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CONTINUOUS DEPENDENCE OF BOUNDARY VALUES FOR SEMIINFINITE INTERVAL ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. Certain elliptic equations arising in catalysis theory can be transformed into ordinary differential equations on the interval $(0,\infty)$. The solutions to these problems usually depend on parameters $\rho \in \mathbb{R}^n$, say $u(t,\rho)$. For certain types of nonlinearities, we show that the boundary value $\dot{u}(\infty,\rho)$ is continuous on compact sets of the variable ρ. As a consequence, bifurcation results for the elliptic equation are obtained.

KEY WORDS AND PHRASES. Continuous dependence, catalysis theory, bifurcation
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1. INTRODUCTION.

Let ε_0 be a positive real number. Let $\ell(\varepsilon)$ be a continuous function with domain $[0,\varepsilon_0]$ and range contained in $[-\infty,0)$. Let $S=\{(\varepsilon,u)\in \mathbb{R}^2: 0<\varepsilon<\varepsilon_0, \ell(\varepsilon)< u\}$. Let $f \in C^2(S)$ have the following properties:

$$\lim_{u\to\ell^+} f(\varepsilon,u) = -\infty \text{ for each } \varepsilon,\ f(\varepsilon,u) = -\infty \text{ on } [0,\varepsilon_0]\times \mathbb{R} - S \tag{1.1}$$

$$f_u(\varepsilon,u) \geqq 0 \text{ and } f_{uu}(\varepsilon,u) \leqq 0 \text{ on } S \tag{1.2}$$

$$\lim_{\varepsilon\to 0^+} f(\varepsilon,u) = u \text{ for each } u \tag{1.3}$$

As a consequence of (1.3), we also have

$$\lim_{u\to\infty} f_u(\varepsilon,u) = L(\varepsilon) \geqq 0 \text{ for each } \varepsilon \geqq 0 \tag{1.4}$$

We consider the semiinfinite interval initial value problem

$$\ddot{u} + \lambda e^{-2t}\exp[f(\varepsilon,u)] = 0,\ 0 < t < \infty,\ \lambda > 0 \tag{1.5}$$

$$u(0) = \alpha,\ \dot{u}(0) = \beta \tag{1.6}$$

where $f(\varepsilon,u)$ has the properties described in (1.1) through (1.4).

Some problems in catalysis theory (in two spatial dimensions) are modeled by (1.5)-(1.6) with the boundary condition $\dot{u}(\infty) = 0$. The classic example is the case $f(\varepsilon,u) = u(1 + \varepsilon u)^{-1}$. The limiting case, $f(0,u) = u$, gives us the Gelfand problem which can be solved explicitly in terms of elementary functions.

We prove that for solutions, $u(t,\lambda,\alpha,\beta,\varepsilon)$, to (1.5)-(1.6), the boundary value $\dot{u}(\infty,\rho)$ is continuous as a function of $\rho = (\lambda,\alpha,\beta,\varepsilon)$ on compact sets with the property that $\lambda \geqq \lambda_0 > 0$. As a consequence, a bifurcation result for (1.5) with boundary data $u(0) = 0$, $\dot{u}(\infty) = 0$, is obtained.

The methods for proving continuous dependence are also applicable to other types of nonlinearities where the bifurcation results (using $f(0,u)$) are much different than in the above problem.

2. PRELIMINARY LEMMAS.

The following lemmas are needed to prove the continuous dependence results for (1.5)-(1.6) at the boundary at ∞.

LEMMA 1. Let $D = \{(\lambda,\beta,\varepsilon): \lambda > 0,\ \beta > 0,\ 0 \leqq \varepsilon \leqq \varepsilon_0\}$. For each $\rho \in D$, $\lim_{t\to\infty} \dot{u}(t,\rho)$ exists.

PROOF. For each $\rho \in D$, define $\omega(\rho) = \sup\{t \in [0,\infty): \ell(\varepsilon) < u(t,\rho)\}$. Since $\ddot{u}(t,\rho) \leqq 0$, $\dot{u}(t,\rho)$ is decreasing. If $\dot{u}(t,\rho) > 0$ for all $t \geqq 0$, then $\dot{u}(t,\rho)$ is bounded below and decreasing. Thus, $\lim_{t\to\omega^-} \dot{u}(t,\rho)$ exists.

