N}}$ is an asymptotic $\mathcal{WMK}$-sequence with respect to this weaker Meir-Keeler cone-type mapping ξ, there exists $\alpha \in \mathbb{N}$ such that

$$d\left(f_i^{\alpha}x, f_j^{\alpha}y\right) \preccurlyeq \phi(d(x,y)), \quad \forall x,y \in X,\ i,j \in \mathbb{N}. \tag{3.5}$$

Given $x_0 \in X$ and we define the sequence $\{x_n\}$ recursively as follows:

$$x_n = f_n^{\alpha} x_{n-1}, \quad \forall n \in \mathbb{N}. \tag{3.6}$$

Hence, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d\left(f_n^{\alpha}x_{n-1}, f_{n+1}^{\alpha}x_n\right) \\ &\preccurlyeq \phi(d(x_{n-1}, x_n)) \\ &= \phi\left(d\left(f_{n-1}^{\alpha}x_{n-2}, f_n^{\alpha}x_{n-1}\right)\right) \\ &\preccurlyeq \phi^2(d(x_{n-2}, x_{n-1})) \\ &\preccurlyeq \cdots \\ &\preccurlyeq \phi^n d(x_0, x_1). \end{aligned} \tag{3.7}$$

Since $\{\phi^n(d(x_0, x_1))\}_{n\in\mathbb{N}}$ is decreasing. Regularity of P guarantees that the mentioned sequence is convergent. Let $\lim_{n\to\infty}\phi^n(d(x_0, x_1)) = \eta$, $\eta \geq \theta$. We claim that $\eta = \theta$. On the contrary, assume that $\theta \ll \eta$. Then, by the definition of the weaker Meir-Keeler cone-type mapping, there exists $\delta \gg 0$ such that for $x_0, x_1 \in X$ with $\eta \preccurlyeq d(x_0, x_1) \ll \delta + \eta$ there exists $n_0 \in \mathbb{N}$ such that $\phi^{n_0}(d(x_0, x_1)) \ll \eta$. Since $\lim_{n\to\infty}\phi^n(d(x, fx)) = \eta$, there exists $m_0 \in \mathbb{N}$ such that $\eta \preccurlyeq \phi^m d(x_0, x_1) \ll \delta + \eta$, for all $m \geq m_0$. Thus, we conclude that $\phi^{m_0+n_0}(d(x_0, x_1)) \ll \eta$. So, we get a contradiction. So, $\lim_{n\to\infty}\phi^n(d(x_0, x_1)) = \theta$, and so $\lim_{n\to\infty}d(x_n, x_{n+1}) = \theta$.

Next, we let $c_m = d(x_m, x_{m+1})$, and we claim that the following result holds:

$$\begin{gathered}\text{for each } \varepsilon \gg \theta, \text{ there is } n_0(\varepsilon) \in \mathbb{N} \text{ such that for all } m, n \geq n_0(\varepsilon), \\ d(x_m, x_{m+1}) \ll \varepsilon.\end{gathered} \tag{$**$}$$

We will prove (3.7) by contradiction. Suppose that (3.7) is false. Then, there exists some $\varepsilon \gg \theta$ such that for all $k \in \mathbb{N}$, there are $m_k, n_k \in \mathbb{N}$ with $m_k > n_k \geq k$ satisfying:

(1) m_k is even and n_k is odd,

(2) $d(x_{m_k}, x_{n_k}) \succcurlyeq \varepsilon$,

(3) m_k is the smallest even number such that the conditions (1), (2) hold.

By (2), we have $\lim_{k\to\infty}d(x_{m_k}, x_{n_k}) = \varepsilon$, and

$$\begin{aligned}\varepsilon &\preccurlyeq d(x_{m_k}, x_{n_k}) \\ &\preccurlyeq d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k}) \\ &\preccurlyeq d(x_{m_k}, x_{m_k+1}) + \phi(d(x_{m_k}, x_{n_k})) + d(x_{n_k+1}, x_{n_k}).\end{aligned} \tag{3.8}$$

Letting $k \to \infty$. Then, by the condition (ii) of this weaker Meir-Keeler cone-type mapping ϕ, we have

$$\varepsilon \preccurlyeq \theta + \lim_{k\to\infty}\phi(d(x_{m_k}, x_{n_k})) + \theta \ll \varepsilon, \tag{3.9}$$

a contradiction. So, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete cone metric space, there exists $\nu \in X$ such that $\lim_{n\to\infty}x_n = \nu$.

We next prove that ν is a unique periodic point of f_j, for all $j \in \mathbb{N}$. Since for all $j \in \mathbb{N}$,

$$\begin{aligned}d\left(\nu, f_j^\alpha \nu\right) &= d(\nu, x_n) + d\left(x_n, f_j^\alpha \nu\right) \\ &= d(\nu, x_n) + d\left(f_n^\alpha x_{n-1}, f_j^\alpha \nu\right) \\ &= d(\nu, x_n) + \phi(d(x_{n-1}, \nu)) \\ &\ll d(\nu, x_n) + d(x_{n-1}, \nu),\end{aligned} \tag{3.10}$$

we have $d(\nu, f_j^\alpha \nu) \to \theta$. This implies that $\nu = f_j^\alpha \nu$. So, ν is a periodic point of f_j, for all $j \in \mathbb{N}$.

Let μ be another periodic point of f_i, for all $i \in \mathbb{N}$. Then,

$$d(\mu,\nu) = d\left(f_i^{\alpha}\mu, f_j^{\alpha}\nu\right) \preccurlyeq \phi(d(\mu,\nu)) \ll d(\mu,\nu). \tag{3.11}$$

Then, $\mu = \nu$.

Since $f_i\nu = f_i(f_i^{\alpha}\nu) = f_j^{\alpha}(f_i\nu)$, we have that $f_i\nu$ is also a periodic point of f_i, for all $j \in \mathbb{N}$. Therefore, $\nu = f_i\nu$, for all $j \in \mathbb{N}$, that is, ν is a unique common fixed point of $\{f_n\}_{n\in\mathbb{N}}$. $\square$

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