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ON COMMON FIXED POINTS OF WEAKLY COMMUTING MAPPINGS AND SET-VALUED MAPPINGS

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ABSTRACT. Our main theorem establishes the uniqueness of the common fixed point of two set-valued mappings and of two single-valued mappings defined on a complete metric space, under a contractive condition and a weak commutativity concept. This improves a theorem of the second author.

KEY WORDS AND PHRASES. Common fixed point, set-valued mapping, weak commutativity.

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1. BASIC PRELIMINARIES.

Let (X,d) be a complete metric space and let $B(X)$ be the set of all nonempty, bounded subsets of X. As in [1], let $\delta(A,B)$ be the function defined by

$$\delta(A,B) = \sup \{d(a,b) : a \in A, b \in B\}$$

for all A, B in $B(X)$.

If A consists of a single point a we write

$$\delta(A,B) = \delta(a,B)$$

and if B also consists of a single point b we write

$$\delta(A,B) = d(a,b) .$$

It follows immediately from the definition that

$$\delta(A,B) = \delta(B,A) \geq 0, \quad \delta(A,A) = \text{diam } A,$$
$$\delta(A,B) \leq \delta(A,C) + \delta(C,B)$$

for all A , B , C in $B(X)$.

We say that a subset A of X is the limit of a sequence $\{A_n\}$ of nonempty subsets of X if each point a in A is the limit of a convergent sequence $\{a_n\}$, where a_n is in A_n for $n = 1,2, \dots$, and if for arbitrary $\epsilon > 0$, there exists an integer N such that $A_n \subseteq A_\epsilon$ for $n > N$, where A_ϵ is the union of all open spheres with centres in A and radius ϵ .

LEMMA 1. If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded subsets of (X,d) which converge to the bounded sets A and B respectively, then the sequence $\{\delta(A_n,B_n)\}$ converges to $\delta(A,B)$.

This lemma was proved in [2].

Now let F be a mapping of X into $B(X)$. We say that F is continuous at the point x in X if whenever $\{x_n\}$ is a sequence of points in X converging to x , the sequence $\{Fx_n\}$ in $B(X)$ converges to Fx in $B(X)$. If F is continuous at each point x in X, we say that F is a continuous mapping of X into $B(X)$. A point z in X is said to be a fixed point of F if z is in Fz.

For a selfmap I of (X,d), the authors of [3], extending the results of [2] and [4], defined F and I to be weakly commuting on X if

$$\delta(FIx,IFx) \leq \max\{\delta(Ix,Fx), \text{diam } IFx\} \qquad (1.1)$$

for all x in X. Two commuting mappings F and I clearly commute, but two weakly commuting mappings F and I do not necessarily commute as is shown in the following example.

EXAMPLE 1. Let $X = [0,1]$, let δ be the function induced by the euclidean metric d and define

$$Fx = [0, x/(x+a^h)], \qquad Ix = x/a$$

for all x in X , where $h \geq 1$ and $a \geq 2$. Then for any non-zero x in X we have

$$FIx = [0, x/(x + a^{h+1})] \neq [0, x/(ax + a^{h+1})] = IFx$$

but for any x in X we have

$$\delta(FIx, IFx) = x/(x + a^{h+1}) \leq x/a = \delta(Ix, Fx) .$$

Note that if F is a single-valued mapping, then the set $\{IFx\}$ consists of a single point and therefore diam $\{IFx\} = 0$ for all x in X. Condition (1.1) therefore reduces to the condition given in [5], i.e.

$$d(FIx, IFx) \leq d(Ix, Fx) \qquad (1.2)$$

for all x in X.

