REISSNER-MINDLIN PLATE THEORY FOR ELASTODYNAMICS

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Existence and uniqueness of solution are proved for elastodynamics of Reissner-Mindlin plate model. Higher regularity is proved under the assumptions of smoother data and certain compatibility conditions. A mass scaling is introduced. When the thickness approaches zero, the solution of the clamped Reissner-Mindlin plate is shown to approach the solution of a Kirchhoff-Love plate.

1. Introduction

The Reissner-Mindlin (R-M) theory [13, 14, 15] has been popularly applied to thin-walled structures with moderate thickness. Transient response plays an important role in many aspects of structural analysis. The governing equation of the elastodynamics problem of R-M plate is of an evolutionary type with second-order time derivatives. In this paper, we apply a priori estimate to investigate the elastodynamics problem of R-M plate. This method has been successfully used in developing the theory of various partial differential equations, for example, [7, 10, 11]. Following the line of [7, 10, 11], we prove the existence and uniqueness of the H^1 solution. We then apply the approaches of [8] to prove the H^2 regularity and higher regularity when the data is smoother and certain compatibility conditions are satisfied.

For static problem under the assumption of load scaling, it is proved in [3] that the solution of the clamped R-M plate approaches the solution of the Kirchhoff-Love (K-L) plate when the thickness approaches zero. This fact has been employed to investigate the finite element method of R-M plate, such as locking-free and uniform convergence, cf. [2, 3, 5, 12, 16, 18]. For dynamics problem, with the introduction of mass scaling [4], we prove that when thickness approaches zero, the H^2 strong solution of the clamped R-M plate approaches the H^2 weak solution of K-L plate (whose classical solution requires H^4 smoothness).

In what follows, we describe the system of equations in Section 2 and prove the existence, uniqueness, and regularity in Section 3. Then we discuss the relation between R-M plate and K-L plate in Section 4. This is followed by a summary in Section 5.

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2. Governing equations of Reissner-Mindlin plate for elastodynamics

For elastodynamic bending shear problem modeled by R-M plate theory [13, 14, 15], the displacement components of a generic point at a distance z to the midsurface are expressed by the deflection w at the midsurface and the rotations (β_1, β_2) of the normal to the midsurface,

$$U_1 = -z\beta_1, \qquad U_2 = -z\beta_2, \qquad U_3 = w, \tag{2.1}$$

$|z| \le \zeta/2$. ζ is the thickness of the plate. For dynamics problems, the velocity and acceleration, traditionally denoted by $\dot{U}_i$ and $\ddot{U}_i$, respectively, have the same format of (2.1) after differentiation with respect to time. The motion equation of R-M plate can be derived from the general three-dimensional elastodynamics by integration through thickness, or from the energy method using Hamilton's principle, for example, [9]

$$\begin{aligned} I\ddot{\beta}_1 + EA_1(\boldsymbol{\beta}) - \lambda\zeta^{-2}(w_{,1} - \beta_1) &= m_1, \\ I\ddot{\beta}_2 + EA_2(\boldsymbol{\beta}) - \lambda\zeta^{-2}(w_{,2} - \beta_2) &= m_2, \\ \rho\zeta^{-2}\ddot{w} - \lambda\zeta^{-2}\nabla\cdot(\nabla w - \boldsymbol{\beta}) = g &= f_3\zeta^{-2}, \end{aligned} \tag{2.2}$$

where we define

$$\begin{aligned} A_1(\boldsymbol{\beta}) &= \frac{-\left((1+\nu)(\beta_{\alpha,\alpha})_{,1} + (1-\nu)\nabla^2\beta_1\right)}{24(1-\nu^2)}, \\ A_2(\boldsymbol{\beta}) &= \frac{-\left((1+\nu)(\beta_{\alpha,\alpha})_{,2} + (1-\nu)\nabla^2\beta_2\right)}{24(1-\nu^2)}. \end{aligned} \tag{2.3}$$

Here, E is the Young's modulus, ρ is the density, and ν is the Poisson ratio. We denote $I = \rho/12$ and $\lambda = G\kappa$, with the shear modulus G and a shear correction factor κ, which is introduced to balance the zero shear stress at the top and bottom surfaces. As analyzed for static problem [2, 3, 5], the lateral loading force f_3 (per unit volume) is scaled to $\zeta^2 g$. The convention of summation on repeated indices is also applied, with the Greek index running over the range from 1 to 2. The bold-faced variables are used to denote a two-dimensional vector, for example, $\boldsymbol{\beta} = (\beta_1, \beta_2)$, and β_α is used to indicate all of the two components involved when the indication is clear. Here, $w_{,1}$ in (2.2) indicates the partial derivative $\partial w/\partial x_1$. The same applies for all the similar cases.

