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# RECURRENT AND WEAKLY RECURRENT POINTS IN $\beta G$

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ABSTRACT. It is shown in this paper that if $\beta G$ is the Stone-Čech compactification of a group G, and G satisfying a certain condition, then there is a weakly recurrent point in $\beta G$ which is not almost periodic, and if another condition will be added, then there is a recurrent point in $\beta G$ which is not almost periodic point.

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## 1. INTRODUCTION.

Let G be infinite group denoted by B(G) the spaces of all bounded real-valued functions with the usual sup norm, and by B(G)* it's conjugate. An g-mean is a function $\phi' \varepsilon B(G)^*$ such that $\|\phi'\| = 1$, $\phi'(u) = 1$ where u is the unit function, i.e. $u(g) = 1$ for all $g\varepsilon G$., $\phi'({}_g f) = \phi'(f)$ for all $f \varepsilon B(G)$ where ${}_g f(s) = f(gs)$, $s \varepsilon G$, and $\phi'(f) \geq 0$ if $f \geqq 0$. If such g-mean exists we call G amenable group.

If G is amenable group with the discrete topology, G be discrete set, as completely regular topological space G has a Stone-Čech Compactification $\beta G$. In W. Rudin [1] the space of real-valued continuous functions on $\beta G$ and the space of bounded real-valued functions on G with the usual sup norm are isomorphic as Banach spaces. Any g-mean $\phi'$ as a functional on $C(\beta G)$ is represented by Riesz representation theorem as a measure $\phi$ defined on the Borel sets of $\beta G$. The correspondence being characterized by $\phi'(f) = \int_{\beta G} \bar{f} d\phi$.

For any $g\varepsilon G$ we have a continuous mapping $\tilde{g}$ of G into $\beta G$ defined by $\tilde{g}(g_1) = gg_1$, $g_1 \varepsilon G$, $\tilde{g}$ has a unique continuous extension to $\beta G$, the extension mapping will also be denoted by $\tilde{g}$. If A subset of G is any subset denote by $\hat{A}$ the open-closed subset of $\beta G \setminus G = \hat{G}$ obtained as $\hat{G} \cap \bar{A}$, where $\bar{A}$ is the closure of A in $\beta G$. If G is infinite left cancellation semigroup, then for $s \varepsilon G$ and B subset of G, $\tilde{s}\hat{B} = \widehat{(sB)}$ Chou [2], $\tilde{g}$ is a homeomorphism of the compact Hausdorff space $\hat{G}$ onto itself denote by $M^{\tilde{g}}$ the set of all $\tilde{g}$-invariant probability measures on $\beta G$, and the upper density of a subset A of G by $\bar{d}_{\tilde{g}}(A) = \sup \{\mu(\hat{A}) : \mu \varepsilon M^{\tilde{g}}\}$.

## 2. THIN AND STRONGLY DISCRETE POINT.

DEFINITION 2.1. A subset A of G is said to be thin if $g_1 A \cap g_2 A$ is finite subset of G for each pair of distinct elements $g_1, g_2 \varepsilon G$.

DEFINITION 2.2. $\omega\varepsilon\beta G\setminus G$ is said to be discrete if the orbit of $\omega$, $O(\omega) = \{g\omega : g\varepsilon G\}$ is discrete with respect to the relative topology that is if and only if there exists a neighborhood $U$ of $\omega$ such that $g\omega\notin U$ if $g \neq e$. Denote by $D^G$ the set of all discrete points in $\hat{G}$.

DEFINITION 2.3. $\omega\varepsilon\hat{G}$ is said to be strongly discrete if there exists a neighborhood $U$ of $\omega$ such that $g_1U\cap g_2U = \phi$ if $g_1 \neq g_2$. Denote by $SD^G$ the set of all strongly discrete points in $\hat{G}$.

REMARK. $SD^G$ is a subset of $D^G$. For take $g_1 = e$ the unit element in $G$, $g_2 = g \neq e$ so $\omega\varepsilon SD^G$ implies there exists a neighborhood $U$ of $\omega$ such that $U\cap gU = \phi$ implies $g\omega\notin U$ implies $\omega\varepsilon D^G$.