An extensive literature exists about (common) fixed points of set-valued mappings satisfying contractive conditions controlled from non-negative real functions f from $[0,\infty)$ into $[0,\infty)$. Suitable properties of f guarantee the convergence to the (common) fixed point of the sequence of successive approximations: see for example the papers of Barcz [6], Chen and Shih [7], Guay, Singh, and Whitfield [8], Miczko and Palezewski [9], Nhan [10], Papageorgiou [11], Popa [12], Sharma [13] and Wegrzyk [14]. In this paper we consider the family $\mathcal{F}$ of functions f from $[0,\infty)$ into $[0,\infty)$ such that

(α) f is non-decreasing,

($\alpha\alpha$) f is continuous from the right,

($\alpha\alpha\alpha$) $f(t) < t$ for all $t > 0$.

LEMMA 2. For any $t > 0$, $\lim_{n\to\infty} f^n(t) = 0$.

The proof of this lemma is obvious but see also [15].

Further details about the usage of functions with properties similar to (α), (αα), and (ααα) can be found in the papers of Benedykt and Matkowski [16], Browder [17], Conserva and Fedele [18], Hegedüs and Szilágyi [19], Hikida [20], Park and Rhoades [21], Rhoades [22], and Singh and Kasahara [23].

2. RESULTS IN COMPLETE METRIC SPACES.

Let F, G be two set-valued mappings of X into $B(X)$ and let I, J be two selfmaps of X such that

$$F(X) \subseteq I(X), \qquad G(X) \subseteq J(X) \ . \tag{2.1}$$

Let x_o (Resp. y_o) be an arbitrary point in X and define inductively a sequence $\{x_n\}$ (resp. $\{y_n\}$) such that, having defined the point x_{n-1} (resp. y_{n-1}), choose a point x_n (resp. y_n) with Ix_n (resp. Jy_n) in Fx_{n-1} (resp. Gy_{n-1}) for $n = 1,2,\ldots$. This can be done since the range of I (resp. J) contains the range of F (resp. G).

Further, assume that

$$\sup\{\delta(Fx_n,Gy_0),\ \delta(Gy_n,Fx_0) \ : \ n = 1,2,\ \ldots\} < \infty \ . \tag{2.2}$$

REMARK 1. IF X is bounded then (2.2) will always be satisfied for all x, y in X.

We consider the following conditions:

(γ_1) I continuous,

(γ_2) F continuous and $IFx \subseteq FIx$ for all x in X .

(λ_1) J continuous,

(λ_2) G continuous and $JGx \subseteq GJx$ for all x in X .

Modifying the proof of theorem 1 of [1] we are now able to prove the following:

THEOREM 1. Let F, G be two set-valued mappings of X into $B(X)$ and let I, J be two selfmaps of X satisfying (2.1) and

$$\delta(Fx,Gy) \leq f(\max\{d(Ix,Jy),\ \delta(Ix,Gy),\ \delta(Jy,Fx)\ \}) \tag{2.3}$$

for all x, y in X , where f is in $\mathcal{F}$. Further let F and G weakly commute with I and J respectively. If there exist points x_0 and y_0 in X satisfying (2.2) and if the conditions (γ_i) and (λ_j) with $i, j = 1,2$, hold, then F , G , I and J have a unique common fixed point z . Further, $Fz = Gz = \{z\}$ and z is the is the unique common fixed point of F and I and of G and J .

PROOF. Since

$$\delta(Fx_r,Gy_s) \leq \delta(Fx_r,Gy_0) + \delta(Gy_0,Fx_0) + \delta(Fx_0,Gy_s),$$

it follows from (2.2) that

$$M = \sup\ \{\delta(Fx_r,Gy_s) \ : \ r,\ s = 0,1,2,\ \ldots\}$$

is finite.

If $M > 0$, then for arbitrary $\epsilon > 0$, we can choose an integer p such that $f^p(M) < \epsilon$ by lemma 2. If $M = 0$, then $f^p(M) = 0 < \epsilon$ for any integer p.

