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FAST COMPLETE LOCALLY CONVEX LINEAR TOPOLOGICAL SPACES

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ABSTRACT. This is a study of relationship between the concepts of Mackey, ultra-bornological, bornological, barrelled, and infrabarrelled spaces and the concept of fast completeness. An example of a fast complete but not sequentially complete space is presented.

KEY WORDS AND PHRASES. *Locally convex space, fast complete space, bornological space, barrelled space, Mackey space, Baire space.*
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INTRODUCTION.

1. Locally Convex Spaces (projective point of view).

Throughout the sequel E will denote a Hausdorff, locally convex, linear topological space over a scalar field F (of real or complex numbers), and T(E) will denote its topology.

The basic building blocks for locally convex spaces are normed linear spaces. Let EPI (E,normed) be the collection of all continuous epimorphisms f on E such that the range of f is a normed linear space R(f). Then the inverse images of the topologies T(R(f)) of the normed spaces R(f) constitute a basis for the topology T(E).

2. Complete Locally Convex Spaces. Complete locally convex spaces have Banach spaces as their building blocks in the following sense: if EPI (E, Banach) is the collection of all f in EPI(E, normed) such that R(f) is a Banach space, then E is complete if (and only if) the inverse images of all the topologies T(R(f)), where f runs through EPI(E,Banach), is a basis for the topology T(E).

3. Locally Convex Spaces (injective point of view). Thus the topology on E is projective, arising from normed spaces, and it is complete if it arises from Banach spaces. There is a dual way of looking at the situation however, in terms of bounded sets and injective maps, instead of open sets and projective maps. This dual point of view, while not quite so simple, provides the natural context for studying fast complete spaces and will be the point of view of the sequel.

Let MON(normed,E) denote the collection of all continuous monomorphisms f into E such that the domain D(f) is a normed linear space. It is immediate that the image f(A) by any such f of a bounded subset A of D(f), is bounded in E. Furthermore, such sets f(A) constitute a basis for the family of bounded subsets of E. This fact is obvious upon the introduction of a certain standard useful instrument which we directly proceed to define.

A bounded, balanced, radial, convex subset of E will be called a cask. The linear span FC of a cask C is a linear subspace of E and is equipped with the Minkowski functional M_c induced by C. Since the Hausdorff property of E entails that no ray in E can be bounded, the Minkowski functional M_c is actually a norm on FC (rather than just a seminorm). We denote by T(C) the topology on FC induced by the norm M_c. The open unit ball of FC is a subset of FC and thus perforce bounded--hence T(C) is finer than the topology on FC relativized from T(E). Consequently, if i_c denotes the identity monomorphism injecting FC into E, i_c is a member of MON(normed,E).

Now, if B is any bounded subset of E, then the balanced, radial, convex hull C of B is a cask, and so B is a subset of the image of the closed unit ball of the normed space FC by the member i_c of MON(normed,E).

4. Fast Complete Locally Convex Spaces. Let MON(Banach,E) be the collection of all members f of MON(normed,E) such that D(f) is a Banach space. We shall say that a bounded subset B of E is fast complete if B is a subset of the image of the unit ball of D(f) for some f in Mon(Banach,E). If the fast complete subsets of E constitute a basis for the collection of all bounded subsets of E, then we say that E is fast complete.

5. History. Fast complete locally convex spaces seem to have been introduced by De Wilde in his doctoral dissertation [2] and used in conjunction with webbed spaces in his work [3] on the closed graph theorem. Valdivia in [8] introduced a similar (but formally at least) stronger notion called a "locally complete space". Ruess in [6] independently introduced the same notion, calling it "completing". A locally convex space is called locally complete, or completing, if every closed cask is fast complete.

II. FAST COMPLETENESS OF COMPLETE SPACES.

1. Sequential Completeness. It is shown in the introduction above that fast completeness, viewed in the context of injective maps, is analogous to completeness, viewed in the context of projective maps. There is more justification than this however for inclusion of the term "complete" in the name "fast complete".

THEOREM 1. Let C be a sequentially complete cask. Then C is fast complete.

