ON THE LINKS BETWEEN LIMIT CHARACTERISTIC ZEROS AND STABILITY PROPERTIES OF LINEAR TIME-INVARIANT SYSTEMS WITH POINT DELAYS AND THEIR DELAY-FREE COUNTERPARTS

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We investigate the relationships between the infinitely many characteristic zeros (or modes) of linear systems subject to point delays and their delay-free counterparts based on algebraic results and theory of analytic functions. The cases when the delay tends to zero or to infinity are emphasized in the study. It is found that when the delay is arbitrarily small, infinitely many of those zeros are located in the stable region with arbitrarily large modulus, while their contribution to the system dynamics becomes irrelevant. The remaining finite characteristic zeros converge to those of the delay-free nominal system. When the delay tends to infinity, infinitely many zeros are close to the origin. Furthermore, there exist two auxiliary delay-free systems which describe the relevant dynamics in both cases for zero and infinite delays. The maintenance of the delay-free system stability in the presence of sufficiently small delayed dynamics is also discussed in light of H_∞-theory. The main mathematical arguments used to derive the results are based on the theory of analytic functions.

1. Introduction

The objective of this paper is to investigate the relationships between the infinitely many modes of linear and time-invariant systems with point delays and their delay-free counterparts based on algebraic basic results and theory of analytic functions [1, 2, 6, 9, 10, 12, 13, 17]. Special interest is devoted to the cases when either the delay or the delay-free dynamics contribution tends to zero, and to the case when the delay tends to infinity. The main technical problem in the above first two cases is that a transcendent characteristic equation with infinitely many zeros tends to a polynomial with a finite number of zeros, but that limit problem for the characteristic zeros has not been addressed in the literature [3, 4, 5, 8, 10, 11, 14, 15, 16]. The dynamic behavior of the system in those limit cases approaches that of a delay-free system as is also deduced from intuition. A question that immediately arises is what in fact happens with infinitely many characteristic modes when the delay converges to zero so that the system becomes a delay-free one at the limit. This is a gap that has not been covered in the existing literature. The basic results in this

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paper are useful to interpret what happens with infinitely many modes as the system approaches a delay-free one then being of finite dimension. Such results in fact corroborate that the behavior is similar to that of the limit delay-free system since infinitely many modes tend to the boundary of the stable region in the left half-plane, while only a finite number of results are relevant in the dynamics. Another interesting feature is that a method based on perturbation theory is provided for calculating balls including the relevant modes for any delays.

The basic result obtained is that infinitely many characteristic modes diverge within the stable region as the delay tends to zero, while the relevant ones converge to those of the resulting delay-free system. Another parallel obtained result is that as the delays tend to infinity, infinitely many characteristic modes are located in small balls centered at zero. The proof of those issues is made using tools from linear algebra and analytic functions of complex variable. A complementary set of parallel results of stability "independent of" and "dependent on" delay is also presented. Such results are based on H_∞-theory for the case when the delay-free system is stable. Finally, a perturbation method is presented to calculate, up to any desired order of approximation, the characteristic roots of the system. The calculation method is exact when the approximation order in the perturbation calculation is infinite. A numerical example is presented to corroborate the obtained results. The paper is organized as follows. Section 2 is devoted to some previous results. It is proved that infinitely many zeros cannot be close to those of the delay-free system for zero delay, which is a key point for obtaining the remaining results. The main results of the paper concerning stability delay-independent/delay-dependent results as well as related properties for the cases when the delay tends to zero or to infinity are given in Section 3. The perturbation method to calculate the characteristic zeros from those associated with the delay-free system is presented in Section 4. Finally, an example related to the application of the perturbation theory to compute the characteristic zeros is presented in Section 5 and compared to Pade's approximation, and conclusions end the paper.

Notation. (1) det, tr, Adj, and superscript T are notational abbreviations for determinant, trace, adjoint, and matrix transpose of any real or complex matrix. I denotes the identity matrix and $\|M\|_2 = \lambda_{\max}^{1/2}(M^T M)$ is the ℓ_2-norm of the M-matrix or vector, where $\lambda_{\max}(\cdot)$ stands for the maximum eigenvalue of the $(\cdot)$-matrix.

(2) $P = P^T > 0$ stands for a real symmetric positive definite matrix P. A positive semidefinite matrix is denoted by the nonstrict inequality "$\geq$" while a negative definite (semidefinite) matrix is denoted by "$<$" ("$\leq$").

(3) $\mathbb{R}$ and $\mathbb{C}$ are the sets of real and complex numbers, $\mathbb{R}^+$ is the set of positive real numbers, $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$, $\mathbb{C}^+$ is the set of complex numbers of real part in $\mathbb{R}^+$, $\mathbb{C}_0^+ = \mathbb{C}^+ \cup \mathbb{C}_{\mathrm{im}}$ with $\mathbb{C}_{\mathrm{im}} = \{s \in \mathbb{C} : s = j\omega,\ \omega \in \mathbb{R}\}$ is the set of purely imaginary complex numbers, $\mathbb{C}^- = \{s \in \mathbb{C} : \operatorname{Re} s < 0\}$ is the stable region, and $\mathbb{C}_a = \{s \in \mathbb{C} : \operatorname{Re} s \geq a\}$.

(4) If f and g are any real or complex functions, $f = O(g)$, in Landau's notation, "f is Big-O of g," if there are nonnegative bounded real constants $K_{1,2}$ such that $|f| \leq K_1 g + K_2$. Also, $f = o(g)$, in Landau's notation, "f is Small-o of g," if $f = O(g)$ and, in addition, there exists $\lim_{g \to 0} f/g = 0$. This notation may be extended in a natural way to real or complex vector functions. If the functions f and g grow at the same rate, that is, $f = O(g)$ and $g = O(f)$, the abbreviated notation $f \approx g$ is used.

Consider the following linear and time-invariant system of state vector $x(t)$ with delayed dynamics with a point delay $h \geq 0$:

$$\dot{x}(t) = Ax(t) + \varepsilon A_0 x(t-h) \tag{1.1}$$

under initial conditions given by the absolutely continuous n-vector real function $\varphi : [-h,0] \to \mathbb{R}^n$, where A and A_0 are square n-real matrices which represent, respectively, the contributions of the delay-free and delayed dynamics. A real parameter ε is introduced to quantify the contribution of the delayed dynamics for a given A_0-matrix. The ε-parameter is only introduced for technical reasons to facilitate the stability study through a possible modification of the amount of contribution of the delayed dynamics for a prescribed A_0-matrix. Note also that system (1.1) may be extended without difficulty to include the presence of any set of distinct point delays. For the purposes of this paper, it is sufficient to consider only one delay h with no loss in generality. The properties of the following two delay-free systems are related to that of (1.1) as the delay tends to zero or to infinity [4, 6, 9, 10, 11, 13, 15, 16]:

(i) *delay-free system ($h = 0$):*

$$\dot{x}(t) = (A + \varepsilon A_0)x(t), \tag{1.2a}$$

(ii) *auxiliary delay-free system (h infinite and/or $\varepsilon = 0$):*

$$\dot{x}(t) = Ax(t). \tag{1.2b}$$

The three characteristic equations of interest in this paper are set as follows in order to be related to each other for derivation of stability properties:

$$\begin{aligned} p(s,\varepsilon) &= \det\left(sI - A - \varepsilon A_0 e^{-hs}\right) = 0, \\ p_{A+\varepsilon A_0}(s) = p_0(s,\varepsilon) &= \det\left(sI - A - \varepsilon A_0\right) = 0, \\ p_A(s) = p(s,0) &= \det(sI - A) = 0. \end{aligned} \tag{1.3}$$

They are, respectively, associated with the current delay system (1.1), the auxiliary delay-free system obtained from (1.1) for zero delay $\dot{x}(t) = (A + \varepsilon A_0)x(t)$, and the auxiliary delay-free system $\dot{x}(t) = Ax(t)$ obtained from (1.1) when $\varepsilon = 0$ and/or for the delay being infinite. Throughout the paper, the zeros of the characteristic equations are called characteristic roots (or modes) of the corresponding dynamic system.

