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ON THE DISTRIBUTIONAL STIELTJES TRANSFORMATION

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ABSTRACT. This paper is concerned with some general theorems on the distributional Stieltjes transformation. Some Abelian theorems are proved.

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1. REGULARLY VARYING FUNCTIONS

Throughout the paper, r will denote a positive continuous function on an interval (X,∞), $X \geqslant 0$, such that the limit

$$\lim_{t\to\infty} \frac{r(pt)}{r(t)}$$

exists for every $p > 0$. Such functions are called regularly varying functions (r.v.f.) at infinity and it is well known ([7]) that they are of the form $r(t) = t^a L(t)$ for some $a \in R$ (called the order or index of r) and some slowly varying function (s.v.f.) L. This means that the function $L : (X,\infty) \to (0,\infty)$ is continuous and that

$$\lim_{t\to\infty} \frac{L(pt)}{L(t)} = 1$$

for every $p > 0$.

2. QUASIASYMPTOTIC BEHAVIOUR AT INFINITY RELATED TO r

The quasiasymptotic behaviour (q.a.b.) at infinity of tempered distributions with support in $[0,\infty)$ (denoted by S'_+) was defined by Zavijalov (see, for instance, [2]). In this paper we use a somewhat more general concept of q,a,b, related to a r.v.f. as defined and analysed in [10].

Definition 1. Let $T \in S'_+$ and r be some r.v.f. . The distribution T has q.a.b. at infinity related to r if there exists the limit in the sense of S' :

$$\lim_{t\to\infty} \frac{T(kt)}{r(k)} = g(t)$$

provided that $g \not\equiv 0$.

If the order of r is a, then $g(t) = A f_{a+1}(t)$ for some $A \neq 0$ (from now on we take A = 1 for simplicity), where

$$f_{a+1}(t) = H(t)t^a/\Gamma(a+1) \text{ for } a \geq 0 \quad \text{and} \quad f_{a+1}(t) = D^n f_{a+n+1}(t)$$

for $a < 0$ and $n+a > 0$, $n \in N$. As usual, H is the characteristic function of the interval $(0,\infty)$, and D stands for the distributional derivative.

It is easy to see that a continuous function on $[0,\infty)$ having ordinary asymptotic behaviour of order $a > -1$ related to r has also q.a.b. of the same order and conversely. However for $a \leq -1$ this may not be true. This follows from the following

Structural Theorem. ([10]) A distribution $T \in S'_+$ has q.a.b. at infinity related to a r.v.f. r of order at iff there exist a natural number n, $n+a > 0$, and a continuous function F on R such that

$$F = T * f_n \quad \text{and} \quad F(t) \sim \frac{1}{\Gamma(n+a+1)} t^n r(t) \quad \text{as} \quad t \to \infty .$$

The proof of this important theorem is analogous to the one of Theorem I in [2], p. 373.

3. EQUIVALENCE AT INFINITY

The other "asymptotic behaviour" of distributions at infinity given in the following definition was used in [3], [1] and [6]; however, this notion goes back to Sebastiao e Silva ([8]).

Definition 2. A distribution $T \in S'_+$ is equivalent at infinity to $r(t) = t^a L(t)$, $a \notin Z_-$, if for some X', $X' \geq X$, and some nonnegative integer n, $n+a > 0$, there exists a continuous function F on $[X',\infty)$ such that $T = D^n F$ on (X',∞) and

$$F(t) \sim t^n r(t)/(a+1)(a+2)\ldots(a+n) \tag{3.1}$$

in the ordinary sense as $t \to \infty$.

It seems to be of interest to compare these two asymptotics; for our purposes it is enough to prove

Lemma 1. Let $T \in S'_+$ be equivalent at infinity to $r(t) = t^a L(t)$ for $a > -1$. Then it has q.a.b. of order a related to r.

