BLOWUP OF SOLUTIONS WITH POSITIVE ENERGY IN NONLINEAR THERMOELASTICITY WITH SECOND SOUND

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This work is concerned with a semilinear thermoelastic system, where the heat flux is given by Cattaneo's law instead of the usual Fourier's law. We will improve our earlier result by showing that the blowup can be obtained for solutions with "relatively" positive initial energy. Our technique of proof is based on a method used by Vitillaro with the necessary modifications imposed by the nature of our problem.

1. Introduction

Results concerning existence, blowup, and asymptotic behaviors of smooth, as well as weak, solutions in classical thermoelasticity have been established by several authors over the past two decades. See in this regard [1, 2, 3, 4, 5, 6, 7, 8, 11, 12, 13, 14, 17, 18, 20].

For thermoelasticity with second sound, Tarabek [21] considered problems related to the one-dimensional system

$$\begin{aligned} u_{tt} - a(u_x,\theta,q)u_{xx} + b(u_x,\theta,q)\theta_x &= \alpha_1(u_x,\theta)qq_x, \\ \theta_t + g(u_x,\theta,q)q_x + d(u_x,\theta,q)u_{tx} &= \alpha_2(u_x,\theta)qq_t, \\ \tau(u_x,\theta)q_t + q + k(u_x,\theta)\theta_x &= 0 \end{aligned} \tag{1.1}$$

in both bounded and unbounded situations and established global existence results for small initial data. He also showed that these "classical" solutions tend to equilibrium as t tends to infinity; however, no rate of decay has been discussed. In his work, Tarabek used the usual energy argument and exploited some relations from the second law of thermodynamics to overcome the difficulty arising from the lack of Poincare's inequality in the unbounded domains. Relations from thermodynamics have been also used by Hrusa & Tarabek [4] to prove a global existence for the Cauchy problem to a classical thermoelasticity system and then by Hrusa & Messaoudi [3] to establish a blowup result for a thermoelastic system. Saouli [19] used the nonlinear semigroup theory to prove a local existence result for a system similar to the one considered by Tarabek.

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Concerning the asymptotic behavior, Racke [15] discussed lately (1.1) and established exponential decay results for several linear and nonlinear initial boundary value problems. In particular, he studied the system (1.1) for a rigidly clamped medium with temperature held constant on the boundary, that is,

$$u(t,0) = u(t,1) = \theta(t,0) = \theta(t,1) = \bar{\theta}, \quad t \geq 0, \tag{1.2}$$

and showed that, for small enough initial data and for $\alpha_1 = \alpha_2 = 0$, classical solutions decay exponentially to the equilibrium state. We should note here that, although the dissipative effects of heat conduction induced by Cattaneo's law are usually weaker than those induced by Fourier's law, a global existence as well as exponential decay results for small initial data have been established. For a discussion in this direction, see Racke [15]. Messaoudi and Said-Houari [10] extended lately the decay result of [15] to (1.1) for α_1 and α_2 that are not necessarily zero.

Regarding the multidimensional case ($n = 2,3$), Racke [16] established an existence result for the n-dimensional problem

$$\begin{gathered} u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla \operatorname{div} u + \beta\nabla\theta = 0, \\ \theta_t + \gamma \operatorname{div} q + \delta \operatorname{div} u_t = 0, \\ \tau q_t + q + \kappa\nabla\theta = 0, \quad x \in \Omega, \quad t > 0, \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, \quad \theta(\cdot,0) = \theta_0, \quad q(\cdot,0) = q_0, \quad x \in \Omega, \\ u = \theta = 0, \quad x \in \partial\Omega, \quad t \geq 0, \end{gathered} \tag{1.3}$$

where Ω is a bounded domain of $\mathbb{R}^n$, with a smooth boundary $\partial\Omega$, $u = u(x,t) \in \mathbb{R}^n$ is the displacement vector, $\theta = \theta(x,t)$ is the difference temperature, $q = q(x,t) \in \mathbb{R}^n$ is the heat flux vector, and μ, λ, β, γ, δ, τ, κ are positive constants, where μ, λ are Lame moduli and τ is the relaxation time, a small parameter compared to the others. In particular, if $\tau = 0$, (1.3) reduces to the system of classical thermoelasticity in which the heat flux is given by Fourier's law instead of Cattaneo's law. He also proved, under the conditions $\operatorname{rot} u = \operatorname{rot} q = 0$, an exponential decay result for (1.3). This result includes the radially symmetric solution, as it is on only a special case. Messaoudi [9] investigated the situation where a nonlinear source term is competing with the damping caused by the heat conduction and established a local existence result. He also showed that solutions with negative energy blow up in finite time. His work generalized an earlier one in [7, 8] to thermoelasticity with second sound.