For simplicity, we consider the equations defined on a smooth bounded domain Ω in R^2, with homogeneous Dirichlet boundary conditions and general initial conditions

$$\begin{aligned} &\beta_\alpha(t,\mathbf{x}) = 0; \qquad w(t,\mathbf{x}) = 0 \quad \text{on } \partial\Omega, \\ &\beta_\alpha(0,\mathbf{x}) = \beta_\alpha^0(\mathbf{x}); \qquad w(0,\mathbf{x}) = W^0(\mathbf{x}), \\ &\dot{\beta}_\alpha(0,\mathbf{x}) = \beta_\alpha^1(\mathbf{x}); \qquad \dot{w}(0,\mathbf{x}) = W^1(\mathbf{x}). \end{aligned} \tag{2.4}$$

We adopt the usual notations of Sobolev spaces. The Galerkin method yields the following variational equation. For any $t \in [0,T]$, find β_α, $w \in V = H_0^1(\Omega)$ such that

$$I\langle \ddot{\beta}_\alpha,\eta_\alpha\rangle+\rho\zeta^{-2}\langle \ddot{w},v\rangle+Ea(\boldsymbol{\beta},\boldsymbol{\eta})+\lambda\zeta^{-2}(w_{,\alpha}-\beta_\alpha,v_{,\alpha}-\eta_\alpha)=(m_\alpha,\eta_\alpha)+(g,v), \quad \forall \eta_\alpha,v\in V. \tag{2.5}$$

Here, $(\cdot,\cdot)$ denotes the usual L^2 inner product, and $\langle\cdot,\cdot\rangle$ denotes the duality on $V'\otimes V$. $a(\cdot,\cdot)$ is a bilinear form on $V\otimes V$ defined as

$$a(\boldsymbol{\beta},\boldsymbol{\eta}) = \frac{1}{24(1-\nu^2)}\big((1+\nu)(\beta_{\alpha,\alpha},\eta_{\alpha,\alpha})+(1-\nu)(\nabla\beta_\alpha,\nabla\eta_\alpha)\big). \tag{2.6}$$

It is associated with the operators A_1 and A_2 such that

$$a(\boldsymbol{\beta},\boldsymbol{\eta}) = \langle A_1(\boldsymbol{\beta}),\eta_1\rangle + \langle A_2(\boldsymbol{\beta}),\eta_2\rangle = \langle \mathbf{A}(\boldsymbol{\beta}),\boldsymbol{\eta}\rangle, \quad \forall \beta_\alpha,\eta_\alpha \in V. \tag{2.7}$$

In fact, $a(\cdot,\cdot)$ is symmetric, the same as the two-dimensional elasticity operator with a scalar factor. With Dirichlet boundary conditions, $a(\cdot,\cdot)$ is equivalent to the H^1-norm on V [6] and there exist constants $\alpha_1,\alpha_2>0$ such that

$$\begin{aligned} \alpha_1\|\boldsymbol{\eta}\|_1^2 &\le a(\boldsymbol{\eta},\boldsymbol{\eta}), \quad \forall \eta_\alpha \in V,\\ a(\boldsymbol{\beta},\boldsymbol{\eta}) &\le \alpha_2\|\boldsymbol{\beta}\|_1\|\boldsymbol{\eta}\|_1, \quad \forall \beta_\alpha,\eta_\alpha \in V. \end{aligned} \tag{2.8}$$

We denote $\|\boldsymbol{\eta}\|_x^2 = \|\eta_1\|_x^2+\|\eta_2\|_x^2$ for η_α of a functional space X. Note that for time-dependent problems, the norm $\|v\|_X$ of a function $v:[0,T]\to X$ is a function of time. We use the following notations for the functional spaces and the measure in time:

$$\begin{aligned} L^2(X) &= L^2(0,T;X)\\ &= \left\{v:[0,T]\longrightarrow X \mid v(t,\mathbf{x})\in X, \|v\|_{L^2(0,T;X)} = \left(\int_0^T (\|v\|_X)^2 dt\right)^{1/2} < \infty\right\},\\ L^\infty(X) &= L^\infty(0,T;X)\\ &= \left\{v:[0,T]\longrightarrow X \mid v(t,\mathbf{x})\in X, \|v\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0\le t\le T}(\|v\|_X) < \infty\right\}. \end{aligned} \tag{2.9}$$

3. Existence, uniqueness, and regularity

For linear hyperbolic equations of the second order in time with one function, the existence and uniqueness are proven (see, e.g., [7, 10, 11]) using the method of a priori estimate. The method is also employed in [17] for Navier-Stokes problem whose steady-state case has close relation to the static problems of R-M plate, (cf. [5]). We extend the scheme to the dynamic problems of R-M plate. For the time being, we keep the material parameters explicitly expressed for later use in Section 4.

Theorem 3.1. *If $m_\alpha, g \in L^2(L^2)$; $B_\alpha^0, W^0 \in H_0^1$; and $B_\alpha^1, W^1 \in L^2$, then there exists a solution (β_α, w) of (2.5) (a weak solution of (2.2)) with initial conditions (2.4), $\beta_\alpha, w \in L^\infty(H_0^1)$, $\dot{\beta}_\alpha, \dot{w} \in L^\infty(L^2)$, and $\ddot{\beta}_\alpha, \ddot{w} \in L^\infty(H^{-1})$. Moreover, there exists a constant $C>0$, independent* *of the material parameters, such that*

$$\begin{aligned} I\|\dot{\boldsymbol{\beta}}\|_0^2+\rho\zeta^{-2}\|\dot{w}\|_0^2+E\|\boldsymbol{\beta}\|_1^2+\lambda\zeta^{-2}\|\nabla w-\boldsymbol{\beta}\|_0^2 \\ \leq C\Big(I\|\mathbf{B}^1\|_0^2+\rho\zeta^{-2}\|W^1\|_0^2+E\|\mathbf{B}^0\|_1^2 \\ +\lambda\zeta^{-2}\|\nabla W^0-\mathbf{B}^0\|_0^2+I^{-1}\|\mathbf{m}\|_{L^2(L^2)}^2+\rho^{-1}\zeta^2\|g\|_{L^2(L^2)}^2\Big). \end{aligned} \tag{3.1}$$