However, if $\dot{u}(T,\rho) = 0$ for some finite $T \in [0,\omega)$, then $u(t,\rho)$ attains a maximum value at $u(T,\rho)$. But $f(\varepsilon,u)$ is increasing in u, so it is true that $f(\varepsilon,u(t,\rho)) \leqq f(\varepsilon,u(T,\rho)) =: \ell n\, k$. Equation (1.5) implies that

$$\ddot{u}(t,\rho) = -\lambda e^{-2t}\exp[f(\varepsilon,u(t,\rho))] \geqq -\lambda k e^{-2t}$$
$$\dot{u}(t,\rho) \geqq \beta + \tfrac{1}{2}\lambda k(e^{-2t} - 1) \tag{2.1}$$

So $\dot{u}(t,\rho)$ is bounded below and decreasing. Thus, $\lim_{t\to\omega^-} u(t,\rho)$ exists.

Notice that if $\omega(\rho) < \infty$, then $u(\omega,\rho) = \ell(\varepsilon)$, (1.5) becomes $\ddot{u} = 0$ for $t \geqq \omega$, and $\dot{u}(\infty,\rho) = \dot{u}(\omega,\rho)$. In all cases, define $m(\rho) = \dot{u}(\omega,\rho)$.

LEMMA 2. $L(\varepsilon)$ is upper semicontinuous on $[0,\varepsilon_0]$.

PROOF. Let $\eta > 0$ and $\varepsilon_1 \in [0,\varepsilon_0]$ be given. There exists a number $u_1 > 0$ such that $f_u(\varepsilon_1,u_1) \leqq L(\varepsilon_1) + \tfrac{1}{2}\eta$ for $u > u_1$ since $f_u(\varepsilon_1, u) \to L(\varepsilon_1)$ as $u \to \infty$. There also is a number $\delta > 0$ such that $f_u(\varepsilon,u_1) - \tfrac{1}{2}\eta \leqq f_u(\varepsilon_1,u_1)$ for $|\varepsilon-\varepsilon_1| < \delta$ since $f_u(\varepsilon,u_1) \to f_u(\varepsilon_1,u_1)$ as $\varepsilon \to \varepsilon_1$. Finally, $L(\varepsilon) \leqq f_u(\varepsilon,u_1)$ since $f_{uu} \leqq 0$ and since $f_u(\varepsilon,u) \to L(\varepsilon)$ as $u \to \infty$. Combining these facts gives us

$$L(\varepsilon) - \tfrac{1}{2}\eta \leqq f_u(\varepsilon,u_1) - \tfrac{1}{2}\eta \leqq f_u(\varepsilon_1,u_1) \leqq L(\varepsilon_1) + \tfrac{1}{2}\eta \tag{2.2}$$

or, $L(\varepsilon) \leqq L(\varepsilon_1) + \eta$ for all ε such that $|\varepsilon-\varepsilon_1| < \delta$. Thus, $\overline{\lim}_{\varepsilon\to\varepsilon_1} L(\varepsilon) \leqq L(\varepsilon_1) + \eta$. But η can be chosen arbitrarily small, so $\overline{\lim}_{\varepsilon\to\varepsilon_1} L(\varepsilon) \leqq L(\varepsilon_1)$; that is, $L(\varepsilon)$ is upper semicontinuous at ε_1. Since ε_1 was also arbitrary, $L(\varepsilon)$ is upper semicontinuous on the interval $[0,\varepsilon_0]$.

LEMMA 3. The value $m(\rho) = \dot{u}(\omega,\rho)$ is upper semicontinuous on compact sets of the variable ρ.

PROOF. Let C be a compact subset of D and let $\rho_0 \in C$. From lemma 1, for a given $\eta > 0$, there exist numbers $\delta > 0$ and $T > 0$ such that $\dot{u}(T,\rho) \leqq m(\rho_0) + \tfrac{1}{2}\eta$ and $\dot{u}(T,\rho) - \tfrac{1}{2}\eta \leqq \dot{u}(T,\rho_0)$ for $|\rho-\rho_0| < \delta$ since $\dot{u}(T,\rho)$ is continuous in ρ by standard continuous dependence. Also, $m(\rho) \leqq \dot{u}(T,\rho)$ since $\ddot{u}(t,\rho) \leqq 0$. Thus,

$$m(\rho) - \tfrac{1}{2}\eta \leq \dot{u}(T,\rho) - \tfrac{1}{2}\eta \leq \dot{u}(T,\rho_0) \leq m(\rho_0) + \tfrac{1}{2}\eta \tag{2.3}$$

or, $m(\rho) \leq m(\rho_0) + \eta$ for $|\rho-\rho_0| < \delta$. As in the proof of lemma 2, it follows that $\overline{\lim}_{\rho\to\rho_0} m(\rho) \leq m(\rho_0)$; that is, $m(\rho)$ is upper semicontinuous on compact sets of the variable ρ.