DEFINITION 2.4. A point $\omega\varepsilon\beta G\setminus G = \hat{G}$ is said to be almost periodic if for every neighborhood $U$ of $\omega$ there is a subset $A$ of $\hat{G}$ which satisfy: (i) $A\omega$ is a subset of $U$, (ii) there exists a finite subset $K$ of $G$ such that $G = KA$ or equivalently for each neighborhood $U$ of $\omega$ the set $A = \{g\varepsilon G: g\omega\varepsilon U\}$ is relatively dense, in the sense there exists $g_1, g_2, \ldots, g_n\ \varepsilon G$ such that $g_1A\cup g_2A\cup \ldots \cup g_nA = G$. Denote by $A^G$ the set of all almost periodic points in $\beta G$.

PROPOSITION 2.5. $D^G\cap A^G = \phi$

PROOF. If $\omega\varepsilon D^G$, then there is a neighborhood $V$ of $\omega$ in $\beta G$ such that $V\cap o(\omega) = \{\omega\}$, hence $\omega$ is not almost periodic point, otherwise there exists a subset $A$ of $G$ such that $A\omega$ is a subset of $V$ which is a contradiction to the conclusion $V\cap o(\omega) = \{\omega\}$. Then $\omega\notin A^G$ and so $D^G\cap A^G = \phi$.

REMARK. If $A$ is a subset of $C$, $\hat{A}$ is empty if and only if $A$ is finite, also $g\hat{A} = (gA)\hat{}$ for $g\varepsilon G$.

THEOREM 2.6. (1) If $A$ is a thin subset of the group $G$ then $\bar{d}(A) = 0$. (2) $SD^G = \cup\{\hat{A}: A$ is a thin subset of $G\}$.

PROOF. (1) Suppose that $A$ is thin so $g_1A\cap g_2A$ is finite for each distinct pair of elements $g_1, g_2\ \varepsilon G$. But

$$\begin{aligned} cl(g_1A\cap g_2A)\cap\hat{G} &= (clg_1A\cap clg_2A)\cap\hat{G} = (clg_1A\cap\hat{G}) \cap (clg_2A\cap\hat{G}) \\ &= (g_1A)\hat{} \cap (g_2A)\hat{} = g_1\hat{A}\cap g_2\hat{A}. \end{aligned}$$

If $A$ is thin and $\phi\varepsilon M$ the set of all invariant probability measures on $\hat{G}$. So $\phi(\hat{G}) = 1$, hence for any distinct elements $g_1, g_2 \ldots, g_n\varepsilon G$, $g_1\hat{A}, g_2\hat{A}, \ldots, g_n\hat{A}$ are distinct and

$$1 = \phi(\hat{G}) \geq \phi(\dot{\bigcup_{i=1}^{n}} (g_i\hat{A})) = \sum_{i=1}^{n} \phi(g_i\hat{A}) = n\,\phi(\hat{A}) \text{ implies}$$

$$\phi(\hat{A}) \leq \frac{1}{n} \text{ for all } n \longrightarrow \phi(\hat{A}) = 0 \text{ which implies } \bar{d}(A) = 0$$

(2)
$$\begin{aligned} SD^G &= \{\omega\varepsilon\hat{G}: \text{There exists neighborhood } U \text{ of } \omega,\ g_1\hat{U}\cap g_2\hat{U} = \phi \text{ for } g_1 \neq g_2\} \\ &= \{\omega\varepsilon\hat{G}: \text{There exists neighborhood } \hat{U} \text{ of } \omega,\ g_1\hat{U}\cap g_2\hat{U} = \phi \text{ for } g_1 \neq g_2\} \\ &= \cup\{cl\,A\cap\hat{G}:\ g_1\hat{U}\cap g_2\hat{U} = \phi \text{ for all distinct pair of elements } g_1, g_2\ \varepsilon\ G\} \\ &= \cup\{cl\,A\cap\hat{G}:\ g_1A\cap g_2A \text{ is finite}\} \\ &= \cup\{\hat{A}: A \text{ is a thin subset of } G\}. \end{aligned}$$

## 3. WEAKLY RECURRENT AND RECURRENT POINTS.

DEFINITION 3.1. $\omega\varepsilon\beta G$ is said to be $\tilde{g}$-recurrent point if, for each neighborhood $V$ of $\omega$ the set $\{i\varepsilon N: \tilde{g}^i\omega\varepsilon V\}$ is infinite. Denote by $_R\tilde{g}$ the set of all $\tilde{g}$-recurrent points, and by $R^G$ = the complement of $D^G$ in $\hat{G}$, to be the set of all recurrent points. So $R^G \supseteq \cup_{g\varepsilon G}\ _R\tilde{g}$.