In this paper, we are concerned with the nonlinear problem

$$\begin{gathered} u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla \operatorname{div} u + \beta\nabla\theta = |u|^{p-2}u, \\ \theta_t + \gamma \operatorname{div} q + \delta \operatorname{div} u_t = 0, \\ \tau q_t + q + \kappa\nabla\theta = 0, \quad x \in \Omega, \quad t > 0, \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, \quad \theta(\cdot,0) = \theta_0, \quad q(\cdot,0) = q_0, \quad x \in \Omega, \\ u = \theta = 0, \quad x \in \partial\Omega, \quad t \geq 0, \end{gathered} \tag{1.4}$$

for $p > 2$. This is a similar problem to (1.3), with a nonlinear source term competing with the damping factor. We will extend the blowup result of [9] to situations where the energy can be positive. Our technique of proof follows carefully the techniques of Vitillaro [22] with the necessary modifications imposed by the nature of our problem. For the sake of completeness, we state here the local existence of [9]. For this purpose, we introduce the following functional spaces:

$$\begin{aligned}\Pi &:= [H_0^1(\Omega)\cap H^2(\Omega)]^n \times [H_0^1(\Omega)]^n \times H_0^1(\Omega)\times D,\\ D &:= \{q\in [L^2(\Omega)]^n \text{ such that } \operatorname{div} q \in L^2(\Omega)\},\\ H &:= [H_0^1(\Omega)]^n \times [L^2(\Omega)]^n \times L^2(\Omega)\times [L^2(\Omega)]^n.\end{aligned} \tag{1.5}$$

Theorem 1.1. *Assume that*

$$\begin{gathered} 2 < p \le \frac{2(n-3)}{n-4}, \quad n \ge 5,\\ 2 < p, \qquad n \le 4,\end{gathered} \tag{1.6}$$

holds. Then given any $(u_0,u_1,\theta_0,q_0)\in\Pi$, there exists a positive number T small enough such that problem (1.4) has a unique strong solution satisfying

$$(u,u_t,\theta,q)\in C^1([0,T);\Pi)\cap C([0,T);H). \tag{1.7}$$

2. Blowup

In order to state and prove our result we introduce the following: let B_1 be the best constant of the Sobolev imbedding $[H_0^1(\Omega)]^n \hookrightarrow [L^p(\Omega)]^n$ and $B_2 = B_1/\mu$. We set

$$\alpha_1 = B_2^{-p/(p-2)}, \qquad E_1 = \left(\frac{1}{2}-\frac{1}{p}\right)\alpha_1^2, \tag{2.1}$$

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{\mu}{2}\|\nabla u\|_2^2 + \frac{\lambda+\mu}{2}\|\operatorname{div} u\|_2^2 + \frac{\beta}{2\delta}\|\theta\|_2^2 + \frac{\gamma\beta\tau}{2\delta k}\|q\|_2^2 - \frac{1}{p}\|u\|_p^p. \tag{2.2}$$

Lemma 2.1. *Let (u,θ,q) be solution of (1.4). Assume that $E(0) < E_1$ and*

$$\left[\mu\|\nabla u_0\|_2^2 + (\lambda+\mu)\|\operatorname{div} u_0\|_2^2 + \frac{\beta}{\delta}\|\theta_0\|_2^2 \frac{\gamma\beta\tau}{\delta k}\|q_0\|_2^2\right]^{1/2} > B_2^{-p/(p-2)}. \tag{2.3}$$