2. Preliminaries

Proposition 2.1. *Assume that s_0 is any of the zeros of $p_0(s,\varepsilon)$ of multiplicity ν_0. Thus, only ν_0 of the infinitely many zeros of $p(s,\varepsilon)$ converges to s_0 as $h \to 0$ for any real nonzero ε, while the remaining infinitely many ones converge to isolated limit points with $\sigma = \operatorname{Re} s \to -\infty$ as $|\sigma^{-1}| = o(h)$; that is, $|h\sigma| \to \infty$ with $\operatorname{Re} s \to -\infty$. In the same way, assume that s_0 is any of the zeros of $p_A(s) = 0$ of multiplicity $\nu_0 \leq n$. Then, only one zero of $p(s,\varepsilon)$ of multiplicity ν_0 converges to s_0 as $\varepsilon \to 0$, for any finite or infinite delay h, while the remaining infinitely many ones converge to limit isolated points with $\sigma = \operatorname{Re} s \to -\infty$ with $|\sigma^{-1}| = o(h)$; that is, $|h\sigma| \to \infty$.*

Proof. First, note that $p_0(s_0,\varepsilon) = \lim_{s\to s_0}[\lim_{h\to 0}(p(s,\varepsilon))]$ since $p_0(s,\varepsilon)$ is an entire complex function at any zero s_0 of $p_0(s,\varepsilon)$ which has to be finite and of finite multiplicity $\nu_0 \leq n = \deg(p_0(s,\varepsilon))$ since $p_0(s,\varepsilon)$ is a nonconstant polynomial, that is, a nonconstant entire function which diverges as $|s| \to \infty$ from Liouville's theorem [12]. If $p_0(s_0,\varepsilon) = 0$, then either there is a disc $D(s_0,r)$ of radius r, centered at s_0, where $p_0(s,\varepsilon) \neq 0$, for all $s\ (\neq s_0) \in D(s_0,r)$, or there is a disc $D'(s_0,r')$ centered at s_0 such that $p_0(s,\varepsilon) \equiv 0$, for all $s \in D'(s_0,r')$. The second possibility is impossible since $p_0(s,\varepsilon)$ is a nonconstant polynomial. Thus, there is a disc $D(s_0,r)$, where the only zero of $p_0(s,\varepsilon)$ is s_0. Since $\lim_{s\to s_0}[\lim_{h\to 0}(p(s,\varepsilon))]$ contains, as a factor, $p_0(s_0,\varepsilon)$, there is a disc $D_\varepsilon(s_0,r_\varepsilon)$, centered at s_0, which depends on ε and contains an isolated zero of $p(s,\varepsilon)$ for each real ε. Otherwise, there would exist a disc centered at s_0, where $p(s,\varepsilon)$ is identically zero, which is also impossible [12]. Thus, for any sufficiently small h, there exists a disc $D_\ell(s_0,r_\varepsilon) \subset D_\varepsilon(s_0,r_\varepsilon) \cap D(s_0,r)$ which contains only a zero, which is unique except for its multiplicity $s_{01} = s_{01}(\varepsilon)$ of $p(s,\varepsilon)$. Since

$$\begin{gathered}\lim_{h\to 0} p(s_{01},\varepsilon) = \lim_{s\to s_{01}}(p(s,\varepsilon)) = p_0(s_{01},\varepsilon) = \lim_{h\to 0}\Big(\big(s - s_{01}(\varepsilon)\big)^{\nu_{0\varepsilon}} p'_{0\varepsilon}(s_{01})\Big), \\ p(s_0,\varepsilon) = (s-s_0)^{\nu_0} p'_0(s_0,\varepsilon) = 0 \quad \text{as } h \longrightarrow 0,\end{gathered} \tag{2.1}$$

with $p'_0(s_0,\varepsilon) \neq 0$ and $p'_{0\varepsilon}(s_{01}) \neq 0$, *since the zeros s_0 and $s_{01}(\varepsilon)$ are isolated finite zeros,* then, as $h \to 0$,

$$\big(s - s_{01}(\varepsilon)\big)^{\nu_{0\varepsilon}} p'_{0\varepsilon}(s_{01}(\varepsilon)) \longrightarrow (s-s_0)^{\nu_0} p'_0(s_0,\varepsilon) = 0 \Longrightarrow (s-s_0)^{\nu_0} p'_0(s_0,\varepsilon) = 0 \tag{2.2}$$

so that $s_{01}(\varepsilon) \to s_0$ and its multiplicity $\nu_{0\varepsilon} \to \nu_0$ as $h \to 0$ since $p'_{0\varepsilon}(s_{01}) \neq 0$ and $p'_0(s_0,\varepsilon) \neq 0$. Also, for $h = 0$ and $s_{01}(\varepsilon) = s_0$, it is a finite and isolated zero with finite multiplicity $\nu_{0\varepsilon} = \nu_0 \leq n$. It has been proved that $p_0(s_0,\varepsilon)$ is a zero factor of multiplicity ν_0 of $p(s_0,\varepsilon)$ for $h = 0$, and since s_0 is finite and arbitrary, all the finite zeros of $p(s,\varepsilon)$ are those of $p_0(s,\varepsilon)$ as $h \to 0$. However, since $p(s,\varepsilon)$ possesses infinitely many zeros for all nonzero ε, it turns out that, as $h \to 0$,

(a) $p(s,\varepsilon) \to p_0(s,\varepsilon)$ for all finite s and for $\sigma = \operatorname{Re} s \to -\infty$. In particular, $\lim_{s\to s_0} p(s,\varepsilon) \to p_0(s_0,\varepsilon) = 0$ for all the (finite) arbitrary zeros of $p_0(s,\varepsilon)$;

(b) $p(s,\varepsilon) \to \bar{p}_0(s,\varepsilon) = \prod_{i=1}^{\infty}(s - s_i(\varepsilon)) = 0$ for infinitely many zeros $s = s_i(\varepsilon)$ with arbitrary large stable abscissas $\sigma_i = \operatorname{Re} s_i \to -\infty$ with $|\sigma_i^{-1}| = o(h)$ which are all distinct except for their multiplicity. The first part of the result has been proved. The second part related to $\varepsilon \to 0$ follows by following similar technical steps by replacing $p_0(s,\varepsilon)$ with $p_A(s)$ and in the comparisons with $p(s,\varepsilon)$.

□

The classical root locus of system (1.1) is of interest in analyzing the classical behavior when the delayed dynamics is arbitrarily large or small compared to the delay-free one or when the sampling period tends to zero or to infinity. The subsequent simple example illustrates this fact.

Example 2.2. Consider the linear and time-invariant system $\dot{x}(t) = ax(t) + \varepsilon x(t-h)$ with a point delay $h \geq 0$. The characteristic equation is

$$\begin{aligned} p(s,\varepsilon) &= s - a - \varepsilon e^{-hs} \\ &= (s-a)\left(1 - \frac{\varepsilon}{e^{hs}(s-a)}\right) \\ &= (s-a-\varepsilon)\left(1 - \varepsilon\left(\frac{1-e^{hs}}{e^{hs}}\right)\frac{1}{s-a-\varepsilon}\right), \end{aligned} \tag{2.3}$$

the last two identities being applicable for $s \neq a$ and $s \neq a+\varepsilon$, respectively. Thus, from the root locus theory, it turns out from the second identity that a characteristic zero of $p(s,\varepsilon)$ tends to $s = a$ as $\varepsilon \to 0$ for any finite h (which is also a zero of $p_A(s)$), while the remaining infinitely many ones tend to points of abscissas $\sigma_i = \operatorname{Re} s_i \to -\infty$ at a rate $|\sigma_i^{-1}| = o(h)$. Also, there are no finite characteristic zeros as $\varepsilon \to \pm\infty$. Note that the first identity also leads to the same conclusion since $s \to a$ as $\varepsilon \to 0$ and, furthermore, the zeros have to fulfill that

$$\begin{aligned} \lim_{\sigma\to-\infty} \operatorname{Re}\left(p(s,\varepsilon)\right) &= \lim_{\sigma\to-\infty}\left(\sigma - a - \varepsilon e^{-h\sigma}\cos(h\omega)\right) = 0, \\ \lim_{\sigma\to-\infty} \operatorname{Im}\left(p(s,\varepsilon)\right) &= \lim_{\sigma\to-\infty}\left(\omega + \varepsilon e^{-h\sigma}\sin(h\omega)\right) = 0, \end{aligned} \tag{2.4}$$

where $\sigma = \operatorname{Re} s$ and $\omega = \operatorname{Im} s$ which imply $\omega = 0$ and $\varepsilon \to (\sigma - a)e^{h\sigma}/\cos h\omega \to (\sigma - a)e^{h\sigma} \to 0$ as $\sigma \to -\infty$ or when $\sigma \to a$ for $\omega \to 0$. This holds for all $h \in [0,\infty)$. For $h \to \infty$, the same conclusion follows for $\sigma \to a$ and $\omega \to 0$ or with $\sigma < 0$ (including, but not requiring, the case $\sigma \to -\infty$). The third identity leads to the following conclusions.

For any complex $s = \sigma + j\omega$, $|e^{-hs} - 1| = |e^{-h\sigma}(\cos\omega h - j\sin\omega h) - 1| \to 0$ implies $\omega = k\pi/h \to 0$ for any integer k. Thus, $\omega \to 0$ with $k = 0$, which also implies that $\cos\omega h = 1$ is the only valid case leading to $|e^{-hs} - 1| \to 0$ for all finite real ε as $h \to 0$. This, in addition, implies that the root locus gain for the third identity $\varepsilon|e^{-hs} - 1|$ tends to 0 for any finite real ε as $h \to 0$. Thus, one zero of $p(s,\varepsilon)$ tends to $s = a + \varepsilon$ and the infinitely many remaining ones have diverging abscissas $\sigma = \operatorname{Re} s \to -\infty$, at a rate $|\sigma^{-1}| = o(h)$, as $h \to 0$ for any finite real ε.

