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ON THE OPERATOR EQUATION $\alpha + \alpha^{-1} = \beta + \beta^{-1}$

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ABSTRACT. Let α, β be $\star$-automorphisms of a von Neumann algebra M satisfying the operator equation $\alpha + \alpha^{-1} = \beta + \beta^{-1}$. In this paper we use new techniques (which are useful in non-commutative situations as well) to provide alternate proofs of the results:- If α, β commute then there is a central projection p in M such that $\alpha = \beta$ on MP and $\alpha = \beta^{-1}$ on M(1 - P) ; If M = B(H), the algebra of all bounded operators on a Hilbert space H, then $\alpha = \beta$ or $\alpha = \beta^{-1}$.

KEY WORDS AND PHRASES. Automorphisms, central projection, Hilbert-Schmidt operators.
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1. INTRODUCTION.

Suppose that α and β are $\star$-automorphisms of a von Neumann algebra M satisfying the operator equation

$$\alpha + \alpha^{-1} = \beta + \beta^{-1} \tag{1.1}$$

In recent years a lot of work has been done on this operator equation. It has played an important role in the decomposition of a von Neumann algebra ([1] and [2]), in the geometric interpretation of the Tomita-Takesaki theory [3] and its generalization to Jordan algebras [4]. For more details concerning this equation, we refer to [1] and [2]. It has been shown in [1] that if α and β are commuting $\star$-automorphisms of a von Neumann algebra M satisfying equation (1.1) then there exists a central projection p in M such that $\alpha = \beta$ on Mp and $\alpha = \beta^{-1}$ on M(1 - p). If M = B(H) (the algebra of all bounded linear operators on a Hilbert space H) and α, β are $\star$-automorphisms satisfying equation (1.1) then Watatani [5] proved that α and β commute. The above result of [1] can now be applied and we get either $\alpha = \beta$ or $\alpha = \beta^{-1}$ because the center of B(H) consists of scalar multiples of the identity operator only. So in the case of B(H), the additional assumption of commutativity of α and β can be dropped to get the decomposition of M = B(H). The aim of this paper is to provide new proofs of these results. The proofs are relatively simple and the techniques used here can be of independent interest as well. Moreover, the arguments used here can be carried over to obtain results in certain non-commuting situations (see, for instance [6]).

2. MAIN RESULTS.

We first prove the following.

PROPOSITION 2.1 ([1]). Let M be a von Neumann algebra and α,β be commuting $\star$-automorphisms satisfying $\alpha+\alpha^{-1} = \beta+\beta^{-1}$. Then there exists a central projection p in M such that (i) $(\alpha\beta)(p) = p$, (ii) $\alpha=\beta$ on Mp and (iii) $\alpha=\beta^{-1}$ on $M(1-p)$.

PROOF. Since α and β commute, therefore

$(\alpha-\beta)(\alpha\beta-1) = (\alpha-\beta)(1-\beta^{-1}\alpha^{-1})\alpha\beta = (\alpha-\beta-\beta^{-1}+\alpha^{-1})\alpha\beta = 0.$

This implies that $R(\alpha\beta-1) \subseteq N(\alpha-\beta)$ where $R(\alpha\beta-1)$ and $N(\alpha-\beta)$ respectively denote the range space and null space of the operators under consideration. Now $R(\alpha\beta-1) + N(\alpha\beta-1)$ is σ-weakly dense in M and the subalgebra generated by $R(\alpha\beta-1)$ is a two-sided ideal in M ([7]), therefore there exists a central projection p in M such that $(\alpha\beta)(p) = p$ and Mp is the smallest closed $\star$-algebra containing $R(\alpha\beta-1)$. But $R(\alpha\beta-1) \subseteq N(\alpha-\beta)$ and $N(\alpha-\beta)$ is a subalgebra. Therefore $Mp \subseteq N(\alpha-\beta)$. Hence (i) and (ii) are proved. To show (iii), we note that $R(\alpha\beta-1) + N(\alpha\beta-1)$ is dense in M and if we apply $(1-p)$ to $R(\alpha\beta-1) + N(\alpha\beta-1)$, we get

$$(1-p)M \subseteq (1-p)N(\alpha\beta-1) \subseteq N(\alpha\beta-1).$$

This shows that $\alpha = \beta^{-1}$ on $(1-p)M$ and the proof of the result is complete.