Proof. We can write $T = B + D^n F(t)$, where the supports of B and F are, respectively, in $[0,X']$ and $[X',\infty)$, $X' \geq 1$. Let us prove that

$$\lim_{k\to\infty} \frac{B(kt)}{k^a L(k)} \overset{S'}{=} 0$$

In fact, for every $\varepsilon > 0$ there exists a number $n_1 \in N_0$ ($N_0 = N \cup \{0\}$) and a continuous function F_1 on R such that $D^{n_1}F_1 = B$ and supp $F_1 \subset [-\varepsilon, X'+\varepsilon]$. For $\phi \in S$ we have

$$\left\langle \frac{B(kt)}{k^a L(k)}, \phi(t) \right\rangle = \left\langle \frac{F_1(t)}{k^{n_1+a+1}L(k)}, (-1)^{n_1} \phi^{(n_1)}(t/k) \right\rangle =$$

$$= \frac{(-1)^{n_1}}{k^{n_1+a+1}L(k)} \int_{-\varepsilon}^{X'+\varepsilon} F_1(t)\, \phi^{(n_1)}(t/k)\, dt \to 0 \tag{3.1}$$

since $n_1+a+1 \geqslant a+1 > 0$. By supposition F satisfies (3.1), so by the Structural theorem it has q.a.b. of order a related to r.

4. STIELTJES TRANSFORM OF DISTRIBUTIONS

For the sake of completness we rewrite the definition of the distributional Stieltjes transform given in [4]. Let $I'(z)$, $z \in C$, denote the subspace of distributions $t \in S'_+$ such that $T = D^n G$ for some $n \in N$ and some locally integrable function G on R with support in $[0,\infty)$ and

$$\int^{\infty} |G(t)|\; t^{-(z+n+1)}\, dt < \infty.$$

From now on we take $z \in R$ and $z > -1$, though a complex setting is also possible (see [4] or [1]). Obviously $I'(z) \subset S'_+$ and $I'(z_1) \subset I'(z_2)$ for $-1 < z_1 < z_2$.

Definition 3. The Stieltjes transf rm of index z of a distribution $T \in I'(z)$ is the complex valued function

$$S_z\{T\}(s) = \left< T(t), \frac{h(t)}{(t+s)^{z+1}} \right>, \quad s \in C \setminus (-\infty,0], \qquad (4.1)$$

where h is an infinitely differentiable function on R such that $h(t) = 1$ in some neighbourhood of $[0,\infty)$ and $h(t) = 0$ in some interval $(-\infty,-\varepsilon)$, $\varepsilon > 0$.

It is easy to see that (4.1) does not depend on the function h, so it is usually omitted. It is proved in [5] that $S_z\{T\}(s)$ is a holomorphic function of the complex variable s in the domain $C \setminus (-\infty,0]$ provided that $T \in I'(z)$. We shall need the following equality ([5], p. 140)

$$S_{z+n}\{T\}(s) = \frac{1}{(z+1)(z+2)\ldots(z+n)}\, S_z\{D^n T\}(s) \qquad (4.2)$$

for $T \in I'(z)$ and $n \in N$. Observe that $T \in I'(z)$ implies $T \in I'(z+n)$ and $D^n T \in I'(z)$.

5. ABELIAN THEOREMS

The initial value type Abelian theorems for the distributional Stieltjes transform seem to have a satisfactory form. So, we prove only final value type ones. We use first the following result from [6]:

Theorem 1. Let us suppose that $T \in I'(z)$ is equivalent at infinity to a regularly varying function $r(t) = t^a L(t)$ of order $a > -1$. Then

$$S_z\{T\}(s) \sim B(a+1,z-a)\, L(s)\, s^{a-z} \text{ as } s \to \infty,\ s \in R, \qquad (5.1)$$

provided that $z > a > -1$.