If $h \to \infty$ and $a > 0$, then one characteristic zero tends to $s = a$, while the remaining ones tend to zero since $|1/e^{hs}| \to 0$, for $\operatorname{Re} s \geq 0$, at a rate $|\sigma| = o(h^{-1})$, which is qualitatively similar (also from the second identity and the root locus) to $|\varepsilon| \to \infty$. This also occurs for infinitely many zeros with $\operatorname{Re} s < 0$ and all real ε since $|1/e^{hs}| \to \infty$, which leads to the same conclusion as that obtained above for finite delay and $|\varepsilon| \to \infty$. Thus, infinitely many stable zeros tend to disappear and do not contribute in practice to the system response. The remaining zero tends to $s = a + \varepsilon$, for any real ε, which also satisfies $p_{a+\varepsilon}(s) = p_0(s,\varepsilon) = 0$. If $h \to \infty$ and $\operatorname{Re} s < 0$ with $|\varepsilon| < \infty$, or $\operatorname{Re} s = 0$ and $\varepsilon \to 0$, then a zero tends to $s = a + \varepsilon$.

The conclusions for the above example can also be obtained from Proposition 2.1. Similarly, a reasoning based on the root locus may be given to obtain similar results as those obtained in the above scalar example for the general system (1.1). The basic results are as follows.

(1) As $0 \neq \varepsilon \to 0$, for any finite zero or nonzero delay h, n characteristic zeros (or modes) of system (1.1) tend to those of $\dot{x}(t) = Ax(t)$, while the remaining ones have abscissas diverging through the stable region $\operatorname{Re} s < 0$.

(2) As $0 \neq h \to 0$, for any finite zero or nonzero ε, n modes of (1.1) tend to those of $\dot{x}(t) = (A + \varepsilon A_0)_x(t)$, while the remaining ones have abscissas diverging through the stable region $\operatorname{Re} s < 0$. This also includes the above result as $\varepsilon \to 0$.

Theorem 3.1 deduces those results and then uses them in a stability context by comparing the current time-delay system (1.1) and its delay-free counterparts.

3. Main results

Theorem 3.1. *The following items hold.*

(i) *Assume that $\dot{x}(t) = Ax(t)$ has $\mu \leq n$ distinct eigenvalues s_i of multiplicities $\nu_i \leq n$, $i = 1, 2, \ldots, \mu$, with $n = \sum_{i=1}^{\mu} \nu_i$. Thus, for any real $r > 0$, there is a real interval $(-\varepsilon^*, \varepsilon^*)$, with ε^* being dependent on r, such that ν_i modes $s_i^{(\ell)}$ of (1.1) satisfy $|s_i^{(\ell)} - s_i| < r$, $j = 1, 2, \ldots, \nu_i$, $i = 1, 2, \ldots, \mu$, for any delay $h \in [0, \infty)$ and all $\varepsilon \in (-\varepsilon^*, \varepsilon^*)$. As $0 \neq \varepsilon^* \to 0$, infinitely many modes of (1.1) are stable and diverge in the stable region with $\operatorname{Re} s \to -\infty$, while μ are arbitrarily close to the s_i modes of $\dot{x}(t) = Ax(t)$ $(i = 1, 2, \ldots, \mu)$ with their respective multiplicities. If A is a stable (unstable) matrix, then (1.1) is globally asymptotically Lyapunov stable—g.a.s.—(unstable) for all finite delay $h \in [0, \infty)$ and for all $\varepsilon \in (-\varepsilon^*, \varepsilon^*)$ for some sufficiently small ε^*.*

(ii) *If $A + \varepsilon A_0$ is a stable (unstable) matrix, then (1.1) is g.a.s. (unstable) for all finite delay $h \in [0, \infty)$ and for all $\varepsilon \in (-\varepsilon^*, \varepsilon^*)$ for some sufficiently small ε^*.*

(iii) *If (1.1) is g.a.s. (unstable) for zero delay, that is, $A + \varepsilon A_0$ is stable (unstable) for zero delay and a given ε, then there is a delay h^* such that (1.1) is g.a.s. (unstable) for all $h \in [0, h^*)$.*

(iv) *Define H_∞-norms* [11]

$$\begin{aligned} \gamma_0 &:= \left\|(sI - A)^{-1}\right\|_\infty \\ &= \operatorname{Max}\left(z \in \mathbb{R}_0^+ : M = \begin{bmatrix} A & z^{-1}I \\ I & -A^T \end{bmatrix} \text{ has no imaginary eigenvalue}\right) \end{aligned} \tag{3.1}$$

if A is stable so that $\gamma_0 \leq \operatorname{Sup}_{\omega \in \mathbb{R}_0^+}(\|(j\omega I - A)^{-1}\|_2) < \infty$, where $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$, and

$$\begin{aligned} \gamma_{0\varepsilon} &:= \left\|\left(sI - A - \varepsilon A_0\right)^{-1}\right\|_\infty \\ &= \operatorname{Max}\left(z \in \mathbb{R}_0^+ : M = \begin{bmatrix} A + \varepsilon A_0 & z^{-1}I \\ I & -\left(A^T + \varepsilon A_0^T\right) \end{bmatrix} \text{ has no imaginary eigenvalue}\right) \end{aligned} \tag{3.2}$$

provided that $A + \varepsilon A_0$ is stable so that $\gamma_{0\varepsilon} \leq \operatorname{Sup}_{\omega \in \mathbb{R}_0^+}(\|(j\omega I - A - \varepsilon A_0)^{-1}\|_2) < \infty$. Thus,

(1) *system (1.1) is g.a.s. and independent of delay (i.e., for all finite delay h) if A is stable and $\|A_0\|_2 < |\varepsilon^{-1}|\gamma_0^{-1}$,*

(2) *system (1.1) is g.a.s. and independent of delay if $A + \varepsilon A_0$ is stable and $\|A_0\|_2 < (1/2)|\varepsilon^{-1}|\gamma_{0\varepsilon^*}^{-1}$,*

(3) *system (1.1) is globally stable and independent of delay for all* $\varepsilon \in [-\varepsilon^*, \varepsilon^*]$ *and any* $\varepsilon^* > 0$ *if* $A \pm \varepsilon^* A_0$ *are both stable and* $\|A_0\|_2 < (1/\varepsilon^*)\operatorname{Max}(1/\gamma_0, 1/2\gamma_{0\varepsilon^*})$.

(v) *Assume that* $A \pm \varepsilon^* A_0$ *are both stable, for any given real* $\varepsilon^* > 0$, *with larger stability abscissa at least* $(-\rho) < 0$, *namely, the largest real part of all the set of eigenvalues of* $A \pm \varepsilon^* A_0$ *is at most* $(-\rho)$. *Thus, system (1.1) is g.a.s. for all* $\varepsilon \in [-\varepsilon^*, \varepsilon^*]$ *and* $h \in [0, h^*]$ *if, furthermore,*

$$\|A_0\|_2 < \frac{e^{-h^*\rho}}{\varepsilon^*}\operatorname{Max}\left(\frac{1-\rho\gamma_0}{\gamma_0}, \frac{1-2\rho\gamma_{0\varepsilon^*}}{\gamma_{0\varepsilon^*}}e^{-h^*\rho}\right) \tag{3.3}$$

provided that $\gamma_0 < |\rho^{-1}|$ *and* $\gamma_{0\varepsilon^*} < 1/2\rho$.

Proof. Note that the characteristic zeros of system (1.1) satisfy

$$\begin{aligned} p(s,\varepsilon) &= \det\left(sI - A - \varepsilon A_0 e^{-hs}\right) = \det(sI - A)\det\left(I - \varepsilon(sI - A)^{-1}A_0 e^{-hs}\right) = 0 \\ &\Longleftrightarrow \det\left(I - \varepsilon(sI - A)^{-1}A_0 e^{-hs}\right) = 0 \end{aligned} \tag{3.4}$$

for all $s \in \mathbb{C}$ such that $\det(sI - A) \neq 0$, that is, it is not an eigenvalue of A. Note that for all nonzero ε, $\det(sI - A) \neq 0$ for all s being a zero of $p(s,\varepsilon)$. Note that (3.4) holds if and only if

$$p(s,\varepsilon) = 0 \Longleftrightarrow 1 - \varepsilon\frac{\operatorname{tr}\left(\operatorname{Adj}(sI - A)A_0\right)}{e^{hs}\det(sI - A)} + o(\varepsilon) = 0, \quad \forall \varepsilon \neq 0, \tag{3.5}$$

after using expansion in the powers of ε. Thus, as $0 \neq \varepsilon \to 0$, the root locus with respect to the ε-parameter establishes, from (3.5), that the zeros of $p(s,\varepsilon)$ are those infinitely many zeros of e^{hs}, which are stable with infinite real parts in the stable region (see Proposition 2.1) and are n zeros, possibly including some multiple roots which converge to the zeros of $\det(sI - A)$ as $\varepsilon \to 0$. Note that the distinct roots of $p(s,\varepsilon) = 0$ are isolated (Proposition 2.1) and they are continuous functions of ε, for all real ε, convergent to the s_i zeros of $\det(sI - A)$ as $\varepsilon \to 0$ for all $h \in [0, \infty)$. Then, for any given real $r > 0$, there is a set of $\mu \leq n$ open neighborhoods $|s - s_i| < r$, each including ν_i (the multiplicity of s_i) equal or distinct roots of $p(s,\varepsilon) = 0$ for some $\varepsilon^* > 0$, dependent in general on r, and all $\varepsilon \in (-\varepsilon^*, \varepsilon^*)$. If A is a stable matrix with stability abscissa $(-\rho) < 0$, $\varepsilon^* > 0$ may be chosen for $r = \rho/2$ so that the zeros of $p(s,\varepsilon)$ are in the stable region. (i) has been proved. The proofs of (ii) and (iii) follow directly under similar arguments by using

