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TAUBERIAN THEOREM FOR THE DISTRIBUTIONAL STIELTJES TRANSFORMATION

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ABSTRACT. In this paper we use the notion of L-quasiasymptotic at infinity of distributions to obtain a final value Tauberian theorem for the distributional Stieltjes transformation.

KEY WORDS AND PHRASES. Slowly varying function, the quasiasymptotic behaviour at infinity, distributional Stieltjes transform.
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1. NOTIONS AND NOTATION.

Throughout this paper, L will denote real valued, positive and measurable function on $[A,\infty)$ such that

$$\lim_{t\to\infty} \frac{L(t\lambda)}{L(t)} = 1$$

for each $\lambda > 0$. Function L which is regularly varying with index of regular variation $a = 0$, is called slowly varying and the class of such functions is introduced and investigated by J. Karamata.

The quasiasymptotic behaviour at infinity of tempered distributions with supports in $[0,\infty)$ (denoted by S'_+) was defined by Zavijalov (see for instance [1]).

Definition 1. A distribution $f \in S'_+$ has the L-quasiasymptotic at infinity of the power $a \in R$ and with the limit $g \in S'_+$, $g \neq 0$, if for every $\phi \in S$ (S is the the space of rapidly decreasing functions)

$$\lim_{k\to\infty} \left< \frac{f(kt)}{k^a L(k)}, \phi(t) \right> = \left< g(t).\ \phi(t) \right>. \qquad \Delta$$

From the properties of homogeneous distributions it follows ([1]) that if this limit exists then g is a homogeneous distribution of degree a. Namely, $g(t) = Cf_{a+1}(t)$ for some $C \neq 0$, where

$$f_{a+1}(t) = \begin{cases} \dfrac{H(t)t^a}{\Gamma(a+1)} : a>-1 \\ D^n f_{a+n+1}(t) : a\leq -1,\ a+n > -1. \end{cases}$$

As usual, H is the characteristic function of the interval $(0,\infty)$ and D stands for the distributional derivative.

We use the definition of the distributional Stieltjes transform given in [2], [3] in a little different notation ([4]).

Space $J'(\rho)$, $\rho \in R\setminus(-N_o)$ $(N_o = N \cup \{0\})$ is the space of distributions with supports in $[0,\infty)$ such that $f \in J'(\rho)$ iff there exist $k \in N_o$ and a locally integrable function F with the support in $[0,\infty)$ such that

$$\text{(a)} \quad f = D^k F; \qquad \text{(b)} \quad \int_0^\infty |F(t)|(t+\beta)^{-\rho-k}dt < \infty \quad \text{for} \quad \beta > 0 \tag{1.1}$$

(D is the distributional derivative).
If instead of (b) we suppose that there exist $C = C(F)$ and $\varepsilon = \varepsilon(F) > 0$ such that

$$\text{(c)} \quad |F(x)| \leq C(1+x)^{\rho+k-1-\varepsilon} \quad \text{if} \quad x > 0,$$

the corresponding space is denoted by $I'(\rho)$.

Obviously $I'(\rho) \subset J'(\rho)$, $\rho \in R\setminus(-N_o)$.

It was proved in [4] that:

If $f \in J'(\rho)$, $\rho+k > 0$ (k is from (a)), then $f \in I'(\tilde{\rho})$ for $\tilde{\rho} > \rho$ and $\tilde{\rho} \in R\setminus(-N_o)$. (1.2)

If $f \in J'(\rho)$, $\rho+k > 0$, then $f \in I'(\tilde{\rho})$ for $\tilde{\rho} > -k$ (k is from (a)) and $\tilde{\rho} \in R\setminus(-N_o)$. (1.2)

The Stieltjes transform S_ρ of index ρ, $\rho \in R\setminus(-N_o)$ of a distribution $f \in J'(\rho)$ with the properties given in (1.1) is a complex-valued function given by

$$(S_\rho\{f\})(s) = (\rho)_k \int_0^\infty \frac{F(t)dt}{(t+s)^{\rho+k}}, \quad s \in C\setminus(-\infty,0] \tag{1.3}$$

where $(\rho) = \rho(\rho+1) \ldots (\rho+k-1)$, $k > 0$ and $(\rho)_o = 1$.

It is proved in [3] that $(S_\rho\{f\})(s)$ is a holomorphic function of the complex variable s in the domain $C\setminus(-\infty,)]$ provided that $f \in J'(\rho)$.

Observe that $f \in J'(\rho)$ implies that $f \in J'(\rho+n)$, $n \in N$.