In fact, one can improve the above proposition and hence the decomposition theorem of [1] and show that there is a central projection p_1 which gives the decomposition of M as in (ii) and (iii) and also α and β leave p_1 invariant (that is $\alpha(p_1) = \beta(p_1) = p_1$). But this requires an argument.

LEMMA 2.1. Let α and β be automorphisms of a von Neumann algebra M satisfying $\alpha+\alpha^{-1} = \beta+\beta^{-1}$. Then $N(\alpha-\beta)$ is invariant under α and β.

PROOF. Note that $(\alpha-\beta) = \beta^{-1}-\alpha^{-1} = \beta^{-1}(\alpha-\beta)\alpha^{-1}$ or $\beta(\alpha-\beta)\alpha = (\alpha-\beta)$. So if $(\alpha-\beta)(x) = 0$ then $(\alpha-\beta)(\alpha(x)) = 0$ for all $x \in M$. Likewise $(\alpha-\beta)(\beta(x)) = 0$ for all $x \in M$.

Now if p is as in the above proposition, then $\alpha(p) = \beta(p)$ because of (ii) and $(\alpha\beta)(p) = p$. It follows that $\alpha^2(p) = p$ and $\beta^2(p) = p$. Put $p_1 = \alpha(p) \vee p$. Then $\alpha(p_1) = p_1$. Moreover, by the lemma, we have $\alpha = \beta$ on Mp. So $\alpha = \beta$ on $M\alpha(p)$ and hence $\alpha = \beta$ on Mp_1. As $(1-p_1) \leq (1-p)$, we get $\alpha\beta = 1$ on Mp_1. Since $\alpha(p_1) = p_1$, we get that $\beta(p_1) + \beta^{-1}(p_1) = 2p_1$. This shows that $\beta(p_1) = p_1$ ([7]). So we obtain the following:

THEOREM 2.1. Let M be a von Neumann algebra and α,β be $\star$-automorphisms satisfying $\alpha + \alpha^{-1} = \beta + \beta^{-1}$. Then there exists a central projection p_1 in M such that

(i) $\alpha(p_1) = \beta(p_1) = p_1$

(ii) $\alpha = \beta$ on Mp_1

(iii) $\alpha = \beta^{-1}$ on $M(1-p_1)$.

Remark that the above result is more important in the case of a factor in which $\alpha = \beta$ or $\alpha = \beta^{-1}$. Watatani [5] showed that if α and β are $\star$-automorphisms of $M = B(H)$ then α and β commute. By Theorem 2.1 we get that either $\alpha = \beta$ or $\alpha = \beta^{-1}$. However, we provide here an independent and direct proof of this result using Hilbert-Schmidt operator etc. which may be of an independent interest.

THEOREM 2.2. Let α and β be $\star$-automorphisms on B(H) such that $\alpha + \alpha^{-1} = \beta + \beta^{-1}$. Then either $\alpha = \beta$ or $\alpha = \beta^{-1}$.

PROOF. We know that α and β are inner on B(H). So there are unitaries u and v such that

$$\alpha(x) = u x u^*$$
$$\beta(x) = v x v^*$$

for all x in B(H).

Thus $u x u^* + u^* x u = v x v^* + v^* x v$.

We can write the above equation in terms of Hilber-Schmidt operators on $H \otimes \bar{H}$ as

$$u \otimes \bar{u} + u^* \otimes \bar{u}^* = v \otimes \bar{v} + v \otimes \bar{v}^* .$$

Assume that $u^* \neq \lambda u$ for any complex number λ (and similarly for v). Choose w in $B(H)_*$ such that $w(v) = 1$ and $w(v^*) = 0$. Applying $(1 \otimes \bar{w})$ we get

$$\bar{w}(\bar{u})u + \bar{w}(\bar{u}^*)u^* = v.$$

So there exist numbers k_1 and k_2 such that

$$v = k_1 u + k_2 u^*$$
$$v^* = \bar{k}_1 u^* + \bar{k}_2 u .$$

And hence

$$u \otimes \bar{u} + u^* \otimes \bar{u}^* = (|k_1|^2 + |k_2|^2) u \otimes \bar{u} + (k_1\bar{k}_2 + \bar{k}_2 k_1) u \otimes \bar{u}^* + (\bar{k}_1 k_2 + k_2\bar{k}_1) u^* \otimes \bar{u} + (|k_1|^2 + |k_2|^2) u^* \otimes \bar{u}^*.$$

Since u and u* are linearly independent, it follows that $k_1\bar{k}_2 = 0$ and $|k_1|^2 + |k_2|^2 = 1$. If $k_1 = 0$ then $|k_2| = 1$ and if $k_2 = 0$ then $|k_1| = 1$. In the first case $v = k_2 u^*$ and hence

$$\beta^{-1}(x) = v^* x v = \bar{k}_2 u x k_2 u^* = |k_2|^2 u x u^* = \alpha(x) \text{ for all } x \in B(H).$$

In the second case $v = k_1 u$ and by similar calculations we have $\alpha(x) = \beta(x)$ for all x in B(H).