LEMMA 4. If $L(\varepsilon) > 0$, then $m(\rho) < 2/L$ on the set D.

PROOF. Integrating equation (1.5) yields

$$m(\rho) + \int_0^\infty \lambda e^{-2t} \exp[f(\varepsilon,u(t,\rho))]\,dt = \beta \tag{2.4}$$

By our assumptions on f, it is a fact that $f_u(\varepsilon,u) \geq L(\varepsilon)$, so $f(\varepsilon,u) \geq f(\varepsilon,0) + L(\varepsilon)u$ for $u \geq 0$. Suppose that for some $\rho \in D$, $m(\rho) > 2/L$. The conditions that m is finite and $\ddot{u} \leq 0$ imply that $u(t) \geq mt$ for $t \geq 0$. So

$$\int_0^\infty \lambda e^{-2t} \exp[f(\varepsilon,u)]\,dt \geq \int_0^\infty \lambda e^{-2t} e^{f(0)} e^{Lu}\,dt \geq \int_0^\infty \lambda e^{f(0)} e^{-2t} e^{2t}\,dt = \infty \tag{2.5}$$

In (2.4), this would force $m = -\infty$ which contradicts lemma 1. Thus, $m(\rho) < 2/L$ for each $\rho \in D$.

LEMMA 5. Let C be a compact subset of D. Then there exists a number $\delta(C) > 0$ such that $L(\varepsilon)m(\rho) \leq 2 - \delta$ for all ρ C.

PROOF. Suppose that the conclusion is not true. Then there are sequences $\{\delta_n\}_1^\infty$ and $\{\rho_n\}_1^\infty$ such that $\delta_n > 0$, $\delta_n \to 0$, $\rho_n \to \rho_0 \in C$, and $L(\varepsilon_n)m(\rho_n) > 2 - \delta_n$. The last inequality implies that $L(\varepsilon_n)$ and $m(\rho_n)$ are positive. By lemma 4, it is true that $2 - \delta_n < L(\varepsilon_n)m(\rho_n) < 2$. Thus, $\overline{\lim}_{n\to\infty} L(\varepsilon_n)m(\rho_n) = 2$. But by lemmas 2 and 3, we have that

$$2 = \overline{\lim_{n\to\infty}}\, L(\varepsilon_n)m(\rho_n) \leq \overline{\lim_{n\to\infty}}\, L(\varepsilon_n)\; \overline{\lim_{n\to\infty}}\, m(\rho_n) \leq L(\varepsilon_0)m(\rho_0) < 2 \tag{2.6}$$

which is a contradiction. Thus, there exists a $\delta > 0$ such that $L(\varepsilon)m(\rho) \leq 2 - \delta$ for all $\rho \in C$.

3. THE MAIN RESULT.

We now show that the function $m(\rho)$ is actually continuous on compact sets of the variable ρ.

THEOREM. Let C be a compact subset of D. Then $m(\rho)$ is continuous on C.

PROOF. Define $h(t,\rho) = (d/dt)[f(\varepsilon,u(t,\rho)] = f_u(\varepsilon,u(t,\rho))\dot{u}(t,\rho)$. Define $I(\rho) = \{t \in [0,\omega) : h(t,\rho) < 2 - \tfrac{1}{2}\delta\}$ where δ is the number constructed in lemma 5. Then $I(\rho)$ contains an interval $(\tau(\rho),\omega(\rho))$ for some smallest $\tau \in [0,\omega)$. For if $m(\rho) > 0$, then

$$\begin{aligned}\lim_{t\to\omega^-} h(t,\rho) &= \lim_{t\to\omega^-} f_u(\varepsilon,u(t,\rho)) \lim_{t\to\omega^-} \dot{u}(t,\rho) \\ &= L(\varepsilon)m(\rho) \leq 2 - \delta < 2 - \tfrac{1}{2}\delta\end{aligned} \tag{3.1}$$

If $m(\rho) = 0$, then $\dot{u}(t,\rho)$ is positive and so $\lim_{t\to\omega^-} f_u(\varepsilon,u(t,\rho))$ exists and

$$\lim_{t\to\omega^-} h(t,\rho) = m(\rho) \lim_{u\to u(\infty)} f_u(\varepsilon,u) = 0 \tag{3.2}$$

If $m(\rho) < 0$, then $u(\omega,\rho) = \ell(\varepsilon)$ and

$$\lim_{t\to\omega^-} h(t,\rho) = \lim_{u\to\ell^+} f_u(\varepsilon,u)\, m(\rho) \in [-\infty,0] \tag{3.3}$$

In all cases, there is a $\tau(\rho)$ such that $h(t,\rho) < 2 - \frac{1}{2}\delta$ on (τ,ω) and τ is chosen as small as possible.