As in the proof of theorem 1 of [24], we have on using inequality (2.3) p times and property (α):

$$\delta(Fx_m,Gy_n) \leq f^p(\max\{\delta(Fx_r,Gy_q) \ : \ m - p \leq r \leq m \ ; \ n - p \leq q \leq n\})$$

$$< \epsilon$$

for $m, n > p$. Thus

$$\delta(Fx_m,Fx_n) \le \delta(Fx_m,Gy_s) + \delta(Gy_s,Fx_r) < 2\epsilon$$

for $m, n > p$. The sequence $\{z_n\}$ is therefore a Cauchy sequence in the complete metric space X and so has a limit Z in X, where z is independent of the particular choice of each z_n. It follows in particular that the sequence $\{Ix_n\}$ converges to z and the sequence of sets $\{Fx_n\}$ converges to the set $\{z\}$.

Similarly, it can be proved that the sequence $\{Jy_n\}$ converges to a point w and the sequence of sets $\{Gy_n\}$ converges to the set $\{w\}$.

Using (2.3) we have

$$\delta(Fx_n,Gy_n) \le f(\max\{d(Ix_n,Jy_n),\ \delta(Ix_n,Gy_n),\ \delta(Jy_n,Fx_n)\})$$

Letting n tend to infinity and using lemma 1 and properties $(\alpha\alpha)$ and $(\alpha\alpha\alpha)$, it is seen that $w = z$.

Now suppose that (γ_1) holds. Then the sequence $\{I^2x_n\}$ and $\{IFx_n\}$ converge to Iz and $\{Iz\}$ respectively. Let w_n be an arbitrary point in FIx_n for $n = 1,2, \ldots$. Then since I weakly commutes with F we have on using (1.1)

$$\begin{aligned} d(w_n,Iz) &\le \delta(FIx_n,Iz) \\ &\le \delta(FIx_n,IFx_n) + \delta(IFx_n,Iz) \\ &\le \max\{\delta(Ix_n,Fx_n),\ 2\delta(I^2x_{n+1},IFx_n)\} + \delta(IFx_n,Iz)\ . \end{aligned}$$

Letting n tend to infinity and using lemma 1 we see that the sequence $\{w_n\}$ converges to Iz. But Iz is independent of the particular choice of w_n in FIx_n and this means that the sequence of sets $\{FIx_n\}$ converges to the set $\{Iz\}$.

Using inequality (2.3) we have

$$\delta(FIx_n,Gy_n) \le f(\max\{d(I^2x_n,Jy_n),\ \delta(I^2x_n,Gy_n),\ \delta(Jy_n,FIx_n)\})\ .$$

Letting n tend to infinity and using lemma 1 and property $(\alpha\alpha)$, we have

$$d(Iz,z) \le f(d(Iz,z))$$

which implies $Iz = z$ by $(\alpha\alpha\alpha)$.

Since

$$\delta(Fz,Gy_n) \le f(\max\{d(Iz,Jy_n),\ \delta(Iz,\ Gy_n),\delta(Jy_n,Fz)\})$$

we have on letting n tend to infinity and using lemma 1 and property $(\alpha\alpha)$

$$\delta(Fz,z) \le f(\delta(z,Fz))$$

which gives $Fz = \{z\}$ by $(\alpha\alpha\alpha)$.

Similarly, the weak commutativity of G and J and condition (λ_1) implies $Jz = z$ and $Gz = \{z\}$.

Now assume that (γ_2) holds. Then the sequence $\{FIx_n\}$ converges to Fz and using inequality (2.3) we have

$$\begin{aligned} \delta(FIx_n,Gy_n) &\le f(\max\{d(I^2x_n,Jy_n),\ \delta(I^2x_n,Gy_n),\ \delta(Jy_n,FIx_n)\}) \\ &\le f(\max\{\delta(FIx_n,Jy_n),\ \delta(FIx_n,Gy_n),\ \delta(Jy_n,FIx_n)\}) \end{aligned}$$

since f is non-decreasing and Ix_n is in Fx_{n-1} and so I^2x_n is in $IFx_{n-1} \subseteq FIx_{n-1}$.