$$\begin{aligned} p(s,\varepsilon) &= \det\left(sI - A - \varepsilon A_0 e^{-hs}\right) \\ &= \det\left(sI - A - \varepsilon A_0\right)\det\left(I - \varepsilon\left(e^{-hs} - 1\right)\left(sI - A - \varepsilon A_0\right)^{-1}A_0\right) = 0 \\ &\Longleftrightarrow 1 - \varepsilon\left(e^{-hs} - 1\right)\frac{\operatorname{tr}\left(\operatorname{Adj}\left(sI - A - \varepsilon A_0\right)A_0\right)}{\det\left(sI - A - \varepsilon A_0\right)} + o(\varepsilon) = 0 \end{aligned} \tag{3.6}$$

and noting that as $|\varepsilon(e^{-hs} - 1)| \to 0$, which is the case as $\varepsilon \to 0$ and/or as $h \to 0$, n zeros, accounted for with their possible multiplicities, of $p(s,\varepsilon)$ converge to those of $\det(sI - A - \varepsilon A_0)$, while the remaining ones tend to infinity in the stable region (Proposition 2.1).

(iv) is proved as follows. Since A is a stable matrix, $(sI - A)$ is nonsingular for all $s \in \mathbb{C}^-$. Thus, $p(s,\varepsilon) \neq 0$ so that $\det(I - \varepsilon(sI - A)^{-1}A_0 e^{-hs}) \neq 0$ for all $s \in \mathbb{C}^-$ if $1 > \varepsilon\|(sI - A)^{-1}\|_\infty \|A_0\|_2$, since $|e^{-j\omega h}| = 1$ for all real ω, from Banach's perturbation lemma [1], and the continuity of the eigenvalues of a matrix with respect to any parameter, and then (1.1) is g.a.s. This proves the first assertion of (iv). Now, if $(A + \varepsilon A_0)$ is a stable matrix, then $p(s,\varepsilon) \neq 0$ for all $s \in \mathbb{C}^-$ provided that $1 > 2\varepsilon\|(sI - A - \varepsilon A_0)^{-1}\|_\infty \|A_0\|_2$ since $\det(sI - A - A_0) \neq 0$ for all $s \in \mathbb{C}^-$, again from Banach's perturbation lemma, continuity arguments, and the fact that $|e^{-j\omega h} - 1| \leq 2$ for all real ω. This proves the second assertion of (iv). The third assertion of (iv) follows directly by taking, as a sufficient stability condition, the less restrictive condition of the above two assertions.

(v) is proved as follows. First, note that the following Lyapunov matrix inequalities hold for any prefixed real matrix $P = P^T > 0$:

$$(A^T + \rho I \pm \varepsilon^* A_0^T)P + P(A + \rho I \pm \varepsilon^* A_0) < 0, \tag{3.7}$$

since the matrices $(A \pm \varepsilon^* A_0)$ both have stability abscissa of at least $(-\rho) < 0$. Summing up both sides of the above matrix inequality, one gets $(A^T + \rho I)P + P(A + \rho I) < 0$ which implies that $(A + \rho I)$ is a stability matrix so that A is also stable with stability abscissa at least $(-\rho) < 0$ as a result. Now, take the $H_{\infty\rho}$-norm ($H_{\infty 0} \equiv H_\infty$), reached on the boundary of the set $\mathbb{C}_{-\rho}$, of the matrix [7, 18]:

$$\begin{aligned}
&\|I - \varepsilon(sI - A)^{-1}A_0 e^{-hs}\|_{\infty\rho} \\
&\quad = \operatorname*{Max}_{s \in \mathbb{C}_{-\rho}} \left(\|I - \varepsilon(sI - A)^{-1}A_0 e^{-hs}\|_{\infty\rho}\right) \\
&\quad \leq \frac{1}{1 - \|(sI - A)^{-1} |e^{-hs}| \|_{\infty\rho}\|A_0\|_2} \\
&\quad \leq \frac{1}{1 - \operatorname{Max}_{\omega \in \mathbb{R}_0^+}\left(\left\|(-\rho I + (j\omega I - A))^{-1}\right\|\right) e^{h\rho}\|A_0\|_2}
\end{aligned} \tag{3.8}$$

provided that the denominator is positive since $(j\omega I - A)^{-1}$ exists for all real ω, since A is a stability matrix. Since

$$\|(-\rho I + (j\omega I - A)^{-1})\|_2 \leq \|(j\omega I - A)^{-1}\|_2 \|I - \rho(j\omega I - A)^{-1}\|_2 \leq \frac{\gamma_0}{1 - \rho\gamma_0}, \tag{3.9}$$

if $\rho < \gamma_0^{-1}$, then (1.1) is g.a.s. for all $h \in [0, h^*]$ if $\|A_0\|_2 < ((1 - \rho\gamma_0)/\varepsilon^*\gamma_0)e^{-h^*\rho}$. Since both $(A \pm \varepsilon^* A_0)$ are also stability matrices with stability abscissa of at least $(-\rho) < 0$, a similar reasoning leads to $\|A_0\|_2 < ((1 - 2\rho\gamma_{0\varepsilon^*})/\varepsilon^*\gamma_{0\varepsilon^*})e^{-2h^*\rho}$. Combining both conclusions, (i) follows. □

Note that Theorem 3.1(v) may also be established for any given A_0 in terms of sufficient smallness of ε^*. However, it turns out that it is more useful for applications as stated since the gain $\gamma_{0\varepsilon^*}$ depends, in general, on ε^*. The following result proves that all the unstable zeros are always finite for any finite delay but they are still finite even when $h \to 0$ and when $h \to \infty$. In the same way, the stable zeros are finite for bounded delay. However, they diverge on $\mathbb{C}^-$ as $h \to 0$, as has also been proved in Proposition 2.1 and Theorem 3.1(i).

THEOREM 3.2. *The following items hold.*

(i) *All the unstable zeros of $p(s,\varepsilon)$, if any, are finite for all finite or unbounded delay h and all finite ε including the case $h \to 0$.*
(ii) *All the stable zeros of $p(s,\varepsilon)$, if any, are finite for all $h \in (0,\infty)$ and all finite ε.*
(iii) *All the zeros of $p(s,\varepsilon)$ are finite for all $h \in (0,\infty)$ and all finite ε.*
(iv) *Assume that $h \to \infty$ and A has n_u, $0 \le n_u \le n$, distinct strictly unstable zeros, that is, s_i, that is, $\operatorname{Re} s_i > 0$, $i = 1,2,\ldots,n$. Thus, for any $r > 0$, there is a nonnegative real constant $h^*(r)$ such that n_u zeros of $p(s,\varepsilon)$ are in n_u open neighborhoods $B_i(s,r) = \{s \in \mathbb{C} : |s - s_i| < r\}$, for all $h \in [h^*,\infty)$. The infinitely many remaining zeros are in an open neighborhood $B(0,r)$ of zero.*

Proof. To prove (i), note that

$$\begin{aligned}
|p(s,\varepsilon)| &= \left| \sum_{i=0}^{n}\sum_{k=0}^{n} c_{ik}(\varepsilon) s^i e^{-khs} \right| \\
&= \left| \sum_{i=0}^{n} \left[c_{i0} + \varepsilon \sum_{k=0}^{n} c'_{ik}(\varepsilon) e^{-khs} \right] s^i + o(\varepsilon) \right| \\
&\le |p_0(s,\varepsilon)| + |\varepsilon| \, |c'_{ik}(\varepsilon) s^i e^{-khs}| + o(|\varepsilon|) \qquad (3.10)\\
&\le \left(M_0 + \varepsilon(n+1)M'_0 + o(|\varepsilon|)\right) \left| \sum_{i=1}^{n} s^i \right| + |s|^n, \\
|p(s,0)| = |p_A(s)| &\le M_0 \left| \sum_{i=0}^{n} s^i \right| + |s^n|,
\end{aligned}$$

for all $\operatorname{Re} s \ge 0$, since $|e^{-khs}| \le 1$, and all the coefficients $c_{i0} = c'_{i0}$ with $|c_{i0}| \le M_0$, for all $i \ge 0$, and $c_{ik}(\varepsilon) = \varepsilon c'_{ik}(\varepsilon)$, for $k \ge 1$ and all $i \ge 0$, in the expansion of $(p(s,\varepsilon) - p(s,0))$ in terms $s^i e^{-khs}$, for $k \ge 1$, involve powers ε^ℓ ($\ell \ge 1$) provided that the above normalized coefficients satisfy $|c'_{ik}| \le M'_0 < \infty$, $i = 0,1,2,\ldots,n$, $k = 0,1,\ldots,n$. If $|\varepsilon| \le M_0/M'_0$, then $|p(s,\varepsilon)| \le (n+2+|o(\varepsilon)|)M_0 |\sum_{i=0}^{n} s^i|$.