Similarly when $u^* = \lambda u$ for a complex number λ, with $|\lambda| = 1$, we get

$$2(u \otimes \bar{u}) = v \otimes \bar{v} + v^* \otimes \bar{v}^* .$$

Again choosing w in $B(H)_*$ with $w(v) = 1$ and $w(v^*) = 0$, we have $2\bar{w}(\bar{u})u = v$. This is possible only when $|2\bar{w}(\bar{u})| = 1$ and this implies that $\alpha = \beta$.

A similar procedure as in the above paragraphs shows that in any case either $\alpha = \beta$ or $\alpha = \beta^{-1}$ and the proof is complete.

COROLLARY 2.1. Let α and β be two inner $*$-automorphisms on a factor M acting on a Hilbert space H such that $\alpha + \alpha^{-1} = \beta + \beta^{-1}$. Then either $\alpha = \beta$ or $\alpha = \beta^{-1}$.

PROOF. Let u and v be unitaries in M such that

$$\alpha(x) = u x u^*$$
$$\beta(x) = v x v^*$$

for all $x \in M$.

Define $\tilde{\alpha}$ and $\tilde{\beta}$ on B(H) by the same formulae. Choose an element x in the algebra generated by M and M′. Then

$$x = \sum_{i=1}^{n} a_i a_i'$$

with $a_i \in M$ and $a_i' \in M'$. Apply $\tilde{\alpha} + \tilde{\alpha}^{-1}$ on the algebra generated by M and M′ and remark that $\tilde{\alpha}(a_i') = a_i'$ and $\tilde{\alpha}^{-1}(a_i') = a_i'$ because $u \in M$. We get

$$(\tilde{\alpha} + \tilde{\alpha}^{-1})(x) = \sum_{i=1}^{n} \alpha(a_i)a_i' + \sum_{i=1}^{n} \alpha^{-1}(a_i)a_i'$$
$$= \sum_{i=1}^{n} (\alpha + \alpha^{-1})(a_i)a_i' .$$

Since $\alpha + \alpha^{-1} = \beta + \beta^{-1}$ then $\tilde{\alpha} + \tilde{\alpha}^{-1} = \tilde{\beta} + \tilde{\beta}^{-1}$ on the algebra L generated by M and M′. Since M is a factor then L is dense in B(H) and by continuity $\tilde{\alpha} + \tilde{\alpha}^{-1} = \tilde{\beta} + \tilde{\beta}^{-1}$ on B(H). By the theorem above we get either $\tilde{\alpha} = \tilde{\beta}$ or $\tilde{\alpha} = \tilde{\beta}^{-1}$ and hence $\alpha = \beta$ or $\alpha = \beta^{-1}$. This proves the result.

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REFERENCES

1. THAHEEM, A.B. On a Decomposition of a von Neumann Algebra, Ren. Sem. Mat. Univ. Padova 65 (1981), 1-7.

2. THAHEEM, A.B., DAELE A.VAN. and VANHEESWIJCK, L. A Result on Two One-parameter Groups of Automorphisms, Math. Scand. 51 (1982), 261-274.

3. HAAGERUP, U. and SKAU, F. Geometric Aspects of the Tomita-Takesaki Theory II, Math. Scand. 48 (1981), 241-252.

4. HAAGERUP, U. and OLSEN, H. Tomita-Takesaki Theory for Jordan Algebras, J. Operator Theory 11 (1984), 343-364.

5. WATATANI, Y. On Commuting Automorphisms, Amer. Math. Monthly (1981), 449.

6. THAHEEM, A.B. On Pairs of Automorphisms of von Neumann Algebras, (preprint).

7. THAHEEM, A.B. Decomposition of a von Neumann Algebra Relative to a *-Automorphism, Proc. Edinburgh Math. Soc. 22/1 (1979), 9-10.