Then there exists a constant $\alpha_2 > B_2^{-p/(p-2)}$ such that

$$\left[\mu\|\nabla u\|_2^2 + (\lambda+\mu)\|\operatorname{div} u\|_2^2 + \frac{\beta}{\delta}\|\theta\|_2^2 + \frac{\gamma\beta\tau}{\delta k}\|q\|_2^2\right]^{1/2} \ge \alpha_2, \tag{2.4}$$

$$\|u\|_p \ge B_2\alpha_2, \quad \forall t\in[0,T). \tag{2.5}$$

Proof. We first note that, by (2.2) and the Sobolev imbedding, we have

$$\begin{aligned} E(t) &\geq \frac{\mu}{2}\|\nabla u\|_2^2 + \frac{\lambda+\mu}{2}\|\operatorname{div} u\|_2^2 + \frac{\beta}{2\delta}\|\theta\|_2^2 + \frac{\gamma\beta\tau}{2\delta k}\|q\|_2^2 - \frac{1}{p}\|u\|_p^p \\ &\geq \frac{1}{2}\left[\mu\|\nabla u\|_2^2 + (\lambda+\mu)\|\operatorname{div} u\|_2^2 + \frac{\beta}{\delta}\|\theta\|_2^2 + \frac{\gamma\beta\tau}{\delta k}\|q\|_2^2\right] \\ &\quad - \frac{B_2^p}{p}\left[\mu\|\nabla u\|_2^2 + (\lambda+\mu)\|\operatorname{div} u\|_2^2 + \frac{\beta}{\delta}\|\theta\|_2^2 + \frac{\gamma\beta\tau}{\delta k}\|q\|_2^2\right]^{p/2} \\ &= \frac{1}{2}\alpha^2 - \frac{B_2^p}{p}\alpha^p = g(\alpha), \end{aligned} \tag{2.6}$$

where

$$\alpha = \left[\mu\|\nabla u\|_2^2 + (\lambda+\mu)\|\operatorname{div} u\|_2^2 + \frac{\beta}{\delta}\|\theta\|_2^2 + \frac{\gamma\beta\tau}{\delta k}\|q\|_2^2\right]^{1/2}. \tag{2.7}$$

It is easy to verify that g is increasing for $0 < \alpha < \alpha_1$, decreasing for $\alpha > \alpha_1$, $g(\alpha) \to -\infty$ as $\alpha \to +\infty$, and

$$g(\alpha_1) = \left(\frac{1}{2} - \frac{1}{p}\right)B_2^{-2p/(p-2)} = E_1, \tag{2.8}$$

where α_1 is given in (2.1). Therefore, since $E(0) < E_1$, there exists $\alpha_2 > \alpha_1$ such that $g(\alpha_2) = E(0)$.

If we set

$$\alpha_0 = \left[\mu\|\nabla u_0\|_2^2 + (\lambda+\mu)\|\operatorname{div} u_0\|_2^2 + \frac{\beta}{\delta}\|\theta_0\|_2^2 + \frac{\gamma\beta\tau}{\delta k}\|q_0\|_2^2\right]^{1/2}, \tag{2.9}$$

then by (2.6), we have

$$g(\alpha_0) \leq E(0) = g(\alpha_2), \tag{2.10}$$

which implies that $\alpha_0 \geq \alpha_2$.

Now to establish (2.4), we suppose by contradiction that

$$\left[\mu\|\nabla u(t_0)\|_2^2 + (\lambda+\mu)\|\operatorname{div} u(t_0)\|_2^2 + \frac{\beta}{\delta}\|\theta(t_0)\|_2^2 + \frac{\gamma\beta\tau}{\delta k}\|q(t_0)\|_2^2\right]^{1/2} < \alpha_2 \tag{2.11}$$

for some $t_0 > 0$ and by the continuity of

$$\mu\|\nabla u(\cdot)\|_2^2 + (\lambda+\mu)\|\operatorname{div} u(\cdot)\|_2^2 + \frac{\beta}{\delta}\|\theta(\cdot)\|_2^2 \frac{\gamma\beta\tau}{\delta k}\|q(\cdot)\|_2^2, \tag{2.12}$$

we can choose t_0 such that

$$\left[\mu\|\nabla u(t_0)\|_2^2 + (\lambda+\mu)\|\operatorname{div} u(t_0)\|_2^2 + \frac{\beta}{\delta}\|\theta(t_0)\|_2^2 \frac{\gamma\beta\tau}{\delta k}\|q(t_0)\|_2^2\right]^{1/2} > \alpha_1. \tag{2.13}$$