PROOF. Let K be a completion of the normed linear space FC. Since C is sequentially complete, the uniformly continuous restriction of i_c to the subset C of FC has a unique uniformly continuous extension to a function from the closed unit ball of K into C. This extension in turn has a unique linear extension h from K into E. Obviously h is a continuous homomorphism from K into E. If $q|K \longrightarrow K/\ker(h)$ is the canonical quotient epimorphism and $g|K/\ker(h) \longrightarrow E$ is the unique map such that $h = g \circ q$, then evidently g is a member of MON(Banach,E) and C is a subset of the image of g of the closed unit ball of K. Q.E.D.

COROLLARY. If E is sequentially complete (or quasi-complete, or complete), then E is fast complete.

2. Metric Spaces. For metric spaces the connection between fast completeness and completeness is particularly simple.

THEOREM 2. Let E be metrizable. Then E is fast complete if and only if E is complete.

PROOF. Suppose that E is fast complete and let K be a completion of E. Let V(n) be a decreasing fundamental sequence of balanced radial neighborhoods of the origin of K. Let a be an arbitrary element of K, and for each n in the set N of natural numbers, let x(n) be in both E and $a + V(n)/n$. Then, for each n and m in N, $x(n+m) - a$ is in $V(n)/n$ so $n[x(n+m) - x(n)]/2$ is in V(n). It follows that the set $\{n[x(n+m) - x(n)]/2 : n,m \in N\}$ is bounded, and hence it is a subset of the image C of the unit ball B of the domain of some member f of MON(Banach,E).

Let b(n) be a sequence in D(f) such that $f(b(n)) = x(n)$ for each $n \in N$. Then $\|b(n) - b(n+m)\| = 2\, M_C(f(n[x(n+m) - x(n)]) < 2/n$ for all n, $m \in N$. The sequence b(n) is thus Cauchy in D(f) and so has a limit b. Since f is continuous, evidently $a = f(b)$. Hence a is in E, and so E is complete. Q.E.D.

III. EXAMPLES OF FAST COMPLETE SPACES.

1. General Examples. The corollary to Theorem 1 above states that all complete and in fact all quasi-complete spaces are fast complete. Thus a semi-reflexive space always being quasi-complete ([7] IV.5.5 Corollary 1) and a strong dual of a bornological space always being quasi-complete ([7] IV.6.1), these spaces are also fast complete. In particular, the strong dual of a metric space is always fast complete.

It is not true in general that a fast complete space is always complete. If K is any Banach space, then the uniform boundedness principle implies that simply bounded subsets of the conjugate space K' are normed bounded, and so K' is fast complete relative to any locally convex topology intermediate to the weak-* topology $\sigma(K',K)$ and the Mackey topology $\tau(K',K)$. Since K' is dense in the algebraic dual relative to the weak-* topology, it follows that K' is only weak-* complete when K is finite dimensional. Thus there are many non-complete fast complete spaces. In fact, we shall show below that a fast complete space need not even be sequentially complete (III.3).

If K is an infinite dimensional Banach space and B the closed unit ball of K', then K' is the union of the sets nB, where n runs through N, and each of the sets nB is compact relative to the weak-* topology (by Alaoglu's Theorem). It follows that a fast complete space need not be a Baire space.

Fast completeness for an inductive limit E of an increasing nested sequence of Frechet spaces E[n] was studied in [5]. The following theorem was proved there ([5] Remark to Theorem 1).

THEOREM 3. The inductive limit E is fast complete if and only if each bounded subset of E is a bounded subset of some one of the constituent spaces E[n].

In particular, a strict inductive limit of Frechet spaces (an LF-space) is always fast complete.

2. A non-Mackey Fast Complete Space. Let L be the Banach space of all Lebesgue-integrable functions on the unit interval, let E be the space of all locally bounded functions on the unit interval, and let T(E) be the weak-* topology $\sigma(E,L)$ (where E is viewed as the dual space of L in the standard way). It follows from the foregoing that E is fast complete.

The conjugate space of E (relative to T(E)) can be identified with L. ([7] IV.3.3). By the Dunford-Pettis Theorem, ([4] 4.21.3), a subset S of L is $\sigma(L,E)$-relatively compact if and only if S is norm bounded and, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sup\{\int_A |f| d\lambda : f \in S, \lambda(A) < \delta\} < \varepsilon.$$

Consequently the set

$$M = \{f \in L: \sup\{|f(x)| : x \in [0,1]\} < 1\}$$

is $\sigma(L,E)$-compact, as well as a cask.