Proof. We apply the scheme for hyperbolic equations developed in [7, 10, 11]. The space V is separable. We construct an approximation of order n, with a countable basis $\{\psi_i(\mathbf{x}), i=1,2,\ldots\}$ of V:

$$\begin{gathered} \beta_{\alpha n}(t,\mathbf{X})=\sum_{j=1}^{n}\beta_{\alpha n}^{j}(t)\psi_j(\mathbf{X}), \qquad w_n(t,\mathbf{X})=\sum_{j=1}^{n}w_n^{j}(t)\psi_j(\mathbf{X}), \\ I(\ddot{\beta}_{\alpha n},\eta_\alpha)+\rho\zeta^{-2}(\ddot{w}_n,\nu)+Ea(\boldsymbol{\beta}_n,\boldsymbol{\eta})+\lambda\zeta^{-2}(w_{n,\alpha}-\beta_{\alpha n},\nu_{,\alpha}-\eta_\alpha) \\ =(g,\nu)+(m_\alpha,\eta_\alpha), \quad \forall \eta_\alpha,\nu\in V_n=\operatorname{span}\{\psi_1,\ldots,\psi_n\}. \end{gathered} \tag{3.2}$$

The approximation problem (3.2) leads to a linear system of second-order ordinary differential equations. With the approximations of (2.4) for the initial conditions

$$\begin{aligned} \sum_{j=1}^{n}B_{\alpha n}^{0j}\psi_j(\mathbf{X})=B_{\alpha n}^{0}\longrightarrow B_{\alpha}^{0}, \qquad \sum_{j=1}^{n}W_n^{0j}\psi_j(\mathbf{X})=W_n^{0}\longrightarrow W^{0}, \\ \sum_{j=1}^{n}B_{\alpha n}^{1j}\psi_j(\mathbf{X})=B_{\alpha n}^{1}\longrightarrow B_{\alpha}^{1}, \qquad \sum_{j=1}^{n}W_n^{1j}\psi_j(\mathbf{X})=W_n^{1}\longrightarrow W^{1}, \end{aligned} \tag{3.3}$$

we have unique solution

$$\{\beta_{1n}^{j}(t),\beta_{2n}^{j}(t),w_n^{j}(t),\ j=1,\ldots,n\}\in H^2([0,T]). \tag{3.4}$$

Now using $\eta_\alpha=\dot{\beta}_{\alpha n}(t,\mathbf{X})$ and $\nu=\dot{w}_n(t,\mathbf{X})$ in (3.2), then integrating from $t=0$ to T, we have

$$\begin{aligned} I\|\dot{\boldsymbol{\beta}}_n\|_0^2+\rho\zeta^{-2}\|\dot{w}_n\|_0^2+Ea(\boldsymbol{\beta}_n,\boldsymbol{\beta}_n)+\lambda\zeta^{-2}\|\nabla w_n-\boldsymbol{\beta}_n\|_0^2 \\ =I\|\dot{\boldsymbol{\beta}}_n(0)\|_0^2+\rho\zeta^{-2}\|\dot{w}_n(0)\|_0^2+Ea(\boldsymbol{\beta}_n(0),\boldsymbol{\beta}_n(0)) \\ +\lambda\zeta^{-2}\|\nabla w_n(0)-\boldsymbol{\beta}_n(0)\|_0^2+2\int_0^T\big((m_\alpha,\dot{\beta}_{\alpha n})+(g,\dot{w}_n)\big)dt \\ \leq I\|\mathbf{B}_n^1\|_0^2+\rho\zeta^{-2}\|W_n^1\|_0^2+E\alpha_2\|\mathbf{B}_n^0\|_1^2+\lambda\zeta^{-2}\|\nabla W_n^0-\mathbf{B}_n^0\|_0^2 \\ +\int_0^T\Big(I^{-1}\|\mathbf{m}\|_0^2+\rho^{-1}\zeta^2\|g\|_0^2\Big)dt+\int_0^T\Big(I\|\dot{\boldsymbol{\beta}}_n\|_0^2+\rho\zeta^{-2}\|\dot{w}_n\|_0^2\Big)dt. \end{aligned} \tag{3.5}$$

Applying Gronwall inequality, we obtain

$$\begin{aligned}
&I\|\dot{\boldsymbol{\beta}}_n\|_0^2+\rho\zeta^{-2}\|\dot{w}_n\|_0^2+\alpha_1E\|\boldsymbol{\beta}_n\|_1^2+\lambda\zeta^{-2}\|\nabla w_n-\boldsymbol{\beta}_n\|_0^2\\
&\quad\le C\Big(I\|\mathbf{B}_n^1\|_0^2+\rho\zeta^{-2}\|W_n^1\|_0^2+\alpha_2E\|\mathbf{B}_n^0\|_1^2\\
&\qquad+\lambda\zeta^{-2}\|\nabla W_n^0-\mathbf{B}_n^0\|_0^2+I^{-1}\|\mathbf{m}\|_{L^2(L^2)}^2+\rho^{-1}\zeta^2\|g\|_{L^2(L^2)}^2\Big).
\end{aligned}\tag{3.6}$$