Now, from (3.10), one gets

$$|p(s,0)| \ge |s|^n \left(1 - \sum_{i=1}^{n} \frac{M_0}{|s|^i}\right) > |s|^n \left(1 - \frac{M_0}{|s|-1}\right) > 0, \qquad (3.11)$$

for all $s \in \mathbb{C}$, such that $|s| > M' = M_0 + 1$ since $|c_{i0}| \le M_0$, $i = 0,1,2,\ldots,n-1$, and

$$|p(s,\varepsilon)| \ge |s|^n \left(1 - \sum_{i=1}^{n} \frac{M}{|s|^i}\right) > |s|^n \left(1 - \frac{M}{|s|-1}\right) > 0 \qquad (3.12)$$

with $M = (n+2+|o(\varepsilon)|)M_0 \geq M_0 + \varepsilon(n+1)M_0' + |o(\varepsilon)|$ for all $s \in \mathbb{C}$, such that $|s| > M+1$ since $|c_{ik}| \leq M$, $i = 0,1,2,\ldots,n-1$, $k = 0,1,\ldots,n$. As a result, all the unstable zeros of $p(s,\varepsilon)$, if any, are finite for any finite or unbounded delay and any finite ε, and then (i) is proved.

Alternative proof of (i). Proceed by contradiction. Assume that $s_0 = \sigma_0 + j\omega_0$ is a zero of $p(s,\varepsilon)$, $|s_0| \to \infty$, and $h \in [0,\infty)$. Then, from (3.12), $|p(s_0,\varepsilon)| \geq |s_0^n| - \sum_{i=1}^{n-1} O(|s_0^i|)O(|\varepsilon|) \to \infty$, as $|s_0| \to \infty$, for any finite real ε. Thus, s_0 is not a zero of $p(s,\varepsilon)$. □

To prove (ii), first the following equivalent expansion of $p(s,\varepsilon)$ in powers of e^{-hs} becomes $p(s,\varepsilon) = \sum_{i=0}^{n} p_i(e^{-hs},\varepsilon)s^i$:

$$p_n(s,\varepsilon) = c_{nn}e^{-nhs} + \sum_{k=0}^{n-1}\left(\sum_{i=0}^{n} c_{ik}(\varepsilon)\right)e^{-khs}. \tag{3.13}$$

Note that if $|p(s_i,\varepsilon)| \to \infty$, then $|p(s_i,\varepsilon)| \to \infty$ so that s_i cannot be a zero of $p(s,\varepsilon)$. As a result, any complex s which is a zero of $p(s,\varepsilon)$ must satisfy

$$\infty > \left|p_n(s,\varepsilon)\right| \geq \left|c_{nn}\right| e^{nh|s|}\left|1 - \frac{f(s,\varepsilon)}{e^{h|s|}-1}\right| > 0 \tag{3.14}$$

for any bounded real ε, any bounded complex s, for all $h \in (0,\infty)$ and s satisfying $\operatorname{Re} s < 0$, and $e^{h|s|} > (f(s,\varepsilon) - 1)/|c_{nn}|$ with $f(s,\varepsilon) = \overline{M}_0 + \varepsilon(n+1)\overline{M}_0'|s^n| + o(|\varepsilon|)$ for $\overline{M}_0 = \operatorname{Max}(1,M_0)$ and $\overline{M}_0' \geq \operatorname{Max}_{0\leq i\leq n-1}(\operatorname{Max}_{0\leq k\leq n}(|c_{ik}'|))$. Note that if $h > 0$ and $|s| \to \infty$, then $|p(s,\varepsilon)| \to \infty$ for $\operatorname{Re} s < 0$, and then $p(s,\varepsilon)$ cannot be upper-bounded or zero, and then s cannot be a zero, so that all stable zeros, if any, are finite. (i) has been proved.

Remark on the proof of (ii). Note that if $h \to 0$ and $|s| \to \infty$ with $\sigma = \operatorname{Re} s < 0$ at a rate $|\sigma^{-1}| = o(h)$, that is, $|\sigma h| \to \infty$, and $e^{h|\sigma|}$ and $|\sigma|$ diverge at the same rate, then $|p(s,\varepsilon)| \to 0$ and its lower bound in (3.14) fails as it has been proved in Proposition 2.1 and Theorem 3.1(i). Note that $e^{h|\sigma|}$ and $|\sigma|$ diverge at the same rate if $h|\sigma| \approx \ln|\sigma|$.

To prove (iii), note that condition (3.14) is guaranteed for all complex $s = \sigma_0 + j\omega_0$ that satisfies

$$\frac{\sqrt{\sigma_0^2+\omega_0^2}}{\ln\left(\sqrt{\sigma_0^2+\omega_0^2}\right)} > \frac{1}{h}\frac{(M_0+o(|\varepsilon|))}{\ln\left(\sqrt{\sigma_0^2+\omega^2}\right)(|c_{nn}| - |\varepsilon|(n+1)M_0')} > \frac{n}{h} \tag{3.15}$$

provided that $|\varepsilon| < |c_{nn}|(1/(n+1)M_0')$. Define $\bar{\sigma}_0 = \operatorname{Min}(\sigma_0 \in \mathbb{C}^- : (3.14)\text{ holds})$. Thus, all the zeros of $p(s,\varepsilon)$ for all finite real ε and any delay $h \in (0,\infty)$ are bounded and fulfill $-\sigma_0^* < |s| < M+1$ and $-\sigma_0^* < \operatorname{Re} s < M+1$.

To prove (iv), first note that

$$p(s,\varepsilon) = \det\left(sI - A - \varepsilon A_0 e^{-h^*s} - \varepsilon\left(e^{(h^*-h)s} - 1\right)e^{-h^*s}A_0\right) \longrightarrow p(s,0) = p_A(s) \tag{3.16}$$

for all complex s which is not an eigenvalue of A and all finite $\varepsilon \neq 0$ since $|\varepsilon e^{-hs}| \to 0$ for $\operatorname{Re} s > 0$ as $h \to \infty$. Note that for $\varepsilon \neq 0$, a zero of $p(s,\varepsilon)$ is not an eigenvalue of A; that is, $p(s,\varepsilon) \neq p_A(s)$ for $\varepsilon \neq 0$. Furthermore,

$$p(s,\varepsilon) \to p_A(s) \det\left(I - \varepsilon(sI - A)^{-1} A_0 e^{-h^* s}\right) \tag{3.17}$$

as $(h - h^*) \to \infty$, for any complex s which is not an eigenvalue of A, if $|\varepsilon e^{-h^* s}| \to 0$ since $|\varepsilon e^{(h^* - h)s} - 1| \to 0$. Thus, there are open neighborhoods $B_i(s,r)$, $i = 1,2,\ldots,n_u$, for any given $r > 0$ and nonzero ε, which contain all the zeros of $p(s,\varepsilon)$ for all $h \in [h^*,\infty)$, some $h^* = h^*(r)$ for each given ε, from the continuity of the roots. Now, note from the form of $p(s,\varepsilon)$ that $|s_0| \approx |e^{-hs_0}|$ in order for s_0 to be a zero of $p(s,\varepsilon)$ as $h \to \infty$ unless $|e^{-hs_0}|$ tends to zero, which is the case discussed above for $\operatorname{Re} s > 0$. Thus, the infinitely many remaining zeros have to satisfy $|s| \approx |e^{-hs}|$ with $\sigma = \operatorname{Re} s \leq 0$ if $\varepsilon \neq 0$ (otherwise, $e^{h|\sigma|}$ would diverge at a faster rate than $|s|$, and then s is not a zero of $p(s,\varepsilon)$). This implies that $\sigma \to 0$ since $h \approx |\ln \sigma / \sigma| \to \infty$. Thus, it follows from the continuity of the roots that all the remaining infinitely many zeros are in an open neighborhood of zero of any prescribed radius $r > 0$ for all sufficiently large delay depending on r and ε. □

Note that it follows from Proposition 2.1 and Theorems 3.1 and 3.2(iv) that if (1.1) is stable and independent of delay, then the infinitely many zeros which converge to zero are evolving with $\operatorname{Re} s < 0$ with $\operatorname{Re} s \to -\infty$ from $h \to 0$ up to $h \to \infty$ through $\mathbb{C}^-$. It is of interest to compare how close the infinitely many roots of the characteristic equation of the delayed system (1.1), that is, the zeros of $p(s,\varepsilon)$, are to those of $p(s,0) = p_A(s)$ according to the size of the delayed dynamics in (1.1). The following result, whose proof is omitted for space reasons, addresses this point.