Letting n tend to infinity and using lemma 1 and property $(\alpha\alpha)$, we have

$$\delta(Fz,z) \leq f(\delta(Fz,z))$$

which implies $Fz = \{z\}$ by $(\alpha\alpha\alpha)$. Thus by (2.1) there must exist a point u in X such that $Iu = z$.

Using inequality (2.3) we have

$$\delta(Fu,Gy_n) \leq f(\max\{d(Iu,Jy_n), \delta(Iu,Gy_n), \delta(Jy_n,Fu)\}).$$

Letting n tend to infinity and using lemma 1 and property $(\alpha\alpha)$, we obtain the inequality

$$\delta(Fu,z) \leq f(\max\{d(Iu,z), \delta(z,Fu)\}) = f(\delta(z,Fu)) .$$

Thus $Fu = \{z\}$ by $(\alpha\alpha\alpha)$ and since F and I weakly commute, we have

$$\{z\} = Fz = FIu = IFu = \{Iz\}.$$

It follows that $Iz = z$.

Similarly property (λ_2) assures that $Gz = \{z\}$ and $Jz = z$.

We have therefore shown that if the conditions (γ_i) and (λ_j), with $i, j = 1, 2$, hold then $Iz = Jz = z$ and $Fz = Gz = \{z\}$.

That z is the unique common fixed point of F and I and of G and J follows easily. This completes the proof of the theorem.

COROLLARY 1. Let F, G be two set-valued mappings of X into $B(X)$ and let I, J be two selfmaps of X satisfying (2.1) and

$$\delta(Fx,Fy) \leqq c.\max\{d(Ix,Jy),\delta(Ix,Gy),\delta(Jy,Fx)\} \tag{2.4}$$

for all x, y in X, where $0 \leqq c < 1$. Further, let F and G commute with I and J respectively. If F or I and G or J are continuous, then F, G, I and J have a unique common fixed point z. Further, $Fz = Gz = \{z\}$ and z is the unique common fixed point of F and I and of G and J.

PROOF. As in the proof of theorem 1 of [1], it is proved that (2.2) holds for any x_0, y_0 in X. Since F and G commute with I and J respectively, we have $FIx = IFx$ and $GJx = JGx$ for all x in X. The thesis then follows from theorem 1 if we assume that $f(t) = ct$ for all $t \geqq 0$.

The result of this corollary was given in [1].

We now give an example in which theorem 1 holds but corollary 1 is not applicable.

EXAMPLE 2. Let $X = [o,1]$ with δ induced by the euclidean metric d and let

$$Fx = [0,x/(x + 4)], \quad Gx = [0,x/(x + 8)], \quad Ix = Jx = \tfrac{1}{2}x$$

for all x in X.

By example 1, F and G weakly commute with I. Further, we have

$$F(X) = [0,1/5] \subset [0,\tfrac{1}{2}] = I(X),$$

$$G(X) = [0,1/9] \subset [0,\tfrac{1}{2}] = J(X),$$

$$IFx = [0,x/(2x + 8)] \subset [0,x/(x + 8)] = FIx$$

$$JGx = [0,x/(2x + 16)] \subset [0,x/(x + 16] = GJx$$

for all x in X.

Since

$$\delta(Fx,Gy) = \max\{x/(x+4), y/(y+8)\}$$
$$\leq \max\{x/(x+4), y/(y+4)\}$$

$$\leq \tfrac{1}{2}\max\{\tfrac{1}{2}x, \tfrac{1}{2}y\}$$
$$= \begin{cases} \tfrac{1}{2}\delta(Ix,Gy), & \text{if } x \geq y, \\ \tfrac{1}{2}\delta(Jy,Fx), & \text{if } x < y \end{cases}$$

and since X is bounded all the hypotheses of theorem 1 are satisfied if we assume $f(t) = \frac{1}{2}t$ for all $t \geq 0$. Clearly f is in $\mathcal{F}$ and 0 is the unique common fixed point of F, G and I .

Theorem 1 is a stronger result than corollary 1, even if the mappings under consideration are commutative, as is shown in the following example.