The right-hand side has limit as $n\to\infty$ due to (3.3). Therefore, the left-hand side is bounded. Note that $\|w_n\|_1\le C\|\nabla w_n\|_0\le C(\|\nabla w_n-\boldsymbol{\beta}_n\|_0+\|\boldsymbol{\beta}_n\|_0)$. By compactness, we can find convergent subsequences, still denoted by subscript n, such that

$$\begin{aligned}
&\beta_{\alpha n}\longrightarrow\beta_\alpha,\qquad w_n\longrightarrow w\quad\text{weakly star in } L^\infty(H_0^1),\\
&\dot\beta_{\alpha n}\longrightarrow\varphi_\alpha,\qquad \dot w_n\longrightarrow\chi\quad\text{weakly star in } L^\infty(L^2).
\end{aligned}\tag{3.7}$$

It is a straightforward task to verify that $\dot\beta_\alpha=\varphi_\alpha$, $\dot w=\chi$, $\ddot\beta_{\alpha n}\to\ddot\beta_\alpha$, $\ddot w_n\to\ddot w$ weakly star in $L^\infty(H^{-1})$, and $\{\beta_\alpha,w\}$ satisfy the initial conditions (2.4) and the variational equation (2.5), thus form a weak solution of (2.2). □

Theorem 3.2. *Under the conditions of Theorem 3.1, the solution $\{\beta_\alpha,w\}$ is unique, that is, if $g=0$, $m_\alpha=0$, $B_\alpha^0=w^0=B_\alpha^1=w^1=0$, then $\beta_\alpha=w=0$.*

Proof. Following the line of [10, 11], we can prove the uniqueness, but omit the details. □

Theorem 3.3. *Under the conditions of Theorem 3.1, if $\dot m_\alpha,\dot g\in L^2(L^2)$, $B_\alpha^0,W^0\in H^2$, and $B_\alpha^1,W^1\in H_0^1$, then the solution (β_α,w) of (2.2) with the initial conditions (2.4) satisfies $\ddot\beta_\alpha,\ddot w\in L^\infty(L^2)$, $\dot\beta_\alpha,\dot w\in L^\infty(H_0^1)$, $\beta_\alpha,w\in L^\infty(H^2)$, and*

$$\begin{aligned}
&I\|\ddot{\boldsymbol{\beta}}\|_0^2+\rho\zeta^{-2}\|\ddot w\|_0^2+E\|\dot{\boldsymbol{\beta}}\|_1^2+\zeta^{-2}\|\nabla\dot w-\dot{\boldsymbol{\beta}}\|_0^2\\
&\quad\le C\Big(I^{-1}\Big(E^2\|\mathbf{B}^0\|_2^2+\lambda^2\zeta^{-4}\|\nabla W^0-\mathbf{B}^0\|_0^2+\|\mathbf{m}(0)\|_0^2\Big)\\
&\qquad+\rho^{-1}\zeta^2\Big(\lambda^2\zeta^{-4}\|\nabla W^0-\mathbf{B}^0\|_1^2+\|g(0)\|_0^2\Big)\\
&\qquad+E\|\mathbf{B}^1\|_1^2+\lambda\zeta^{-2}\|\nabla W^1-\mathbf{B}^1\|_0^2+I^{-1}\|\dot{\mathbf{m}}\|_{L^2(L^2)}^2+\rho^{-1}\zeta^2\|\dot g\|_{L^2(L^2)}^2\Big),
\end{aligned}\tag{3.8}$$

$$E\|\boldsymbol{\beta}\|_2\le C(I\|\ddot{\boldsymbol{\beta}}\|_0+\lambda\zeta^{-2}\|\nabla w-\boldsymbol{\beta}\|_0+\|\mathbf{m}\|_0),\tag{3.9}$$

$$\lambda\zeta^{-2}\|w\|_2\le C(\rho\zeta^{-2}\|\ddot w\|_0+\lambda\zeta^{-2}\|\beta\|_1+\|g\|_0),\tag{3.10}$$

where the bounds of $\|\nabla w-\boldsymbol{\beta}\|_0$ and $\|\beta\|_1$ are established in (3.1).

Proof. We apply the method for hyperbolic equations demonstrated in [8]. From Theorem 3.1, we have $\dot\beta_{\alpha n}^j,\dot w_n^j\in H^1([0,T])$. Differentiating (3.2) with respect to t, we obtain $\dddot\beta_{\alpha n}^j,\dddot w_n^j\in L^2([0,T])$. The a priori estimate like (3.6) holds:

$$\begin{aligned}
&I\|\ddot{\boldsymbol{\beta}}_n\|_0^2+\rho\zeta^{-2}\|\ddot w_n\|_0^2+\alpha_1E\|\dot{\boldsymbol{\beta}}_n\|_1^2+\lambda\zeta^{-2}\|\nabla\dot w_n-\dot{\boldsymbol{\beta}}_n\|_0^2\\
&\quad\le C\Big(I\|\ddot{\boldsymbol{\beta}}_n(0)\|_0^2+\rho\zeta^{-2}\|\ddot w_n(0)\|_0^2+\alpha_2E\|\dot{\boldsymbol{\beta}}_n(0)\|_1^2\\
&\qquad+\lambda\zeta^{-2}\|\nabla\dot w_n(0)-\dot{\boldsymbol{\beta}}_n(0)\|_0^2+I^{-1}\|\dot{\mathbf{m}}\|_{L^2(L^2)}^2+\rho^{-1}\zeta^2\|\dot g\|_{L^2(L^2)}^2\Big).
\end{aligned}\tag{3.11}$$

From (2.2), we obtain $\ddot{\beta}_\alpha(0), \ddot{w}(0) \in L^2$,

$$\begin{aligned} I\|\ddot{\beta}_\alpha(0)\|_0 &\le E\|\mathbf{B}^0\|_2 + \lambda\zeta^{-2}\|\nabla W^0 - \mathbf{B}^0\|_0 + \|m_\alpha(0)\|_0, \\ \rho\zeta^{-2}\|\ddot{w}(0)\|_0 &\le \lambda\zeta^{-2}\|\nabla W^0 - \mathbf{B}^0\|_1 + \|g(0)\|_0. \end{aligned} \tag{3.12}$$

The argument of boundedness and compactness leads to the conclusion that $\ddot{\beta}_{\alpha n} \to \ddot{\beta}_\alpha$, $\ddot{w}_n \to \ddot{w}$ weakly star in $L^\infty(L^2)$. Hence, (3.11) implies (3.8).