Theorem 3.3. *The following items hold for a given delay h.*

(i) *Consider the closed bounded rectangle $\mathbb{C} \supset \mathbb{C}_1 = \{s \in \mathbb{C} : |\operatorname{Re} s| \leq m_1;\ |\operatorname{Im} s| \leq m_2\}$ of boundary $\Gamma_{\mathbb{C}_1}$ which contains all the roots of $p(s,0) = 0$. Thus, for any given real constant $m > |p(s,0)| > 0$ for all $s \in \mathbb{C}_1$, there is a real constant $\varepsilon^* > 0$ such that $|p(s,\varepsilon) - p(s,0)| < m$ and $|p(s,\varepsilon)| < 2m$ for all $s \in \mathbb{C}_1$ and all $\varepsilon \in [-\varepsilon^*,\varepsilon^*]$. In the same way, for any given real constant $m > |p(s,0)| > 0$ for all $s \in \mathbb{C}_1$, there is a real constant $\varepsilon^{*'} > 0$ such that $|p(s,\varepsilon) - p_{A+\varepsilon B}(s)| < m$ and $|p(s,\varepsilon)| < 2m$ for all $s \in \mathbb{C}_1$ and all $\varepsilon \in [-\varepsilon^{*'},\varepsilon^{*'}]$.*

(ii) *For any given real constants $m > 0$ and $1/2 > \lambda > 0$, there is a polynomial $\bar{p}_0(s,\varepsilon) = p(s,0) + \varepsilon \sum_{i=0}^{N} \bar{p}_{ik} s^i$, of sufficiently large degree N, such that*

$$\begin{aligned} |p(s,\varepsilon) - \bar{p}_0(s,\varepsilon)| &< m, \\ |p(s,0) - \bar{p}_0(s,\varepsilon)| &< \lambda m \quad \text{if } |\varepsilon| < m \operatorname{Min}\left(\frac{\lambda}{2m'}, 2^{N+1}\right). \end{aligned} \tag{3.18}$$

Note that Theorem 3.3(i) may be interpreted as a version of Runge's theorem [12], for approximation of analytic functions in the bounded perfect set $\mathbb{C}_1$ whose complementary in the extended complex plane is an open connected set. In fact, for all $s \in \mathbb{C}_1$ and any real $m > 0$, there is a real ε^*, which depends on m in general, such that for all $\varepsilon \in [-\varepsilon^*,\varepsilon^*]$, there is a polynomial $q(s)$ such that the uniform and locally analytic function $p(s,\varepsilon)$ satisfies $|p(s,\varepsilon)-q(s)|<m$, for all $s\in\mathbb{C}_1$. In this case, $q(s)=p(s,0)=p_A(s)$ is the polynomial which describes the system dynamics as the delayed dynamics vanishes. The second part of Theorem 3.3(i) is developed in a similar way for $q(s)=p_{A+\varepsilon A_0}(s)$ which describes the system dynamics for zero delay. Thus, Theorem 3.3(i) quantifies how close the current dynamics is to the delay-free dynamical systems $\dot{x}(t)=Ax(t)$, that is, $\varepsilon=0$ and/or infinite delay, and $\dot{x}(t)=Ax(t)+\varepsilon A_0x(t)$, respectively. Theorem 3.3(ii) approximates the behavior of the delayed dynamics closely to that of a delay-free system of sufficiently large order since the characteristic quasipolynomial may be approximated, with any prescribed accuracy degree, by a polynomial of sufficiently large degree over a relevant finite open region. This may be considered as an ad hoc version of Montel's theorem concerning time-delay systems with point delays for the approximation of uniform locally analytic functions by appropriate polynomials [12].

4. Calculation of the characteristic zeros from perturbation theory

Define $\mathbf{A}(s,\varepsilon)=A+\varepsilon(\mathbf{s})A_0$ with $\varepsilon(s)=\varepsilon e^{-hs}$ and consider $\mathbf{A}(s,\varepsilon)$ as a perturbed matrix of A. The following simple preliminary result, which is concerned with the existence of a simple factorization for the characteristic quasipolynomial of (1.1), holds.

Proposition 4.1. *The characteristic quasipolynomial $p(s,\varepsilon)$ of (1.1) may be factorized as $p(s,\varepsilon)=\prod_{i=1}^{\mu_\varepsilon}(s-\lambda_i(s,\varepsilon))^{\nu_{i\varepsilon}}$ with $1\le\nu_{i\varepsilon}\le\mu_\varepsilon\le n=\sum_{i=1}^{\mu_\varepsilon}\nu_{i\varepsilon}$, $i=1,2,\dots,\mu_\varepsilon$.*

Proof. The proof is made constructively as follows. For each fixed complex s and real ε, consider the square n-complex matrix $X_{s\varepsilon}=A+\varepsilon e^{-hs}A_0$ so that there is a nonsingular square n-complex matrix Y such that $X_{\Lambda x_{s\varepsilon}}=Y_{s\varepsilon}^{-1}X_{s\varepsilon}Y_{s\varepsilon}$ is a Jordan diagonal matrix with identical eigenvalues of identical multiplicities to those of $X_{s\varepsilon}$. Consider, for each real ε, a complex matrix function $T_\varepsilon: s\in\mathbb{C}\to\mathbb{C}^{n\times n}$ defined by $\mathbf{T}(s,\varepsilon):=T_\varepsilon(s)=Y_{s\varepsilon}$, which is trivially everywhere nonsingular. It turns out that $\Lambda(\mathbf{s},\varepsilon)=\mathbf{T}^{-1}(\mathbf{s},\varepsilon)\mathbf{A}(\mathbf{s},\varepsilon)\mathbf{T}(\mathbf{s},\varepsilon)$ is the image of a complex matrix function $\Lambda_\varepsilon: s\in\mathbb{C}\to\mathbb{C}^{n\times n}$ which is a complex diagonal matrix for all s whose distinct eigenvalues are μ distinct complex functions of images $\lambda_i(s,\varepsilon)$ of multiplicities $\nu_{i\varepsilon}$ satisfying

$$1\le\nu_{i\varepsilon}\le\operatorname*{Inf}_{s\in\mathbb{C}}\left(\nu_i(s,\varepsilon)\right)\le\mu_\varepsilon=\operatorname*{Inf}_{s\in\mathbb{C}}\left(\mu_i(s,\varepsilon)\right)\le\mu\le n=\sum_{i=1}^{\mu_\varepsilon}\nu_{i\varepsilon}=\sum_{i=1}^{\mu}\nu_i. \tag{4.1}$$

□

Note that, typically, $n=\mu_\varepsilon$ and $\nu_i=1$ $(i=1,2,\dots,n)$ despite the fact that eigenvalues with multiplicities larger than unity can occur at particular values of the complex plane since the $\lambda_i(s,\varepsilon)$ are complex functions, as s takes values in $\mathbb{C}$ for each real ε whose particular values are the eigenvalues of the corresponding complex matrix $\mathbf{A}(s,\varepsilon)$. No particular characterization as complex functions is made for the $\lambda_i(s,\varepsilon)$ and the entries of $\mathbf{T}(\mathbf{s},\varepsilon)$, while we are only interested in the existence of such a factorization. Note also that if $h=0$, then $\varepsilon(s)=\varepsilon$ and $\mathbf{A}=A+\varepsilon A_0$ is independent of s. Thus, the infinitely many eigenvalues of $\mathbf{A}(\mathbf{s},\varepsilon)$ may be exactly calculated as a perturbation of the $\mu\le n$ distinct eigenvalues of A using Kato's formula [1], as $\lambda_i(\mathbf{A}(\mathbf{s},\varepsilon))=\sum_{k=0}^{\infty}\delta_i^{(k)}\varepsilon^{\mathbf{k}}(\mathbf{s})$ with the following cases.

Case 1 (all the eigenvalues λ_i of A are distinct).

$$\lambda_i(\mathbf{A}(\mathbf{s},\varepsilon)) = \sum_{k=0}^{\infty} \delta_i^{(k)} \varepsilon^{\mathbf{k}}(\mathbf{s}),$$
$$\delta_i^{(k)} = (-1)^k k^{-1} \operatorname{tr}\left(\sum_{\sum_{l=1}^{k} k_i = k-1\,(k_i \geq 0)} \prod_{l=1}^{k} [A_0 S_i^{k_l}] \right), \tag{4.2}$$

for $k \geq 1$ with $\delta_i^{(0)} = \lambda_i^{(0)} = \lambda_i$ (ith eigenvalue of A), where the S_i are constant complex matrices defined by $S_i = \sum_{k=1,\ i\neq k}^{n} E_k/(\lambda_k - \lambda_i)$ with the E_k-matrices being the components of A which are defined such that $A = \sum_{i=1}^{n} \lambda_i E_i$ [1, 17]. From Proposition 4.1,

$$p(s,\varepsilon) = \det\left(sI - \mathbf{A}(\mathbf{s},\varepsilon)\right) = \prod_{i=1}^{n} [s - \lambda_i(s,\varepsilon)]. \tag{4.3}$$

All the zeros of $p(s,\varepsilon)$ are got by zeroing each factor in that expression. Each of those factors generates infinitely many zeros $s_i = \lambda_i^{(0)} = \lambda_i = \sigma_i + j\omega_i$, for any order of approximation $k \geq 1$,

$$s_i - \lambda_i(s,\varepsilon) = \sigma_i + j\omega_i - \sum_{k=0}^{\infty} \delta_k^{(i)} \varepsilon^k e^{-kh\sigma}(\cos k\omega h - j\sin k\omega h) = 0 \tag{4.4}$$

for $i = 1,2,\ldots,n$, with the above matrix product defined through expansion to the right as l increases. From the above formula, it is easy to obtain low-order approximations. For instance, the first-order approximation for the ith eigenvalue is stated as the infinitely many solutions of the following equations:

$$\begin{aligned} &\sigma_i^{(1)} - \operatorname{Re}\lambda_i - \varepsilon e^{-h\sigma_i}\left(\operatorname{Re}\lambda_i^{(1)} \cos\omega_i^{(1)} h + \operatorname{Im}\lambda_i^{(1)} \sin\omega_i^{(1)} h\right) = 0, \\ &\omega_i^{(1)} - \operatorname{Im}\lambda_i + \varepsilon e^{-h\sigma_i}\left(\operatorname{Re}\lambda_i^{(1)} \sin\omega_i^{(1)} h - \operatorname{Im}\lambda_i^{(1)} \cos\omega_i^{(1)} h\right) = 0, \end{aligned} \tag{4.5}$$

for $i = 1,2,\ldots,n$. The following result holds.