The following equality holds:

$$(\rho)_n(S_{\rho+n}\{f\})(s) = (S_\rho\{D^n f\})(s), \quad f \in J'(\rho) \quad \text{and} \quad n \in N.$$

We shall need the following theorem ([5], p. 339, Macaev and Palant)

THEOREM A. Let us suppose that for some $m > 0$ and $x \to \infty$

$$\int_0^\infty \frac{d\phi(\lambda)}{(\lambda+x)^{m+1}} \sim \int_0^\infty \frac{d\psi(\lambda)}{(\lambda+x)^{m+1}}$$

and the following conditions are satisfied:

1) Functions ϕ and ψ are defined for $x > 0$ and are non-decreasing;
2) $\lim_{x\to\infty} (x) = \infty$;
3) For any $C > 1$ there are constants γ and N , $0 < \gamma < m$, $N > 0$, such that for any $x < y < N$ is

$$\frac{\phi(x)}{\phi(x)} \leq C \left(\frac{x}{y}\right)\gamma .$$

Then, for $\lambda\to\infty$, $\phi(\lambda) \sim \psi(\lambda)$. (This means $\left|\frac{\phi(\lambda)}{\psi(\lambda)} - 1\right| < \varepsilon$ if $\lambda < \lambda_o(\varepsilon)$.) Δ

2. TAUBERIAN THEOREM.

THEOREM. Let us suppose that $s > 1$, $\rho+k-s-1 > 0$, $f \in I'(\rho)$ and F (from (1.1)(a)) is non-decreasing function. Moreover, let

$$(S_\rho\{f\})(x) \quad \frac{1}{x^s L(x)} \quad , \quad x\to\infty$$

where L is slowly varying function in some interval $[A,\infty)$, such that $x^{\rho+k-s-1}L(x)$ is non-decreasing function.

Then f has the L-quasiasymptotic of the power $\rho-1-s$ and with the limit

$$\frac{(\rho-s)_k}{(\rho)_k}\, x^{\rho-1-s} \quad .$$

PROOF. Let us put

$$\phi(x) = \begin{cases} x^{\rho+k-s-1}L(x) & ; \quad x > A \\ 0 & ; \quad x \le A \end{cases} \tag{2.1}$$

Then ϕ has the L-quasiasymptotic of the power $\rho+k-s-1$ ([1] Theorem 1) and with the limit $x^{\rho+k-s-1}$. By [6] it is

$$\int_0^\infty \frac{d\phi(t)}{(x+t)^{\rho+k-1}} = (\rho+k-1)\int_0^\infty \frac{\phi(t)dt}{(x+t)^{\rho+k}} \sim \frac{(\rho+k-1)}{x^s L(x)} \quad , \ x\to\infty \quad . \tag{2.2}$$

Now we show that the conditions of Theorem A hold for ϕ and F. In fact we have only to show that for some $0 < \gamma < \rho+k-2$ and every $C > 1$ there exists $N > 0$ such that

$$\frac{\phi(\lambda y)}{\phi(y)} < C\,\lambda^\gamma \qquad \text{for} \quad \lambda > 1 \quad \text{and} \quad y > N \ . \tag{2.3}$$

Let us put $\gamma = \rho+k-s-1+\varepsilon$ where we choose $\varepsilon > 0$ such that $\gamma > 0$ and $\varepsilon < s-1$. After substituting (2.1) in (2.3) we obtain

$$L(\lambda y) \le C\,\lambda^\varepsilon L(y)$$

and this inequality is true if $\lambda > 1$ and $y > N$ where N depends on C (see[6]).

From the assumption that $f \in I'(\rho)$ and (2.2) we have

$$(S_\rho\{f\})(x) = (\rho)_k \int_0^\infty \frac{dF(t)}{(x+t)^{\rho+k}} = (\rho)_{k-1}\int_0^\infty \frac{dF(t)}{(x+t)^{\rho+k-1}} \sim \frac{1}{x^s L(x)} \quad , \qquad x\to\infty \ .$$

It implies

$$(\rho)_k \int_0^\infty \frac{dF(t)}{(x+t)^{\rho+k-1}} \sim \int^\infty \frac{d\phi}{(x+t)^{\ \ +k-1}} \quad , \quad x\to\infty$$

and by Theorem A it implies

$$(\rho)_k F \sim \phi \ , \quad x\to\infty \ .$$

Thus, we obtain that $F(x) \sim \frac{1}{(\rho)_k}\, x^{\rho+k-s-1}L(x)$, $x\to\infty$.

Since $\rho+k-s-1 > 0$, it follows ([1]) that F has the L-quasiasymptotic of the power $\rho+k-s-1$ and with the limit $\frac{1}{(\rho)_k}\, x^{\rho+k-s-1}$. Since $f = D^k F$ it easily follows ([1]) that f has the L-quasiasymptotic of the power $\rho-s-1$ and with the limit

$$\frac{(\rho+k-s-1)\ldots(\rho-s)}{(\rho)_k}\, x^{\rho-s-1} \ . \qquad \Delta$$

COROLLARY. Let us suppose that $f \varepsilon J'(\rho)$ and for some $\tilde{\rho} > \rho$ $\tilde{\rho} \varepsilon R \setminus (-N_o)$

$$(S_{\tilde{\rho}}\{f\}(x) \sim \frac{1}{x^s L(x)}, \quad x \to \infty, \quad s > 1,$$

where L is slowly varying function on $[A,\infty)$. Further, suppose $\tilde{\rho}-s+k>0$ and $x^{\tilde{\rho}-s+k}L(x)$ is non-decreasing in $[A,\infty)$. (Consequently, $f \varepsilon I'(\tilde{\rho})$ and $f(x) = D^{k+1}(\int_o^x F(t)dt)$).

Then f has the L-quasiasymptotic of the power $\tilde{\rho}-1-s$ and with the limit

$$\frac{(\tilde{\rho}-s)_{k+1}}{(\tilde{\rho})_{k+1}}$$

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