On the other hand, we rewrite (2.2):

$$\begin{aligned} EA_1(\boldsymbol{\beta}) &= m_1 - I\ddot{\beta}_1 + \lambda\zeta^{-2}(w_{,1} - \beta_1), \\ EA_2(\boldsymbol{\beta}) &= m_2 - I\ddot{\beta}_2 + \lambda\zeta^{-2}(w_{,2} - \beta_2), \\ -\lambda\zeta^{-2}\nabla^2 w &= g - \rho\zeta^{-2}\ddot{w} - \lambda\zeta^{-2}\beta_{\alpha,\alpha}. \end{aligned} \tag{3.13}$$

For any fixed time t, the right-hand sides of these equations are in L^2. We have the elasticity operator and the Laplace operator in the left-hand side. According to the theory of elliptic equations, with a smooth domain Ω, we have $\beta_\alpha, w \in H^2$, and the bounds (3.9) and (3.10). □

We are ready to extend the method for higher regularity of hyperbolic equation [8] to the transient dynamics of R-M plate. For simplicity, the dependence on the material parameters is not explicitly expressed and will have more discussion in Section 4.

Theorem 3.4. *Assume for any integer $P \ge 0$,*

$$\begin{gathered} B_\alpha^0, W^0 \in H^{P+1} \cap H_0^1, \qquad B_\alpha^1, W^1 \in H^P \cap H_0^1, \\ \frac{\partial^k m_\alpha}{\partial t^k}, \quad \frac{\partial^k g}{\partial t^k} \in L^2(H^{P-k}), \quad k = 0, 1, \dots, P, \end{gathered} \tag{3.14}$$

and that the following compatibility conditions hold for $P \ge 2$:

$$\left.\begin{aligned} B_\alpha^{k+2} &= I^{-1}\left(\frac{\partial^k m_\alpha(0)}{\partial t^k} - EA_\alpha(\mathbf{B}^k) + \lambda\zeta^{-2}(\nabla W^k - \mathbf{B}^k)\right) \in H_0^1, \\ W^{k+2} &= (\rho\zeta^{-2})^{-1}\left(\frac{\partial^k g(0)}{\partial t^k} + \lambda\zeta^{-2}\nabla\cdot(\nabla W^k - \mathbf{B}^k)\right) \in H_0^1, \end{aligned}\right. \quad k = 0, 1, \dots, P-2. \tag{3.15}$$

Then the solution of (2.2) with (2.4) satisfy, for $k = 0, 1, \dots, P+1$,

$$\frac{\partial^k \beta_\alpha}{\partial t^k}, \quad \frac{\partial^k w}{\partial t^k} \in L^\infty(H^{P+1-k}), \tag{3.16}$$

$$\begin{aligned} &\left\|\frac{\partial^k \beta_\alpha}{\partial t^k}\right\|_{P+1-k} + \left\|\frac{\partial^k w}{\partial t^k}\right\|_{P+1-k} \\ &\quad\le C\left(\sum_{j=0}^{P}\left(\left\|\frac{\partial^j m_\alpha}{\partial t^j}\right\|_{L^2(H^{P-j})} + \left\|\frac{\partial^j g}{\partial t^j}\right\|_{L^2(H^{P-j})}\right)\right. \\ &\qquad\left. + \|\mathbf{B}^0\|_{P+1} + \|W^0\|_{P+1} + \|\mathbf{B}^1\|_P + \|W^1\|_P\right). \end{aligned} \tag{3.17}$$

Proof. The cases of $P = 0$ and $P = 1$ are proved in Theorems 3.1 and 3.3, respectively. Using the method of induction, we assume that the theorem is true for $P \le Q$ and assume that the conditions (3.14) and (3.15) are valid for $P = Q+1$. Denote

$$\begin{aligned} \tilde{B}_\alpha &= \dot{\beta}_\alpha, \qquad \tilde{w} = \dot{w}, \\ \tilde{m}_\alpha &= \dot{m}_\alpha, \qquad \tilde{g} = \dot{g}, \\ \tilde{B}_\alpha^k = B_\alpha^{k+1}, &\quad \tilde{W}^k = W^{k+1}, \quad k = 0, 1, \dots, Q. \end{aligned} \tag{3.18}$$

Then $\tilde{B}_\alpha^k, \tilde{W}^k$, $k = 0, 1, \dots, Q$, satisfy (3.15) for $P = Q$. $\tilde{B}_\alpha^0 = B_\alpha^1$ and $\tilde{W}^0 = W^1 \in H^{Q+1} \cap H_0^1$. For $k = 0, 1, \dots, Q$, $\partial^k \tilde{m}_\alpha / \partial t^k = \partial^{k+1} m_\alpha / \partial t^{k+1}$ and $\partial^k \tilde{g} / \partial t^k = \partial^{k+1} g / \partial t^{k+1} \in L^2(H^{Q-k})$. From (3.14) and (3.15) with $P = Q+1$,

$$\begin{aligned} ||\tilde{\mathbf{B}}^1||_Q = ||\mathbf{B}^2||_Q &\le C(||\mathbf{m}(0)||_Q + ||\mathbf{B}^0||_{Q+2} + ||W^0||_{Q+1} + ||\mathbf{B}^0||_Q) \le \infty, \\ ||\tilde{W}^1||_Q = ||W^2||_Q &\le C(||g(0)||_Q + ||W^0||_{Q+2} + ||\mathbf{B}^0||_{Q+1}) \le \infty. \end{aligned} \tag{3.19}$$