Theorem 4.2. *The following items hold.*

(i) *The unstable eigenvalues of $A(s,\varepsilon)$, if any, are inside the circles $|\lambda_i(\mathbf{A}(\mathbf{s},\varepsilon)) - \lambda_i| \leq \operatorname{Sup}_{0\leq k<\infty}(|\delta_i^{(k)}|)(\varepsilon/(1-\varepsilon))$ independent of any finite delay h provided that $|\varepsilon| < 1$. As a result, if A is a stability matrix with stability abscissa $(-\rho_0) < 0$, then $A(s,\varepsilon)$ remains a stability matrix if $|\varepsilon| < \operatorname{Min}(1, \rho_0/(\rho_0 + \operatorname{Sup}_{0\leq k<\infty}(|\delta_i^{(k)}|)))$.*

(ii) *Any eigenvalues of $A(s,\varepsilon)$ (including all the unstable ones) satisfying $\operatorname{Re} s \geq -\rho_0$, if any, are inside circles $|\lambda_i(\mathbf{A}(\mathbf{s},\varepsilon)) - \lambda_i| \leq \operatorname{Sup}_{0\leq k<\infty}(|\delta_i^{(k)}|)(\varepsilon e^{h\rho_0}/(1-\varepsilon e^{h\rho_0}))$ independent of any finite delay h provided that $|\varepsilon| < e^{-h\rho_0}$.*

(iii) *The kth approximations of the expressions for the eigenvalues of $A(s,\varepsilon)$ of (i) and (ii) are, respectively,*

$$\begin{aligned}
\left|\lambda_i^{(k)}(\mathbf{A}(\mathbf{s},\varepsilon))-\lambda_i\right| &\le \operatorname*{Sup}_{0\le l\le k}\left(\left|\delta_i^{(l)}\right|\right)\frac{\varepsilon(1-\varepsilon^k)}{1-\varepsilon} \quad \text{if } |\varepsilon|<1,\\
\left|\lambda_i^{(k)}(\mathbf{A}(\mathbf{s},\varepsilon))-\lambda_i\right| &\le \operatorname*{Sup}_{0\le l\le k}\left(\left|\delta_i^{(l)}\right|\right)\frac{\varepsilon e^{h\rho_0}(1-\varepsilon^k e^{kh\rho_0})}{1-\varepsilon^{h\rho_0}} \quad \text{if } |\varepsilon|<e^{-h\rho_0}.
\end{aligned} \tag{4.6}$$

Proof. The proof of (ii) follows directly from the identity $\sum_{k=1}^{\infty}\delta_i^{(k)}\varepsilon^k(s)=\sum_{k=0}^{\infty}\delta_i^{(k)}\varepsilon^k(s)-\delta_i^{(0)}$ since $\sum_{k=0}^{\infty}\varepsilon^k(s)$ converges for $\operatorname{Re}s\ge-\rho_0$ since $|\varepsilon(s)|\le|\varepsilon e^{h\rho_0}|<1$ for $|\varepsilon|<e^{-h\rho_0}$. The proof of (ii) is similar to that of (i) for the particular case with $\operatorname{Re}s\ge(-\rho_0)\equiv 0$. The proof of (iii) follows in the same way as those of (i) and (ii) after using

$$\left|\sum_{\ell=1}^{k}\delta_i^{(\ell)}\varepsilon^{\ell}(s)\right| = \left|\sum_{l=1}^{\infty}\delta_i^{(l)}\varepsilon^k(s)-\sum_{l=k+1}^{\infty}\delta_i^{(l)}\varepsilon^l(s)\right| \le \frac{\varepsilon e^{h\rho_0}(1-\varepsilon^k e^{kh\rho_0})}{1-\varepsilon^{h\rho_0}} \tag{4.7}$$

for $\operatorname{Re}s\ge-\rho_0$ if $|\varepsilon|<e^{-h\rho_0}$. □

In general, repeated eigenvalues in the A-matrix may not generate repeated eigenvalues in $A(\mathbf{s},\varepsilon)$. The above method may be generalized for the case of eigenvalues with multiplicities larger than one in two alternative ways as follows.

Case 2 (there exist some eigenvalues of A with multiplicity larger than unity). Note that $f(A)=\sum_{k=1}^{\mu}\sum_{l=0}^{\nu_k-1}f^{(l)}(\lambda_k)E_{kl}$ for f being any analytic function at the eigenvalues λ_k of A of multiplicity ν_k, $k=1,2,\dots,\mu$, and E_{kl}, $l=0,1,\dots,\nu_k-1$, are the components of A [17]. If $\mu=n$, all the multiplicities are unity, and then $E_{k0}=E_k$ (Case 1). Since $A=f(A)$ for the complex function $f(\lambda)=\lambda$, it follows that

$$A=\sum_{k=1}^{\mu}\left(\lambda_k E_{k0}+E_{k1}\right)=\sum_{k=1}^{n}\lambda_k' E_k' \tag{4.8}$$

since $f^{(1)}(\lambda)=df(\lambda)/d\lambda=1$ for λ taking values at all the eigenvalues, $E_i'=E_{i0}$ if $\lambda_i=0$ and/or $\nu_i=1$, and $E_k'=E_{i1}/(\nu_i-1)\lambda_i$ if $\nu_i\ge 2$ with $\lambda_k'=\lambda_i$ for all $i=1,2,\dots,\mu$ and each ν_i value for the integer $k\in(\sum_{l=1}^{i-1}\nu_l,\sum_{l=1}^{i}\nu_l)$ that generates ν_i auxiliary components E_k' for the eigenvalue λ_i of A. Note that the total number of auxiliary components is n and that the second identity in (4.8) is obtained constructively from the first one by repeating each eigenvalue according to its multiplicity. Now, proceed as follows. First calculate E_{i0}, E_k, and S_k $(i=1,2,\dots,\mu,\ k=1,2,\dots,n)$ satisfying [1, 17]

$$\begin{gathered}
\sum_{k=1}^{\mu}E_{i0}=I,\quad E_{i0}^2=E_{i0}\quad \text{(idempotent matrix)},\\
E_{kl}E_{\mathrm{im}}=0\quad \forall 0\le l\le\nu_k-1,\ 0\le m\le\nu_i-1,\\
S_k'=\sum_{\substack{l=1\\ l\ne k}}^{n}\frac{E_l'}{\lambda_l-\lambda_k}\quad (k=1,2,\dots,n).
\end{gathered} \tag{4.9}$$

The eigenvalues of $\mathbf{A}(\mathbf{s},\varepsilon)$ are calculated as

$$\lambda_i(\mathbf{A}(\mathbf{s},\varepsilon)) = \sum_{k=0}^{\infty} \delta_i^{'(k)} \varepsilon^{\mathbf{k}}(\mathbf{s}), \qquad \delta_k^{'(0)} = \lambda_l,$$
$$\delta_i^{'(k)} = (-1)^k k^{-1} \operatorname{tr}\left(\sum_{\sum_{i=1}^{k} k_i = k-1\,(k_i \geq 0)} \left[\prod_{l=1}^{k} A_0 S_i^{'k_l} \right] \right), \tag{4.10}$$

with $\varepsilon(\mathbf{s}) = \varepsilon e^{-hs}$ for $l = 1,2,\dots,\mu$, $i = 1,2,\dots,n$, $k \in (\sum_{l=1}^{i-1} \nu_l, \sum_{l=1}^{i} \nu_l)$. In other words, the method works in the same way as in the case of single eigenvalues by repeating each eigenvalue of A according to its multiplicity to use it as a generator in the perturbation calculation while using the set of n auxiliary components E'_k of the A-matrix. Those components are calculated from the standard ones E_{k0} and E_{k1} $(k = 1,2,\dots,\mu)$ in such a way that the identities in (4.8) hold.