Let $\rho_0 \in C$ and suppose that $\tau_0 = \tau(\rho_0) > 0$. By the construction, $h(\tau_0,\rho_0) = 2 - \frac{1}{2}\delta$. But $h_t(\tau_0,\rho_0) = f_u(\varepsilon_0,u_0)\ddot{u}_0 + f_{uu}(\varepsilon_0,u_0)(\dot{u}_0)^2$ where $u_0 = u(\tau_0,\rho_0)$, $\dot{u}_0 = \dot{u}(\tau_0,\rho_0)$, and $\ddot{u}_0 = \ddot{u}(\tau_0,\rho_0)$. Also, $2 - \frac{1}{2}\delta = f_u(\varepsilon_0,u_0)\dot{u}_0$. Thus, $f_{uu}(\varepsilon_0,u_0) \leqq 0$, $f_u(\varepsilon_0,u_0) > 0$, and $\ddot{u}_0 < 0$ imply that $h_t(\tau_0,\rho_0) < 0$. Consequently, $h(t,\rho_0) > 2 - \frac{1}{2}\delta$ on $[0,\tau_0)$. By the implicit function theorem, there exists a continuous function $t(\rho)$ and a number $\eta > 0$ such that $t(\rho_0) = \tau_0$ and $h(t(\rho),\rho) = 2 - \frac{1}{2}\delta$ for $|\rho-\rho_0| < \eta$. In fact, $t(\rho) = \tau(\rho)$ whenever $t(\rho) > 0$ (guaranteed by the uniqueness condition in the implicit function theorem). It follows immediately that the function, $\tau(\rho) = t(\rho)$ when $t(\rho) > 0$ and 0 otherwise, is continuous on C. Since C is compact, $\tau^* = \sup\{\tau(\rho) : \rho \in C\}$ is finite.

Thus, $h(t,\rho) < 2 - \frac{1}{2}\delta$ for $t \geqq \tau(\rho)$ since the t-derivative of h is negative at a point where $h = 2 - \frac{1}{2}\delta$, and by the previous argument, $h(t,\rho) < 2 - \frac{1}{2}\delta$ for $t \geqq \tau^*$. On the interval $[0,\tau^*]$, by continuous dependence of u and by continuity of f_u, $f(\varepsilon,u(t,\rho)) \leqq M = M(C)$. For $t \geqq \tau^*$, $f_u(\varepsilon,u(t,\rho))\dot{u}(t,\rho) < 2 - \frac{1}{2}\delta$ implies that

$$f(\varepsilon,u(t,\rho)) \leqq f(\varepsilon,u(\tau^*,\rho)) + (2 - \tfrac{1}{2}\delta)t \leqq K + (2 - \tfrac{1}{2}\delta)t \tag{3.4}$$

where K is a uniform bound (again by continuous dependence of solutions u on compact sets in the variable (t,ρ)).

In the equation (2.4) we had $m(\rho) = \beta - \int_0^\infty \lambda e^{-2t}\exp[f(\varepsilon,u(t,\rho))]\,dt$. Since the integrand is continuous on $[0,\infty)\times C$ and is uniformly bounded on the set C by the integrable function $K\exp(-\delta t)$, $m(\rho)$ is a continuous function on C.

4. APPLICATIONS.

Consider the Dirichlet problem

$$\Delta u + \lambda \exp[f(\varepsilon,u)] = 0, \quad x \in \Omega \tag{4.1}$$

$$u(x) = 0, \quad x \in \partial\Omega \tag{4.2}$$

where Ω is the unit ball of $\mathbb{R}^2$ with center 0, and where Δ is the Laplace operator. A typical example of a nonlinearity in applications (for catalysis problems) is $f(\varepsilon,u) = u/(1 + \varepsilon u)$. Using a result by Gidas, Ni, and Nirenberg [1], all solutions to (4.1)-(4.2) are radially symmetric; that is, $u = u(r)$ where $r = |x|$. Equations (4.1)-(4.2) then can be rewritten as

$$u'' + \frac{1}{r}u' + \lambda \exp[f(\varepsilon,u)] = 0, \quad 0 < r < 1 \tag{4.3}$$