DEFINITION 3.2. Denote by $WR^G$ the set of all weakly recurrent points in $\hat{G}$, it is the complement of $SD^G$ in $\hat{G}$.

Since $D^G \cap A^G = \phi$ proposition 2.5 which implies $A^G \subseteq R^G \subseteq \omega R^G$.

DEFINITION 3.3. We call a subset A, a C-subset of G provided that

(i) $\bar{d}(A) > 0$

(ii) $\bar{d}(K^{-1}A) < 1$ for every finite subset K of G. equivalently.

(ii)' For every finite number k, $\bar{d}(A \cup g_1 A \cup \ldots \cup g_{k-1}A) < 1$.

REMARK. C stands for Chou. Denote by AC the class of all amenable semigroup which has a C-subset. This class contains the semigroup N of positive integers, the group Z of integers, all countably infinite locally finite groups, all infinite abelian cancellation semigroups, and all infinite solvable groups, with the discrete topology for more details see Fairchild [3].

One reason for studying the C-subset is the following result.

PROPOSITION 3.4. Suppose G contains a C-subset A then

$$\hat{A} \cap \bar{A}^G = \phi$$

PROOF. Suppose $\hat{A} \cap \bar{A}^G \neq \phi$ say $\omega \in \hat{A} \cap \bar{A}^G$, since $\hat{A}$ is open subset contains ω. Let $B = \{g \in G : g\omega \in \hat{A}\}$ so there exists a finite subset K of G such that $G = K^{-1}B$, $B\omega$ is a subset of $\hat{A}$, hence $o(\omega) = \{g\omega : g \in G\} = G\omega \subseteq K^{-1}\hat{A} = (K^{-1}A)\hat{}$ implies $\hat{A} \to \bar{O}(\omega) \subseteq (K^{-1}A)\hat{}$. But $\bar{O}(\omega)$ is closed invariant set implies there exists $\phi$ a probability measure such that $\text{supp}\ \phi \subseteq (K^{-1}A)\hat{}$ implies $\phi'(I_{K^{-1}A}) = 1$ which contradicts the definition of C-subset. Then,

$$\hat{A} \cap A^G = \phi.$$

REMARK. If A is a subset of G, $I_A$ denote the function 1 on A and 0 otherwise.

THEOREM 3.5. If $G \in AC$ then there exists a weakly recurrent point in βG which is not almost periodic, in other words $A^G \subsetneq WR^G$.

PROOF. Theorem 2.6 shows that $SD^G = \cup\{\hat{A} : A \text{ is a thin subset of } G\} \subseteq \cup\{\hat{A} : \bar{d}(A) = 0\}$, but $\bar{d}(A) > 0$ where A is a C-subset of G, then A is not thin subset implies $\hat{A} \not\subseteq SD^G$, so $\hat{A} \cap WR^G \neq \phi$. In Proposition 3.4 we proved that if A is a C-subset then $\hat{A} \cap \bar{A}^G = \phi$, hence we get $A^G \subsetneq WR^G$. So there exists a weakly recurrent point in βG which is not almost periodic. Moreover $A^G \cup SD^G \neq \hat{G}$.

The only known method to find $\tilde{g}$-recurrent points is to apply Zorn's lemma to find a $\tilde{g}$-minimal set K, then show that each $\omega \in K$ is $\tilde{g}$-almost periodic and therefore $\tilde{g}$-recurrent.

In theorem 3.8 we are going to produce many other $\tilde{g}$-recurrent points for a reasonable class of semigroups.

Chou [4] has proved that

THEOREM (Chou): Let $\phi$ be a homemorphism of a compact Hausdorff space X onto itself. Suppose that $T_1 \supset T_2 \supset \ldots$ is a sequence of non-empty closed subsets of X such that a sequence of positive integers $k_1 < k_2 < \ldots$ can be found to satisfy $\phi^{k_n} T_{n+1} \subset T_n$.