5. Numerical example

Consider the system defined by the subsequent equations

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix} x(t) + \varepsilon \begin{pmatrix} 0 & 0 \\ 4 & 6 \end{pmatrix} x(t-h). \tag{5.1}$$

In order to determine the evolution in the complex plane of the poles depending on the variation of the delay for two different values of the parameter ε, the third-order Pade approximation is considered [14]. This approximate approach allows the approximate location of a finite set of poles graphically. On the other hand, the first-order perturbations method is applied for the determination of the stability of the system for the different values of the parameter ε. Comparing with the results obtained with Pade's approximation, the validity of the theoretical results proposed in this paper is tested. According to the developments in Section 4, the first-order approximation for the characteristic equation of the system is

$$\left(s + 1 + 6\varepsilon e^{-hs}\right)\left(s + 3 + 6\varepsilon e^{-hs}\right) = 0. \tag{5.2}$$

Using this equation, depending on ν, the stability results of the above sections can be used. First, the case $\varepsilon = 0.1$, when the system is stable for delay $h = 0$, is considered. Figure 5.1 shows the evolution of the poles when the delay h takes values 0.1, 0.5, 1, and 5. From this figure, the theoretical conclusions obtained are confirmed, since the poles introduced by the delay are in the left half-plane, and the distance from the origin to those poles grows to infinity when the delay goes to zero.

Note that this system is stable for any delay $h \geq 0$. This property is observed from Figure 5.2, which shows the maximal distance between the poles of the delay-free system ($s = -1$ and $s = -3$) and the poles in the right half-plane for any delay. The approximate characteristic equation (3.1) is used for such a purpose. The resulting region is inside the left half-plane, where all the poles are stable and then the system is stable for any delay.

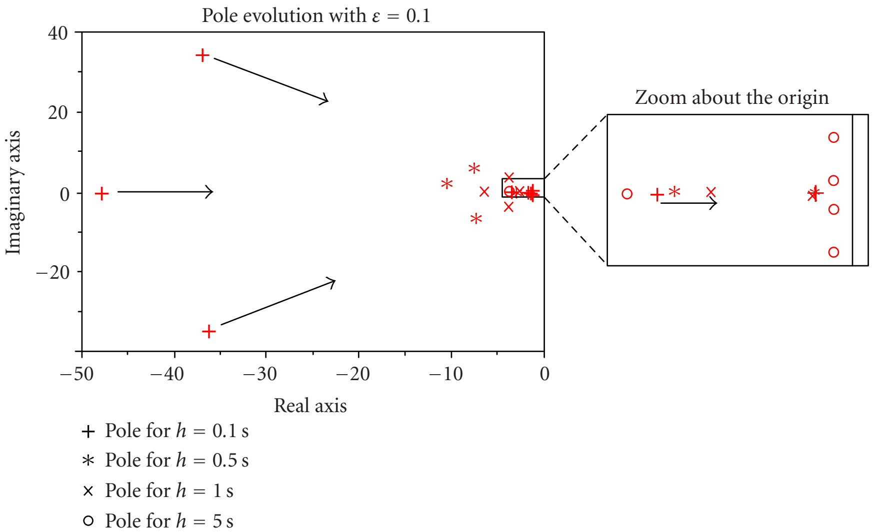

Figure 5.1. Third-order Pade's approximation: the arrows show the pole motion as the delay h increases for $\varepsilon = 0.1$.

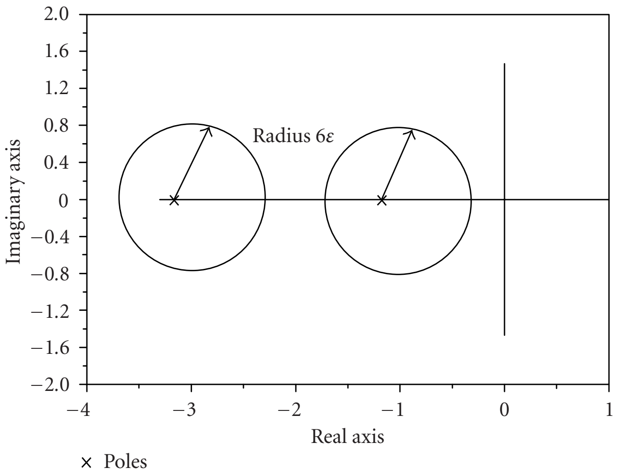

Figure 5.2. Possible situation of the first-order approximation unstable poles around the eigenvalues of A for $\varepsilon = 0.1$.

In Figure 5.3, the evolution of the poles is shown for $\varepsilon = 1$. In this case, the system is unstable when $h = 0$ and also for any delay $h > 0$. As in the previous case, the same conclusions about the evolution of the poles, which diverge within the left half-plane when the delay goes to zero, are obtained. However, the absolute values of those poles in the right half-plane go to zero asymptotically on increasing the delay, since the system is unstable. This confirms the foreseen conclusions from the theoretical results in Section 3.

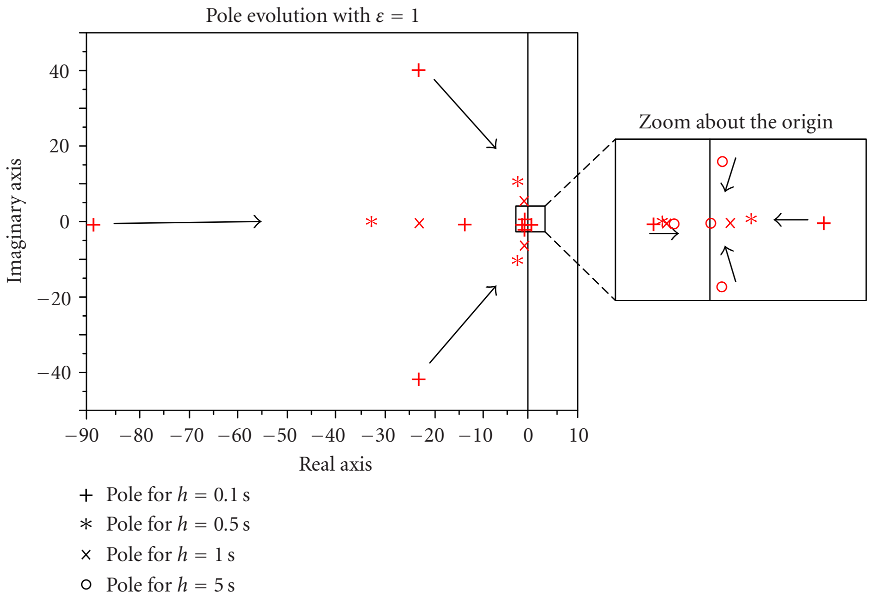

Figure 5.3. Third-order Pade's approximation: the arrows show the pole motion as the delay h increases for $\varepsilon = 1$.

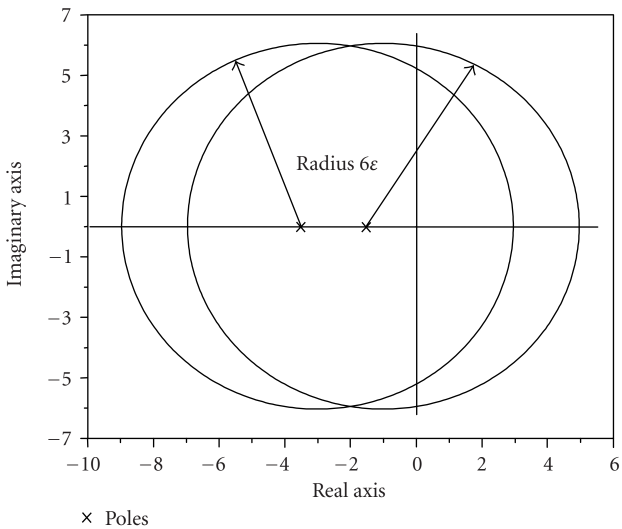

Figure 5.4. Possible situation of the first-order approximation unstable poles around the eigenvalues of A for $\varepsilon = 1$.

It follows from Figure 5.2 that for a first-order perturbation method, the approximate delayed system is stable for $\varepsilon = 0.1$, as follows from Theorem 4.2(iii). For $\varepsilon = 1$, note that the radii of the circles centred at the poles of the delay-free part of the system intersect the right half-plane, which indicates the potential possibility of existence of unstable poles, as is shown in Figure 5.4.

6. Concluding remarks

This paper has mainly addressed an analytic study related to the behavior of the limit zeros of time-invariant delay systems with point delays. Also, a computational method based on perturbation theory via Kato's formula has been given to calculate regions inside which it is possible to locate the characteristic zeros, and an example has been given to compare the computational efficiency with that obtained via Pade's approximation. The main obtained results are as follows.

(i) Infinitely many zeros of the delay system cannot be close to those of its delay-free counterpart as the delay tends to zero.
(ii) When the delay tends to zero, infinitely many zeros are located in the stable region with arbitrarily large absolute values, while the (finite) remaining ones converge to those of the delay-free system.
(iii) As the delay tends to infinity, infinitely many zeros converge to a small region around the origin.
(iv) All the unstable zeros are finite for all delays even if such a delay converges to zero or tends to infinity.
(v) Some complementary stability results independent of and others dependent on the delay size have been proved based on H_∞ stability theory.

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