Let W be the directed family of all finite subsets of L and let D be the product directed set WxN. For $(Y,n) \in D$, let $h[Y,n]$ be an element of E such that $\int h[Y,n]\, f\, d\lambda = 0$ for all $f \in Y$ and such that

$$\int |h[Y,n]| d\lambda = n .$$

Thus, for each $(Y,n) \in D$, the function

$$g[Y,n] = \overline{h[Y,n]} / |h[Y,n]|$$

is in M and

$$\int g[Y,n]\, h[Y,n]\, d\lambda = n .$$

Hence the net $g|D \rightarrow E$ does not tend to zero uniformly on M, but evidently does tend to zero relative to the weak-* topology T(E). Since M is a $\sigma(L,E)$-compact cask, it follows that g does not converge to zero relative to the Mackey topology on E.

It follows that E is not a Mackey space. Hence a fast complete space need not be Mackey (and so need not be bornological or barreled).

3. A non-Sequentially Complete Fast Complete Space. Let L be the Banach space of Lebesgue-integrable functions on R, and let E be the set of all continuous F-valued functions on R which vanish at infinity. Using the integral to induce the standard duality between L and E, let T(E) be the weak topology $\sigma(E,L)$. Since E is a Banach space relative to the supremum norm, the topology induced by this norm is finer than T(E), and thus each $\sigma(E,L)$-bounded subset of E is norm-bounded--it follows that E is fast complete.

For each $n \in N$, let g(n) be the element in E equal to 1 on the interval $[-n,n]$, equal to $1 - |x-n|$ for $x \in [-n-1, -n] \cup [n, n+1]$, and equal to 0 elsewhere. Then $\{g(n)\}$ is a Cauchy sequence in E, but has no limit since the constant function 1 is not contained in E. Thus E is not sequentially complete.

IV. EXAMPLES OF SPACES WHICH ARE NOT FAST COMPLETE.

1. Baire spaces and Fast Completeness. It was mentioned above that a fast complete space is not necessarily a Baire space. Since there are Baire spaces which are metrizable and not complete ([1] p. 3, ex. 6) it follows from Theorem 2 that a Baire space need not be fast complete.

2. Ultra-bornological Spaces and Fast Completeness. A locally convex space E is ultra-bornological provided that it is an inductive limit of Banach spaces. This definition is rather suggestive of the definition of fast completeness. However an ultra-bornological space is always a Mackey space so the example of

III.2 above shows that a fast complete space need not be ultra-bornological. We now present an example of an ultra-bornological space which is not fast complete.

Let E[1] be the space of all double sequences on $N \times N$ vanishing at infinity, and let T(E[1]) be the topology induced by the supremum norm (which we shall here denote by $\| \ \|_1$). For $n = 2,3,\ldots$ let E[n] be the set

$$\{s | N \times N \longrightarrow F : \lim_{J\to\infty} s_{m,J} = 0 \ (\forall m \in N) \text{ and } \lim_{J\to\infty} s_{m,J}/J = 0 \ (\forall m = 1,2,\ldots,n-1)\},$$

and let T(E[n]) be the topology induced by the norm

$$\|s\|_n = \max \{\sup\{|s_{m,J}|/J: m = 1,2,\ldots,n-1;\ J \in N\},\ \sup\{|s_{m,J}| : m = n,\ n+1,\ldots;\ J \in N\}\}$$

for all $s \in E[n]$. Evidently E[n] is an increasing nested sequence of Banach spaces. Let E be the union of this nest and let T[E] be the locally convex inductive limit topology on E. Thus E is, by its definition, ultra-bornological.

To demonstrate that E is not fast complete, we shall exhibit a bounded subset of E which is a subset of none of the constituent spaces. Then by Theorem 3 above E cannot be fast complete.

For each $n \in N$, let x[n] be the element in $E[n+1] \setminus E[n]$ such that $x[n]_{n,J} = 1$ for all $J \in N$ and $x[n]_{m,J} = 0$ whenever $m \neq n$. Let B be the set of all the elements x[n], $n \in N$. Since B is contained in no single E[n], it remains only to show that B is bounded.