$$u'(0) = 0, \ u(1) = 0 \tag{4.4}$$

Making the change of variables $r = e^{-t}$, we have

$$\ddot{u} + \lambda e^{-2t}\exp[f(\varepsilon,u)] = 0, \quad 0 < r < 1 \tag{4.5}$$

$$u(0) = 0, \; \dot{u}(\infty) = 0 \tag{4.6}$$

Equation (4.5) with initial conditions $u(0) = \alpha$ and $\dot{u}(0) = \beta$ gives us equations (1.5)-(1.6). Let $\varepsilon = 0$. Then $f(0,u) = u$ and we have

$$\ddot{u} + \lambda e^{-2t} e^{u} = 0, \quad 0 < t < \infty \tag{4.7}$$

$$u(0) = \alpha, \; \dot{u}(0) = \beta \tag{4.8}$$

The solution to this is given by

$$u(t,\lambda,\alpha,\beta,0) = \ell n\left(\frac{2kA}{\lambda}\right) + (2-\sqrt{A})t - 2\,\ell n\left(1 + ke^{-t\sqrt{A}}\right) \tag{4.9}$$

where $A = (\beta-2)^2 + 2\lambda e^{\alpha}$ and $k = [\sqrt{A} + (\beta-2)]/[\sqrt{A} - (\beta-2)]$. The boundary conditions $u(0) = 0$ and $\dot{u}(\infty) = 0$ imply that $2 - \sqrt{A} = 0$, or $\lambda = \frac{1}{2}(4\beta - \beta^2)$. The bifurcation curve is given in figure 1.

A result by Dancer [2] shows the bifurcation curve to (4.3)-(4.4) is a 1-dimensional C^1-manifold which is connected for each $\varepsilon \geq 0$. The manifold has a boundary point at $(\lambda,u) = (0,0)$. In terms of the variables (λ,β), the theorem shows that given a compact set C in D and a number $\eta > 0$ (but small), there is an interval $[0,\varepsilon_1]$ contained in $[0,\varepsilon_0]$ such that $|m(\lambda,\beta,\varepsilon) - m(\lambda,\beta,0)| < \eta$ whenever $(\lambda,\beta,\varepsilon)$ is in the appropriate set. But $m(\lambda,\beta,0) = 2 - \sqrt{A}$, so

$$2 - \sqrt{A} - \eta < m(\lambda,\beta,\varepsilon) < 2 - \sqrt{A} + \eta \tag{4.10}$$

In the region $\{(\lambda,\beta) : 2 - \sqrt{A} + \eta < 0\}$, $m(\lambda,\beta,\varepsilon)$ is negative and in the region $\{(\lambda,\beta) : 2 - \sqrt{A} - \eta > 0\}$, $m(\lambda,\beta,\varepsilon)$ is positive. The zeros of m must occur in the parabolic strip between these two regions. See figure 2.

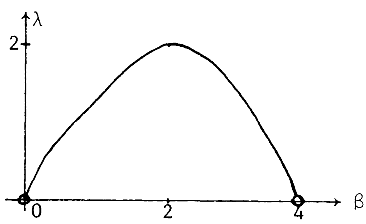

Figure 1

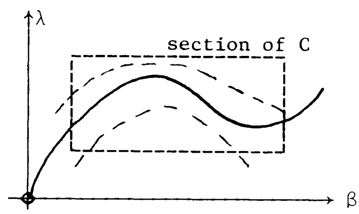

Figure 2

5. OBSERVATIONS AND CONCLUSIONS.

The condition $f(\varepsilon,u) \to u$ as $\varepsilon \to 0$ was only needed to illustrate the example above. Similar results could be obtained if there is knowledge of a bifurcation result for other nonlinearities. For example, in Eberly [3], the nonlinearity e^u-1 is analyzed with similar results, although there are an infinite number of branches of solutions to the condition $\dot{u}(\infty) = 0$.

The important condition used is that $f_u(\varepsilon,u) \to L(\varepsilon)$ as $u \to \infty$. We conjecture that the condition $f_u(\varepsilon,u) \geqq 0$ is technical and that the results on continuous dependence should hold for those nonlinearities $\exp[f(u)]$ where $f_{uu} \leqq 0$. For example, the nonlinearity $g(\varepsilon,\kappa,\rho,u) = (1-\kappa\varepsilon u)^{\rho}\exp[u/(1+\varepsilon u)]$, where ε, κ, and ρ are positive constants, also occurs in catalysis theory and this function has the property that $(d^2/du^2)[\ell n\ g(\varepsilon,\kappa,\rho,u)] \leqq 0$.

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