Hence $\tilde{B}_\alpha^k, \tilde{W}^k$, $k = 0$ and 1, satisfy (3.14) for $P = Q$. We apply the assumption of induction and obtain from (3.16) and (3.17) with $P = Q$, for $k = 0, 1, \dots, Q+1$,

$$\frac{\partial^k \tilde{B}_\alpha}{\partial t^k}, \quad \frac{\partial^k \tilde{w}}{\partial t^k} \in L^\infty(H^{Q+1-k}), \tag{3.20}$$

$$\begin{aligned} &\left\| \frac{\partial^k \tilde{\beta}_\alpha}{\partial t^k} \right\|_{Q+1-k} + \left\| \frac{\partial^k \tilde{w}}{\partial t^k} \right\|_{Q+1-k} \\ &\quad \le C\Bigg(\sum_{j=0}^{Q} \left(\left\| \frac{\partial^j \overleftrightarrow{m}_\alpha}{\partial t^j} \right\|_{L^2(H^{Q-j})} + \left\| \frac{\partial^j \tilde{g}}{\partial t^j} \right\|_{L^2(H^{Q-j})} \right) \\ &\qquad + ||\tilde{\mathbf{B}}^0||_{Q+1} + ||\tilde{W}^0||_{Q+1} + ||\tilde{\mathbf{B}}^1||_Q + ||\tilde{W}^1||_Q \Bigg). \end{aligned} \tag{3.21}$$

It implies that, for $k = 1, \dots, Q+2$,

$$\frac{\partial^k \beta_\alpha}{\partial t^k}, \quad \frac{\partial^k w}{\partial t^k} \in L^\infty(H^{Q+2-k}), \tag{3.22}$$

$$\begin{aligned} &\left\| \frac{\partial^k \beta_\alpha}{\partial t^k} \right\|_{Q+2-k} + \left\| \frac{\partial^k w}{\partial t^k} \right\|_{Q+2-k} \\ &\quad \le C\Bigg(\sum_{j=0}^{Q+1} \left(\left\| \frac{\partial^j m_\alpha}{\partial t^j} \right\|_{L^2(H^{Q+1-j})} + \left\| \frac{\partial^j g}{\partial t^j} \right\|_{L^2(H^{Q+1-j})} \right) \\ &\qquad + ||\mathbf{B}^1||_{Q+1} + ||W^1||_{Q+1} + ||\mathbf{B}^2||_Q + ||W^2||_Q \Bigg). \end{aligned} \tag{3.23}$$

We can use (3.19) to estimate $\mathbf{B}^2$ and W^2 in (3.23) with

$$\begin{aligned} ||\mathbf{m}(0)||_Q &\le C\|\mathbf{m}\|_{C^0(H^Q)} \le C\Big(\|\mathbf{m}\|_{L^2(H^Q)} + \|\dot{\mathbf{m}}\|_{L^2(H^Q)} \Big), \\ ||g(0)||_Q &\le C\|g\|_{C^0(H^Q)} \le C\Big(\|g\|_{L^2(H^Q)} + \|\dot{g}\|_{L^2(H^Q)} \Big). \end{aligned} \tag{3.24}$$

Therefore, (3.17) is true for $P = Q+1$ and $k = 1,\dots,Q+2$. Now the right-hand sides of (3.13) are bounded in H^Q. We have

$$\|\boldsymbol{\beta}\|_{Q+2}^2 \le C\|\mathbf{I}\ddot{\boldsymbol{\beta}} - \lambda\zeta^{-2}(\nabla w - \boldsymbol{\beta}) - \mathbf{m}\|_Q^2 \le C(\|\ddot{\boldsymbol{\beta}}\|_Q^2 + \|w\|_{Q+1}^2 + \|\boldsymbol{\beta}\|_Q^2 + \|\mathbf{m}\|_Q^2) \le \infty,$$
$$\|w\|_{Q+2}^2 \le C\|\rho\zeta^{-2}\ddot{w} + \lambda\zeta^{-2}\beta_{\alpha,\alpha} - g\|_Q^2 \le C(\|\ddot{w}\|_Q^2 + \|\boldsymbol{\beta}\|_{Q+1}^2 + \|g\|_Q^2) \le \infty. \tag{3.25}$$

Therefore,

$$\begin{aligned}\|\boldsymbol{\beta}\|_{Q+2} + \|w\|_{Q+2} &\le C\Bigg(\sum_{j=0}^{Q+1}\left(\left\|\frac{\partial^j m_\alpha}{\partial t^j}\right\|_{L^2(H^{Q+1-j})} + \left\|\frac{\partial^j g}{\partial t^j}\right\|_{L^2(H^{Q+1-j})}\right) \\ &\quad + \|\mathbf{B}^0\|_{Q+2} + \|W^0\|_{Q+2} + \|\mathbf{B}^1\|_{Q+1} + \|W^1\|_{Q+1}\Bigg).\end{aligned} \tag{3.26}$$