Then $\bigcap_{n=1}^{\infty} T_n$ contains a $\phi$-recurrent point.

LEMMA 3.6. Suppose that A is a subset of G, $\bar{d}_{\tilde{g}}(A) > 0$, and $n \in N$. Then there exists a subset of B of A, $s \in N$, $s \geq n$ such that $\bar{d}_{\tilde{g}}(B) > 0$ and $\tilde{g}^s B \subseteq A$.

PROOF. By definition of upper $\tilde{g}$-density, there exists $\mu \in M^{\tilde{g}}$ such that $\mu(\hat{A}) > 0$. If for each $s \geq n$, $\mu(\hat{A} \cap \tilde{g}^{-s}\hat{A}) = 0$. Then

$$\sum_{i=0}^{\infty} \mu(\tilde{g}^{-in}\hat{A}) = \mu(\bigcup_{i=0}^{\infty} g^{-in}\hat{A}) \leq 1$$

This contradicts the fact that $\mu$ is a $\tilde{g}$-variant $(\mu(\hat{A}) = \mu(\tilde{g}^{-in}\hat{A}))$.

Therefore there exists $s \geq n$ such that $\mu(\hat{A} \cap \tilde{g}^{-s}\hat{A}) > 0$. But since $\tilde{g}\hat{A} = (gA)^\wedge$ so $(\hat{A} \cap \tilde{g}^{-s}\hat{A}) = \hat{A} \cap (g^{-s}A)^\wedge = (A \cap g^{-s}A)^\wedge$. Take $B = A \cap (g^{-s}A)$ then $\mu(\hat{B}) > 0$ and $g^sB \subseteq A$, but $\mu \in M^{\tilde{g}}$ so $d^-_{\tilde{g}}(B) > 0$.

DEFINITION 3.7. The group G is said to be nontorsion group if G contains an element of infinite order.

THEOREM 3.8. If G is nontorsion group, $G \in AC$, then there is a recurrent point in $\beta G$ which is not almost periodic. In other words $A^G \subsetneq R^G$.

PROOF. Since G has a C-subset we may assume that A to be a C-subset hence by proposition 3.4 $\hat{A} \cap \bar{A}^G = \phi$. Therefore it remains to produce a recurrent point in $\hat{A}$. By lemma 3.6 it is easy to construct $s_1 < s_2 < \ldots$ and $A = A_1 \supseteq A_2 \supseteq \ldots$, inductively, such that $d^-(A_i) > 0$ and $\tilde{g}^{\,i-1}A_i \subseteq A_{i-1}$, $i = 2,3 \ldots$ therefore $\hat{A}$ contains a recurrent point by applying Chou's theorem to the case $\phi = \tilde{g}$, $X = G$, $k_i = s_i$ and $T_n = \hat{A}_n$ noting that g is an element of G of infinite order, so the function $\tilde{g}$ is nonperiodic and hence there is a recurrent point which is not almost periodic, and since $A^G \subseteq R^G$ we get $A^G \subsetneq R^G$. In fact $R^G$ is much bigger than $A^G$.

The above theorem tells us that $A^G \cup D^G \neq \hat{G}$, this answers the question raised by Nilsen [5].

CONJECTURE. If G is amenable group then there is a recurrent point in $\beta G$ which is not almost periodic point. In otherwords: $A^G \subsetneq R^G$.

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## REFERENCES

1. RUDIN, W. Homogeneity Problems in the Theory of Čeck Compactifications, Duke Math., J. 23 (1956), 409-419.

2. CHOU, C. On the Size of Left Invariant Means on Semigroup, Proc. A.M.S. 23 (1969), 199-205.

3. FAIRCHILD, L. Extreme invariant Means Without Minimal Support, Trans. A.M.S. 172 (1972), 83-93.

4. CHOU, C. Minimal Sets, Recurrent Points and Discrete Orbits on $\beta N \setminus N$, Illinois, J., of Math. 22 (1978), 54-63.

5. NILLSEN, R. Discrete Orbits in $\beta N \setminus N$, Coloq. Math. 33 (1975), 71-81.