EXAMPLE 3. Let X be the reals with δ induced by the euclidean metric d , let

$$Fx = \begin{cases} \{0\}, & \text{if } x \leq 0, \\ [0, x/(1+3x)], & \text{if } 0 < x \leq 1, \\ [0, 1/4], & \text{if } x > 1 \end{cases}$$

$$Gx = \begin{cases} \{0\}, & \text{if } x \leq 0, \\ [0, x/(1+2x)], & \text{if } 0 < x \leq 1, \\ \{1/3\}, & \text{if } x > 1, \end{cases}$$

$$Ix = \begin{cases} 0, \text{ if } x \leq 0, \\ x, \text{ if } 0 < x \leq 1, \\ 1, \text{ if } x > 1, \end{cases} \qquad Jx = \begin{cases} 0, \text{ if } x \leq 0, \\ x, \text{ if } x > 0, \end{cases}$$

for all x in X and let f in $\mathcal{F}$ be given by

$$f(t) = \begin{cases} t/(1+2t), & \text{if } 0 \leq t \leq 1, \\ t/3, & \text{if } t > 1. \end{cases}$$

We have

$$\delta(Fx,Gy) = 0 = f(d(Ix,Jy)), \text{ if } x,y \leq 0,$$
$$\delta(Fx,Gy) = y/(1+2y) = f(y) = f(d(Ix,Jy)), \text{ if } x \leq 0 \text{ and } 0 < y \leq 1,$$
$$\delta(Fx,Gy) = 1/3 < y/3 = f(y) = f(d(Ix,Jy)), \text{ if } x \leq 0 \text{ and } y > 1,$$
$$\delta(Fx,Gy) = x/(1+3x) < x/(1+2x) = f(x) = f(d(Ix,Jy)), \text{ if } 0 < x \leq 1 \text{ and } y \leq 0,$$
$$\delta(Fx,Gy) = \max\{x/(1+3x), y/(1+2y)\}$$
$$< \max\{x/(1+2x), y/(1+2y)\}$$
$$= \begin{cases} f(y) = f(\delta(Fx,Jy)), & \text{if } x \leq y, \\ f(x) = f(\delta(Ix,Gy)), & \text{if } x > y, \end{cases} \text{ and if } 0 < x, y \leq 1,$$
$$\delta(Fx,Gy) = 1/3 < y/3 = f(y) = f(\delta(Jy,Fx)), \quad \text{if } 0 < x \leq 1 \text{ and } y > 1,$$
$$\delta(Fx,Gy) = 1/4 < 1/3 = f(1) = f(d(Ix,Jy)), \quad \text{if } x > 1 \text{ and } y \leq 0,$$
$$\delta(Fx,Gy) = \max\{1/4, y/(1+2y)\} \leq 1/3 = f(1) = f(\delta(Ix,Gy)), \text{ if } x > 1 \text{ and } 0 < y \leq 1,$$
$$\delta(Fx,Gy) = 1/3 < y/3 = f(y) = f(\delta(Jy,Fx)), \text{ if } x,y > 1.$$

Condition (2.3) therefore holds in every case since f is non-decreasing. Further

$$F(X) = [0,1.4] \subset [0,1] = I(X),$$
$$G(X) = [0,1/3] \subset [0,\infty] = J(X)$$

and F and G commute with I and J respectively. Since $Fx \subseteq [0,1/4]$ and $Gx \subseteq [0,1/3]$ for all x in X, it is easily seen that $M \leq 1/3$ and so (2.2) holds for any x_0 and y_0 chosen in X. As I and J are continuous, theorem 1 is applicable. However, the conditions of the corollary are not satisfied. Otherwise for x=0 and $0 < y \leq 1$, condition (2.4) should imply

$$\delta(Fx,Gy) = \frac{y}{1+2y} \leq c.\max\{y, \frac{y}{1+2y}, y\} = cy$$

and so $1/(1 + 2y) \leq c$ which as y tends to zero, gives $c \geq 1$, a contradiction.

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