Thus, (3.17) also holds for $P = Q+1$ and $k = 0$. The case of $P = Q+1$ of the induction is true. □

4. Relation to Kirchhoff-Love plate

For static problem, it is understood that when the thickness $\zeta \to 0$, the solution of the clamped R-M plate approaches the solution of a K-L plate (see, e.g., [3] for a proof). The convergence is for the systems with load scaling, in the sense that $\beta_\alpha \to \tilde{\beta}_\alpha$, $w \to \tilde{w}$, and

$$\tilde{\boldsymbol{\beta}} = \nabla\tilde{w}, \tag{4.1}$$
$$D_0\nabla^4\tilde{w} = g, \tag{4.2}$$

where $D_0 = E/12(1-\nu^2) = D\zeta^{-3}$. D is the usual bending stiffness. Due to the load scaling, the K-L equation (4.2) is independent of thickness. Physically, when the thickness approaches zero, the bending stiffness approaches zero faster with a factor of ζ^3. The unscaled loading, which contributes to the external work, is proportional to the thickness and will not give a meaningful solution. This fact is used for investigating the thickness-independent convergence of finite element method, for example, [2, 3, 5, 12, 16, 18] (see [12, 16, 18] for numerical examples).

For dynamic problem, due to the appearance of the inertia term, which contributes to the kinetic energy, the equation of K-L plate is no longer thickness independent. To keep K-L plate as a reference model, a possible approach is then to scale the mass density [4] along with the load. Assume

$$\begin{gathered}\rho = \zeta^2\rho_0, \\ I = \zeta^2 I_0, \quad I_0 = \frac{\rho_0}{12}.\end{gathered} \tag{4.3}$$

We consider the scaled R-M equation (2.2) with $m_\alpha = 0$, which does not appear in K-L plate:

$$\begin{gathered} I_0\zeta^2\ddot{\beta}_1 + EA_1(\boldsymbol{\beta}) - \lambda\zeta^{-2}(w_{,1} - \beta_1) = 0, \\ I_0\zeta^2\ddot{\beta}_2 + EA_2(\boldsymbol{\beta}) - \lambda\zeta^{-2}(w_{,2} - \beta_2) = 0, \\ \rho_0\ddot{w} - \lambda\zeta^{-2}\nabla\cdot(\nabla w - \boldsymbol{\beta}) = g, \end{gathered} \tag{4.4}$$

or the variational equation (2.5):

$$I_0\zeta^2\langle\ddot{\beta}_\alpha, \eta_\alpha\rangle + \rho_0\langle\ddot{w}, v\rangle + Ea(\boldsymbol{\beta}, \boldsymbol{\eta}) + \lambda\zeta^{-2}(w_{,\alpha} - \beta_\alpha, v_{,\alpha} - \eta_\alpha) = (g, v), \quad \forall \eta_\alpha, v \in V. \tag{4.5}$$

As a parallel study to the static problem, we consider a special case of elastodynamics with zero initial conditions:

$$B_\alpha^0 = B_\alpha^1 = W^0 = W^1 = 0. \tag{4.6}$$

Theorem 4.1. *Assume $g \in H^1(L^2)$, $\dot{g} \in L^2(L^2)$, and $(\beta_\alpha, w) \in L^\infty(H_0^1)$ is the solution of (4.4) (or (4.5)) with initial conditions (4.6). Then as $\zeta \to 0$, there exists a sequence of (β_α, w) with the same notation for simplicity such that*

$$\begin{gathered} w \longrightarrow \tilde{w} \quad \text{weakly star in } L^\infty(H^2), \\ \beta_\alpha \longrightarrow \tilde{\beta}, \qquad \dot{w} \longrightarrow \dot{\tilde{w}} \quad \text{weakly star in } L^\infty(H^1), \\ \ddot{w} \longrightarrow \ddot{\tilde{w}} \quad \text{weakly star in } L^\infty(L^2). \end{gathered} \tag{4.7}$$

Moreover,

$$\tilde{\boldsymbol{\beta}} = \nabla\tilde{w} \tag{4.8}$$

and $\tilde{w}$ is the solution of a K-L plate problem of elastodynamics with clamped boundary conditions

$$\begin{gathered} \rho_0\ddot{\tilde{w}} + D_0\nabla^4\tilde{w} = g \qquad \text{or} \\ \rho_0(\ddot{\tilde{w}}, v) + D_0(\nabla^2\tilde{w}, \nabla^2 v) = (g, v), \quad \forall v \in H_0^2, \\ \tilde{w}\big|_{\partial\Omega} = \frac{\partial\tilde{w}}{\partial n}\Big|_{\partial\Omega} = 0, \\ \tilde{w}(0, \mathbf{x}) = \dot{\tilde{w}}(0, \mathbf{x}) = 0. \end{gathered} \tag{4.9}$$

Proof. By Theorems 3.1 to 3.3, we have a unique solution $(\beta_\alpha, w) \in L^\infty(H_0^1) \cap L^\infty(H^2)$ for (4.4) (or (4.5)), where the generic constant $C > 0$ used in the a priori estimates is independent of material parameters. With (4.6), the a priori estimate (3.1) is reduced to

$$\sqrt{I_0}\zeta\|\dot{\boldsymbol{\beta}}\|_0 + \sqrt{\rho_0}\|\dot{w}\|_0 + \sqrt{E}\|\boldsymbol{\beta}\|_1 + \sqrt{\lambda\zeta^{-2}}\|\nabla w - \boldsymbol{\beta}\|_0 \le C\|g\|_{L^2(L^2)}. \tag{4.10}$$