Again the use of (2.6) leads to

$$E(t_0) \geq g\Big(\mu\|\nabla u(t_0)\|_2^2 + (\lambda+\mu)\|\operatorname{div} u(t_0)\|_2^2 + \frac{\beta}{\delta}\|\theta(t_0)\|_2^2 \frac{\gamma\beta\tau}{\delta k}\|q(t_0)\|_2^2\Big) > g(\alpha_2) = E(0). \tag{2.14}$$

This is impossible since $E(t) \leq E(0)$ for all $t \in [0,T)$. Hence (2.4) is established.

To prove (2.5), we exploit (2.2) to see

$$\frac{1}{2}\Big[\mu\|\nabla u\|_2^2 + (\lambda+\mu)\|\operatorname{div} u\|_2^2 + \frac{\beta}{\delta}\|\theta\|_2^2 \frac{\gamma\beta\tau}{\delta k}\|q\|_2^2\Big] \leq E(0) + \frac{1}{p}\|u\|_p^p. \tag{2.15}$$

Consequently,

$$\begin{aligned} \frac{1}{p}\|u\|_p^p &\geq \frac{1}{2}\Big[\mu\|\nabla u\|_2^2 + (\lambda+\mu)\|\operatorname{div} u\|_2^2 + \frac{\beta}{\delta}\|\theta\|_2^2 + \frac{\gamma\beta\tau}{\delta k}\|q\|_2^2\Big] - E(0) \\ &\geq \frac{1}{2}\alpha_2^2 - E(0) \geq \frac{1}{2}\alpha_2^2 - g(\alpha_2) = \frac{B_2^p}{p}\alpha_2^p. \end{aligned} \tag{2.16}$$

Therefore, (2.16) and (2.1) yield the desired result. □

Theorem 2.2. *Suppose that*

$$2 < p \leq \frac{2n}{n-2}, \quad n \geq 3, \tag{2.17}$$

$$\frac{\beta\tau\delta}{\kappa\gamma} < 8. \tag{2.18}$$

Then any solution of (1.4), with initial data satisfying

$$\Big[\mu\|\nabla u_0\|_2^2 + (\lambda+\mu)\|\operatorname{div} u_0\|_2^2 + \frac{\beta}{\delta}\|\theta_0\|_2^2 + \frac{\gamma\beta\tau}{\delta k}\|q_0\|_2^2\Big] > B_2^{-2p/(p-2)} \tag{2.19}$$

and

$$E(0) < E_1, \tag{2.20}$$

blows up in finite time.

Remark 2.3. The condition (2.18) is "physically" reasonable due to the very small value of τ. For instance, in [15], for the isotropic silicon and a medium temperature of 300 K, we have

$$\begin{aligned} \beta \approx 391.62\Big[\frac{m^2}{s^2K}\Big], \qquad \tau \approx 10^{-12}[s], \qquad \delta \approx 163.82[K], \\ \gamma \approx 5.99\times 10^{-7}\Big[\frac{ms^2K}{kg}\Big], \qquad \kappa \approx 148\Big[\frac{W}{mK}\Big]; \end{aligned} \tag{2.21}$$

consequently, we get

$$\frac{\beta\tau\delta}{\kappa\gamma} \approx 72.367\times 10^{-7} < 8. \tag{2.22}$$

Proof. A multiplication of (1.4) by u_t, $(\beta/\delta)\theta$, and $(\beta\gamma/\delta\tau)q$, respectively, integration over Ω, using integration by parts, and addition of equalities yields

$$E'(t) = -\frac{\gamma\beta}{\delta k}\|q\|_2^2 \leq 0. \tag{2.23}$$

We then set

$$H(t) = E_1 - E(t). \tag{2.24}$$

By using (2.2) and (2.23), we get

$$\begin{aligned} 0 &< H(0) \leq H(t) \\ &\leq E_1 - \frac{1}{2}\left(\|u_t\|_2^2 + \mu\|\nabla u\|_2^2 + (\lambda+\mu)\|\operatorname{div} u\|_2^2 + \frac{\beta}{\delta}\|\theta\|_2^2 + \frac{\gamma\beta\tau}{\delta k}\|q\|_2^2\right) + \frac{1}{p}\|u\|_p^p, \end{aligned} \tag{2.25}$$