As usual, $B(p,q)$ stands for the beta function. In view of Lemma 1 we see that this Theorem can be rewritten as

Theorem 1'. Let us suppose that $T \in S'_+$ has q.a.b. of order $a > -1$ related to the r.v.f. $r(t) = t^a L(t)$. Then (5.1) holds if $z > a > -1$.

If T in these two theorems is a continuous function on $[0,\infty)$, then $T(t) \sim t^a L(t)$ as $t \to \infty$ in the ordinary sense. Essentially, we need such a "functional" (i.e. not "distributional") version of them in the following

Abelian Theorem. Let $T \in S'_+$ have q.a.b. of order a related to a r.v.f. $r(t) = t^a L(t)$. Then

i) $T \in I'(z)$ for $z > \max(-1,a)$

and

ii) $S_z\{T\}(s) \sim \frac{\Gamma(z-a)}{\Gamma(z+1)} L(s)\, s^{a-z}$ as $s \to \infty$,

staying on the real line.

Remark. Such a statement was proved in [3] for $r(t) = t^a$, $a > -1$ and in [4] for $r(t) = t^a \log^j t$, $a > -1$. Further on, r.v.f. were used in [6] (again for $a > -1$). In all these papers the equivalence at infinity was used. The q.a.b. was used in [9] for $r(t) = a^a$ (a - arbitrary real number) and now for any r.v.f. . In [1] the results from [3] are given in a complex setting; it might be of interest to prove an analogous statement for our Abelian theorem.

Proof of the Abelian theorem. Part i) follows from the Structural theorem and the estimate $L(t) \leq C_\varepsilon t^\varepsilon$ for $t \geq t_0 = t_0(\varepsilon)$ (ε in $(0,1)$). For ii), we take $n > -a$ and F as in the structural theorem; then

$$F(t) \sim C_n t^{n+a} L(t) \quad \text{as} \quad t \to \infty$$

for $C_n = 1/\Gamma(n+a+1)$, By Theorem 1' we get

$$S_{z+n}\{F\}(s) \sim C_n B(n+a+1, z+n-(n+a))\, L(s)\, s^{a-z} \quad \text{as } s \to \infty,$$

and from (4.2) we have

$$S_z\{T\}(s) \sim (z+1)(z+2)\dots(z+n)\, S_{z+n}\{F\}(s), \text{ so}$$

$$S_z\{T\}(s) \sim C_n \frac{\Gamma(n+a+1)\,\Gamma(z-a)}{\Gamma(z+1)} L(s)\, s^{a-z} . \tag{5.2}$$

This gives the statement ii).

Example. The equivalence at infinity with the distribution

$$T = A(a,j)\, Fp(t^a \log^j_+ t), \quad a \in R,\ j \in N_0 \tag{5.3}$$

for appropriate constant A(a,j) was analysed in [4]: Fp stands for the finite part. Obviously, T is equivalent at infinity to $t^a \log^j t$ for $a \notin Z_-$; we take $A(a,j) = 1$ then. On the other hand, T has q.a.b. of order a related to $t^a \log^j t$ for $a \notin Z_-$ and related to $A(a,j) t^a \log^{j+1} t$ for $a \in Z_-$; we take $A(a,j) = (-1)^{-a-1}/((-a-1)!(j+1))$ then. Computing the Stieltjes transform of T we see that it behaves at infinity as the Abelian theorem predicts (see [4], formulae (2.3) and (2.4)).

Now let $-2 < a < -1$. Then the distribution $S = T + \delta$ has q.a.b. of order -1 related to 1/t and is equivalent at infinity with $t^a \log^j t$. But for $z > -1$

$$S_z\{S\} = S_z\{T\} + S_z\{\delta\} \sim C_{a,j} \log^{j+1} s\, s^{a-z} + s^{-(z+1)} \sim s^{-(z+1)}$$

when $s \to \infty$. This trivial example (which can be generalized easily) shows again that q.a.b. is more appropriate for final value type Abelian theorems for Stieltjes transformation than equivalence at infinity, though the latter seems more "natural".

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