Inequality (3.8) yields

$$\sqrt{I_0}\zeta\|\ddot{\boldsymbol{\beta}}\|_0 + \sqrt{\rho_0}\|\ddot{w}\|_0 + \sqrt{E}\|\dot{\boldsymbol{\beta}}\|_1 + \sqrt{\lambda\zeta^{-2}}\|\nabla\dot{w} - \dot{\boldsymbol{\beta}}\|_0 \le C\left(\|g(0)\|_0 + \|\dot{g}\|_{L^2(L^2)}\right). \tag{4.11}$$

Inequality (3.10) results in

$$\|w\|_2 \le C\left(\left\|g(0)\right\|_0 + \|\dot{g}\|_{L^2(L^2)} + \|g\|_{L^2(L^2)}\right). \tag{4.12}$$

Furthermore, $\|\dot{w}\|_1 \le C(\|\dot{w}\|_0 + \|\nabla \dot{w}\|_0) \le C(\|\dot{w}\|_0 + \|\nabla \dot{w} - \dot{\boldsymbol{\beta}}\|_0 + \|\dot{\boldsymbol{\beta}}\|_0)$. Therefore, the boundedness of w in $L^\infty(H^2)$, $\beta_\alpha, \dot{\beta}_\alpha, \dot{w}$ in $L^\infty(H^1)$, and $\zeta\ddot{\beta}_\alpha, \ddot{w}$ in $L^\infty(L^2)$ are all uniform with respect to the thickness. We can extract the convergent sequences

$$\begin{aligned} &w \longrightarrow \tilde{w} \quad \text{weakly star in } L^\infty(H^2), \\ &\beta_\alpha \longrightarrow \tilde{\beta}_\alpha, \qquad \dot{\beta}_\alpha \longrightarrow \dot{\tilde{\beta}}_\alpha, \qquad \dot{w} \longrightarrow \dot{\tilde{w}} \quad \text{weakly star in } L^\infty(H^1), \\ &\ddot{w} \longrightarrow \ddot{\tilde{w}} \quad \text{weakly star in } L^\infty(L^2), \end{aligned} \tag{4.13}$$

where, for simplicity, no ζ-dependence notation is used for the sequences. The relation with time differentiation is trivial.

The initial conditions $\tilde{w}(0,\mathbf{x}) = \dot{\tilde{w}}(0,\mathbf{x}) = 0$ are a direct result of (4.6). Inequality (4.10) implies $\sqrt{\lambda}\|\nabla w - \boldsymbol{\beta}\|_0 \le C\zeta\|g\|_{L^2(L^2)} \to 0 \Rightarrow \nabla\tilde{w} - \tilde{\boldsymbol{\beta}} = 0$. Meanwhile, the boundary conditions on (β_α, w) lead to $\tilde{\beta}_\alpha|_{\partial\Omega} = \tilde{w}|_{\partial\Omega} = \nabla\tilde{w}|_{\partial\Omega} = 0$. The last equation implies, for smooth domain, $\partial\tilde{w}/\partial n|_{\partial\Omega} = 0$.

On the other hand, by (4.11), $\zeta^2\|\ddot{\boldsymbol{\beta}}\|_0 \to 0$. Thus, the first two equations of (4.4) yield $EA_\alpha(\boldsymbol{\beta}) - \lambda\zeta^{-2}(w_{,\alpha} - \beta_\alpha) \to 0$. That means $\lambda\zeta^{-2}(w_{,\alpha} - \beta_\alpha) \to EA_\alpha(\tilde{\boldsymbol{\beta}}) = EA_\alpha(\nabla\tilde{w})$. Then the third equation of (4.4) gives $\rho_0\ddot{\tilde{w}} - E\nabla\cdot(\mathbf{A}(\nabla\tilde{w})) = \rho_0\ddot{\tilde{w}} + D_0\nabla^4\tilde{w} = g$. The last equality can be easily verified with the definition of the operator $\mathbf{A}$ and considered in the weak sense. Similar statement for the variational equation is straightforward. □

Remark 4.2. The generic constant C involved in the inequalities derived in Theorem 3.4 is thickness dependent. So the boundedness of β and w in higher spaces may not be uniform with respect to the thickness. It is worth noting that the boundary layer is found for static problems of R-M plate [1]. With clamped boundary conditions, as in our case, $\partial^3\boldsymbol{\beta}/\partial n^3 = O(\zeta^{-1})$ near the boundary, that is, $\|\boldsymbol{\beta}\|_3 = O(\zeta^{-1/2})$. The boundary layer is expected for the dynamics problem too, which warrants further investigation. Since $\tilde{\boldsymbol{\beta}} = \nabla\tilde{w}$, it is not optimistic that we can have the convergence of $w \to \tilde{w}$ in the sense of H^4, although the corresponding K-L plate can have a strong solution $\tilde{w}$ in H^4.

5. Summary

Existence and uniqueness of H^1 solution of R-M plate for elastodynamics with homogeneous Dirichlet boundary conditions and general initial conditions were proved. The solution with smoother data was further investigated and proved to be in H^2. Furthermore, with higher smoothness of data and certain compatibility requirements satisfied, higher regularity of the solution was proved. With the introduction of mass scaling, along with the load scaling, the H^2 solution of R-M plate was proved to approach the H^2 weak solution of K-L plate when the thickness approaches zero.

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