and from (2.1) and (2.4), we obtain

$$\begin{aligned} &E_1 - \frac{1}{2}\left(\|u_t\|_2^2 + \mu\|\nabla u\|_2^2 + (\lambda+\mu)\|\operatorname{div} u\|_2^2 + \frac{\beta}{\delta}\|\theta\|_2^2 + \frac{\gamma\beta\tau}{\delta k}\|q\|_2^2\right) \\ &\qquad < E_1 - \frac{1}{2}\alpha_1^2 = -\frac{1}{p}\alpha_1^2 < 0, \quad \forall t \geq 0. \end{aligned} \tag{2.26}$$

Hence

$$0 < H(0) \leq H(t) \leq \frac{1}{p}\|u\|_p^p, \quad \forall t \geq 0. \tag{2.27}$$

We then define

$$L(t) = H^{1-\sigma}(t) + \varepsilon\int_\Omega \left[u\cdot u_t + \frac{\beta\tau}{k}u\cdot q\right](x,t)\,dx, \tag{2.28}$$

for ε small to be chosen later and

$$\sigma = \frac{p-2}{2p}. \tag{2.29}$$

By taking a derivative of (2.28) and using (1.4), we obtain

$$\begin{aligned} L'(t) = {}&(1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\left(\|u\|_p^p + \|u_t\|_2^2 - \mu\|\nabla u\|_2^2 - (\lambda+\mu)\|\operatorname{div} u\|_2^2\right) \\ &- \frac{\varepsilon\beta}{k}\int_\Omega u\cdot q\,dx + \frac{\varepsilon\beta\tau}{k}\int_\Omega u_t\cdot q\,dx. \end{aligned} \tag{2.30}$$

By exploiting (2.2) and (2.24), the estimate (2.30) takes the form

$$\begin{aligned} L'(t) = {}&(1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\left(1-\frac{2}{p}\right)\|u\|_p^p + 2\varepsilon\|u_t\|_2^2 - \frac{\varepsilon\beta}{k}\int_\Omega u\cdot q\,dx \\ &+ \frac{\varepsilon\beta\tau}{k}\int_\Omega u_t\cdot q\,dx + 2\varepsilon H(t) - 2\varepsilon E_1 + \frac{\varepsilon\beta}{\delta}\|\theta\|_2^2 + \frac{\varepsilon\gamma\beta\tau}{\delta k}\|q\|_2^2. \end{aligned} \tag{2.31}$$

Then using (2.5), we obtain

$$L'(t) \geq (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\left(1 - \frac{2}{p} - 2E_1\left(B_2\alpha_2\right)^{-p}\right)\|u\|_p^p + 2\varepsilon\|u_t\|_2^2 - \frac{\varepsilon\beta}{k}\int_\Omega u \cdot q\,dx + \frac{\varepsilon\beta\tau}{k}\int_\Omega u_t \cdot q\,dx + 2\varepsilon H(t) + \frac{\varepsilon\beta}{\delta}\|\theta\|_2^2 + \frac{\varepsilon\gamma\beta\tau}{\delta k}\|q\|_2^2, \tag{2.32}$$

which implies

$$L'(t) \geq (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon c_0\|u\|_p^p + 2\varepsilon\|u_t\|_2^2 + 2\varepsilon H(t) + \frac{\varepsilon\beta}{\delta}\|\theta\|_2^2 + \frac{\varepsilon\gamma\beta\tau}{\delta k}\|q\|_2^2 - \frac{\varepsilon\beta}{k}\int_\Omega u \cdot q\,dx + \frac{\varepsilon\beta\tau}{k}\int_\Omega u_t \cdot q\,dx, \tag{2.33}$$

where $c_0 = 1 - 2/p - 2E_1(B_2\alpha_2)^{-p} > 0$ since $\alpha_2 > B_2^{-p/(p-2)}$.