Let V be an arbitrary convex and balanced neighborhood of 0 in E. Then $V \cap E[1]$ is a neighborhood in E[1] and so there exists $r > 0$ such that $\{s \in E[1] : \|s\|_1 < 2\} \subset rV$. Let $n \in N$ be arbitrary. Let y[n] be the element of E[1] such that $y[n]_{m,J} = 1$ if $m = n$ and $J \leq m$ and $y[n]_{m,J} = 0$ otherwise. Then for each m, $n \in N$ $\|y[m] - x[n]\|_{n+1} = 1/(m+1)$ which converges to 0 as m tends toward infinity. It follows that the sequence y[m] converges to x[n] in E[n+1]. Hence B is in the closure in E of the unit ball W of E[1]. Thus $W \subset (x[n] + E[n+1] \cap rV/2)$. Consequently x[n] is an element of $W \cap r(E[n+1] \cap V)/2$ whence follows that x[n] is in $(rV/2) + (rV/2) \subset rV$. So $B \in rV$, and B is bounded.

V. SOME IMPLICATIONS OF FAST COMPLETENESS.

1. Bornological and Ultra-bornological Spaces. It was shown above that there is no general implication between ultra-bornological spaces and fast complete spaces. The following theorem does hold however.

THEOREM 4. A fast-complete bornological space is ultra-bornological.

PROOF. The topology T(E) of a bornological space E is the locally convex inductive limit topology induced by the family of normed spaces FC, C a cask in E ([7] II.8.4 et supra). If E is fast complete as well, then each cask C is a subset of f(A) where f is a member of MON(Banach,E) and A is the unit ball of D(f). Consequently T(E) is the locally convex inductive topology induced by the collection of maps f in MON(Banach,E). Hence E is ultra-bornological. Q.E.D.

2. Barrelled and Infra-barrelled Spaces.

THEOREM 5. A fast-complete infra-barrelled space is barrelled.

PROOF. Let V be a closed, radial, and balanced subset of E, and let B be a bounded subset of E. Since E is fast complete, there exists a member f of MON(Banach,E) such that, if A is the closed unit ball of D(f), then $B \subset f(A)$. That V is radial implies that

$$A = \bigcup_{n=1}^{\infty} f^{-1}(nV).$$

From the Baire category theorem follows that $A \cap f^{-1}(nV)$ must have non-void interior for some $n \in N$. Since this set is balanced, there evidently exists some $r > 0$ such that $rA \subset A \cap f^{-1}(nV)$. Consequently we have $B \subset f(A) \subset nV/r$, and so V absorbs bounded subsets of E. Since E is infra-barrelled, V must be a neighborhood of 0. Hence E is barrelled. Q.E.D.

3. Fast Completeness and the Second Conjugate Space. The two topologies on E' which historically seem to have been of most interest are the weak-* topology $\sigma(E',E)$ and the strong topology $\beta(E',E)$. A natural question is: "When is the strong dual of E' the same relative to both these topologies?" This is logically equivalent to the question: "When are weak-* bounded subsets of E' always strongly bounded?" We have the following two properties on the matter. As an immediate consequence of ([9], 10-4-5) we have:

COROLLARY. A bounded subset B of E is always $\beta(E,E')$-bounded if and only if a weak-* bounded subset D of E' is always strongly bounded.

THEOREM 6. If E is fast complete, then the properties of the previous corollary are both satisfied.

PROOF. Let D be a weak-* bounded subset of E' and let V be a closed, balanced, circled, convex neighborhood of 0 in E' relative to the strong topology. Then V° is bounded and so, since E is fast complete, there is a member f of MON(Banach,E) such that $V^{\circ} \subset f(A)$ where A is the closed unit ball of the Banach space D(f).

The set of linear functionals in D are pointwise bounded so the set $H = \{h \circ f : h \in D\}$ is pointwise bounded on the Banach space D(f). From the uniform boundedness theorem follows that H is uniformly bounded on A. Consequently D is bounded uniformly on f(A) and so on V° as well. Hence $V^{\circ\circ}$, which equals V, absorbs D. Q.E.D.

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