Next we exploit Young's inequality to estimate the last two terms in (2.33) as follows:

$$\begin{aligned} \left|\int_\Omega u_t \cdot q\,dx\right| &\leq \frac{a}{2}\|u_t\|_2^2 + \frac{1}{2a}\|q\|_2^2, \quad \forall a > 0, \\ \int_\Omega u \cdot q\,dx &\leq \frac{b}{2}\|q\|_2^2 + \frac{1}{2b}\|u\|_2^2, \quad \forall b > 0. \end{aligned} \tag{2.34}$$

Thus (2.33) yields

$$L'(t) \geq (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon c_0\|u\|_p^p + \varepsilon\left(2 - \frac{a\beta\tau}{2k}\right)\|u_t\|_2^2 + 2\varepsilon H(t) + \frac{\varepsilon\beta}{\delta}\|\theta\|_2^2 + \varepsilon\left(\frac{\gamma\beta\tau}{\delta k} - \frac{\beta\tau}{2ak}\right)\|q\|_2^2 - \frac{\varepsilon\beta}{k}\left[\frac{b}{2}\|q\|_2^2 + \frac{1}{2b}\|u\|_2^2\right]. \tag{2.35}$$

At this point, we choose a so that

$$A_1 := 2 - \frac{a\beta\tau}{2k} > 0, \qquad A_2 := \frac{\beta\tau}{2k}\left(\frac{2\gamma}{\delta} - \frac{1}{a}\right) > 0. \tag{2.36}$$

This is possible by virtue of (2.18); consequently, (2.35) becomes

$$L'(t) \geq (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon A_1\|u_t\|_2^2 + \varepsilon A_2\|q\|_2^2 + \varepsilon c_0\|u\|_p^p + \varepsilon A_3\|\theta\|_2^2 + 2\varepsilon H(t) - \frac{\varepsilon\beta}{k}\left[\frac{b}{2}\|q\|_2^2 + \frac{1}{2b}\|u\|_2^2\right], \tag{2.37}$$

where A_1, A_2, A_3 are strictly positive constants.

We also set $b = 2M\gamma H^{-\sigma}(t)/\delta$, for M a large constant to be determined, to deduce from (2.37)

$$\begin{aligned} L'(t) \geq {} & [(1-\sigma) - \varepsilon M]H^{-\sigma}(t)H'(t) + \varepsilon A_1\|u_t\|_2^2 + \varepsilon A_2\|q\|_2^2 \\ & + \varepsilon c_0\|u\|_p^p + \varepsilon A_3\|\theta\|_2^2 + 2\varepsilon H(t) - \frac{C\varepsilon}{4M}H^{\sigma}(t)\|u\|_p^2, \end{aligned} \tag{2.38}$$

where C, here and in the sequel, is a positive generic constant depending on Ω, p, β, γ, δ, k, λ, μ, τ only.

We then use (2.27) to get

$$\begin{aligned} L'(t) \geq {} & [(1-\sigma) - \varepsilon M]H^{-\sigma}(t)H'(t) + \varepsilon A_1\|u_t\|_2^2 + \varepsilon A_2\|q\|_2^2 \\ & + \varepsilon c_0\|u\|_p^p + \varepsilon A_3\|\theta\|_2^2 + 2\varepsilon H(t) - \frac{C\varepsilon}{4pM}\|u\|_p^{2+\sigma p}. \end{aligned} \tag{2.39}$$

By using (2.29) and the inequality

$$z^{\nu} \leq (z+1) \leq \left(1 + \frac{1}{a}\right)(z+a), \quad \forall z \geq 0,\ 0 < \nu \leq 1,\ a > 0, \tag{2.40}$$

we have the following:

$$\|u\|_p^{2+\sigma p} \leq d\left(\|u\|_p^p + H(0)\right) \leq d\left(\|u\|_p^p + H(t)\right), \quad \forall t \geq 0, \tag{2.41}$$

where $d = 1 + 1/H(0)$.

Inserting the estimate (2.41) into (2.39), we arrive at

$$\begin{aligned} L'(t) \geq {} & [(1-\sigma) - \varepsilon M]H^{-\sigma}(t)H'(t) + \varepsilon A_1\|u_t\|_2^2 \\ & + \varepsilon A_2\|q\|_2^2 + \varepsilon\left(c_0 - \frac{Cd}{4pM}\right)\|u\|_p^p + \varepsilon A_3\|\theta\|_2^2 + \varepsilon\left(2 - \frac{Cd}{4pM}\right)H(t). \end{aligned} \tag{2.42}$$

At this point, we choose M large enough so that (2.42) becomes, for some positive constant A_0,

$$L'(t) \geq [(1-\sigma) - \varepsilon M]H^{-\sigma}(t)H'(t) + \varepsilon A_0\left[\|u_t\|_2^2 + \|q\|_2^2 + \|u\|_p^p + H(t)\right]. \tag{2.43}$$

Once M is fixed (hence A_0), we pick ε small enough so that $(1-\sigma) - \varepsilon M \geq 0$ and

$$L(0) = H^{1-\sigma}(0) + \varepsilon \int_{\Omega}\left[u_0 \cdot u_1 + \frac{\beta\tau}{k}u_0 \cdot q_0\right](x,t)\,dx > 0. \tag{2.44}$$

Therefore, (2.43) yields

$$L'(t) \geq \varepsilon A_0\left[\|u_t\|_2^2 + \|q\|_2^2 + \|u\|_p^p + H(t)\right]. \tag{2.45}$$

Consequently, we have

$$L(t) \geq L(0) > 0, \quad \forall t \geq 0. \tag{2.46}$$

Next we estimate

$$\left|\int_\Omega uu_t(x,t)dx\right| \leq \|u\|_2\|u_t\|_2 \leq C\|u\|_p\|u_t\|_2, \tag{2.47}$$

which implies

$$\left|\int_\Omega uu_t(x,t)dx\right|^{1/(1-\sigma)} \leq C\|u\|_p^{1/(1-\sigma)}\|u_t\|_2^{1/(1-\sigma)}. \tag{2.48}$$

Again Young's inequality gives us

$$\left|\int_\Omega uu_t(x,t)dx\right|^{1/(1-\sigma)} \leq C\left[\|u\|_p^{r/(1-\sigma)} + \|u_t\|_2^{s/(1-\sigma)}\right] \tag{2.49}$$

for $1/r + 1/s = 1$. We take $s = 2(1-\sigma)$ to get $r/(1-\sigma) = 2/(1-2\sigma) = p$ by virtue of (2.29). Therefore, (2.49) becomes

$$\left|\int_\Omega uu_t(x,t)dx\right|^{1/(1-\sigma)} \leq C\left[\|u\|_p^p + \|u_t\|_2^2\right], \quad \forall t \geq 0. \tag{2.50}$$

Similarly we have

$$\left|\int_\Omega uq(x,t)dx\right|^{1/(1-\sigma)} \leq C\left[\|u\|_p^p + \|q\|_2^2\right], \quad \forall t \geq 0. \tag{2.51}$$

Finally, by noting that

$$\begin{aligned} L^{1/(1-\sigma)}(t) &= \left(H^{1-\sigma}(t) + \varepsilon\int_\Omega u\left(u_t + \frac{\beta\tau}{\kappa}q\right)(x,t)dx\right)^{1/(1-\sigma)} \\ &\leq C\left(H(t) + \left|\int_\Omega uu_t(x,t)dx\right|^{1/(1-\sigma)} + \left|\int_\Omega uq(x,t)dx\right|^{1/(1-\sigma)}\right) \\ &\leq C\left[H(t) + \|u\|_p^p + \|u_t\|_2^2 + \|q\|_2^2\right], \quad \forall t \geq 0, \end{aligned} \tag{2.52}$$

and combining it with (2.45), we obtain

$$L'(t) \geq a_0 L^{1/(1-\sigma)}(t), \quad \forall t \geq 0, \tag{2.53}$$

where a_0 is a positive constant depending on εA_0 and C. A simple integration of (2.53) over $(0,t)$ then yields

$$L^{(p-2)/(p+2)}(t) \geq \frac{1}{L^{-(p-2)/2}(0) - a_0 t(p-2)/2}. \tag{2.54}$$

Therefore, $L(t)$ blows up in a time

$$T^* \leq \frac{1-\alpha}{\alpha a_0 [L(0)]^{(p-2)/(p+2)}}. \tag{2.55}$$

□

Remark 2.4. The estimate (2.55) shows that the larger $L(0)$ is the quicker the blowup takes place.

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