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A DIGRAPH EQUATION FOR HOMOMORPHIC IMAGES

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ABSTRACT. The definitions of a homomorphism and a contraction of a graph are generalized to digraphs. Solutions are given to the graph equation $\overline{\phi(D)} = \theta_\phi(\bar{D})$.

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By a graph G we mean a finite graph with no multiple edges or loops. If graphs G and H are isomorphic we write G = H. An elementary homomorphism of a graph G is an identification of two non-adjacent vertices of G and a homomorphism is a sequence of elementary homomorphisms. A homomorphism of G onto H preserves adjacency. Likewise, an elementary contraction of G is the identification of two adjacent vertices of G and a contraction is a sequence of elementary contractions [1]. Thus for every homomorphism ϕ of G there is a related contraction θ_ϕ of the complement of G, $\bar{G}$. This contraction is constructed as follows: ϕ is a sequence of elementary homomorphisms $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ so we let θ_ϕ be sequence of elementary contractions $\theta_1, \theta_2, \ldots, \theta_n$ where θ_i identifies the same vertices in $\bar{G}$ that ε_i identifies in G.

Recently [2] the graph equation $\overline{\phi(G)} = \theta_\phi(\bar{G})$ was studied. In this paper, we generalize the definition of a homomorphism and its related contraction to digraphs and find general solutions to this graph equation. In doing so, we find an easier proof of the result given in [2].

A digraph D consists of a finite vertex set V(D) together with a set E(D) of ordered pairs of distinct elements of V(D), called arcs. Again, if D_1 is isomorphic to D_2 we write $D_1 = D_2$. By an elementary homomorphism of D we mean an identification of two mutually non-adjacent vertices of D (neither uv nor vu are in E(D)). Similarly, an elementary contraction is an identification of two mutually adjacent vertices of D (both uv and vu are in E(D)). A homomorphism (contraction) of D is again a sequence of elementary homomorphisms (contractions). The contraction θ_ϕ of $\bar{D}$ related to the homomorphism ϕ of D is defined as for undirected graphs.

We will use the following notation as need arises: Ib(u) is the set of vertices v of D such that vu is an arc of D, Ob(u) is the set of vertices v of D such that uv is an arc of D, and A(u) is the adjacency set of u in the graph G.

THEOREM 1. Let ε be an elementary homomorphism of D identifying vertices u_1 and u_2. Then $\varepsilon(D) = \theta_\varepsilon(\bar{D})$ if and only if $Ib(u_1) = Ib(u_2)$ and $Ob(u_1) = Ob(u_2)$.

PROOF. Let $u = \varepsilon(u_1) = \theta_\varepsilon(u_1)$. First suppose that $Ob(u_1) \neq Ob(u_2)$. Excluding u as a possible endpoint of an arc, we have vv' is an arc of $\overline{\varepsilon(D)}$ if and only if vv' is an arc of $\theta_\varepsilon(\bar{D})$. Hence there is a one to one correspondence of those arcs in $\overline{\varepsilon(D)}$ without u as an endpoint and those of $\theta_\varepsilon(\bar{D})$ without u as an endpoint. The vertex v of the arc uv must be in $Ob(u_1) \cap Ob(u_2)$, $(Ob(u_1) \cup Ob(u_2))^c$, or $Ob(u_1) \nabla Ob(u_2)$, the symmetric difference. In the first case, uv is not an arc of $\overline{\varepsilon(D)}$ or $\theta_\varepsilon(\bar{D})$ and in the second case, uv is an arc of both. The latter case implies that uv is not an arc of $\overline{\varepsilon(D)}$ but is an arc of $\theta_\varepsilon(\bar{D})$. Thus for every vertex in $Ob(u_1) \nabla Ob(u_2)$, $\theta_\varepsilon(\bar{D})$ has one more arc than $\overline{\varepsilon(D)}$. The same holds for vertices in $Ib(u_1) \nabla Ib(u_2)$. Thus if $Ob(u_1) \neq Ob(u_2)$ or $Ib(u_1) \neq Ib(u_2)$, $|E(\theta_\varepsilon(\bar{D}))| > |E(\overline{\varepsilon(D)})|$ and hence $\overline{\varepsilon(D)} \neq \theta_\varepsilon(\bar{D})$. Now let $Ib(u_1) = Ib(u_2)$ and $Ob(u_1) = Ob(u_2)$. We will use the identity map from $V(\overline{\varepsilon(D)})$ onto $V(\theta_\varepsilon(\bar{D}))$ and hence need only consider arcs to and from u. If uv is in $E(\theta_\varepsilon(\bar{D}))$ then u_1v and u_2v are arcs in $\bar{D}$. Thus u_1v and u_2v are not arc of D and subsequently uv is in $E(\overline{\varepsilon(D)})$. By the same argument, if uv is an arc of $\overline{\varepsilon(D)}$, uv will be an arc of $\theta_\varepsilon(\bar{D})$. This holds for arcs vu, so $\overline{\varepsilon(D)} = \theta_\varepsilon(\bar{D})$.

COROLLARY 1: $\overline{\phi(D)} = \theta_\phi(\bar{D})$ if and only if ϕ is a sequence of elementary homomorphisms, each of which satisfies the conditions of Theorem 1.

A digraph D is pseudo-complete n-partite if there is a partition $V_1, V_2, \ldots, V_n$ such that u, u' in V_i for some i implies u and u' are mutually non-adjacent, if u is an element of V_i and v is an element of V_j, $i \neq j$, then either uv or vu is an arc of D, and finally if u and u' are in V_i, v and v' are in V_j, $i \neq j$, and uv is an arc then uv', u'v, and u'v' are also.

THEOREM 2. $\overline{\phi(D)} = \theta_\phi(\bar{D})$ for all homomorphisms ϕ of D if and only if D is pseudo-complete n-partite.

PROOF. If D is pseudo-completely n-partite, every elementary homomorphism identifies two vertices u_1 and u_2 in the same partition set and thus $Ib(u_1) = Ib(u_2)$ and $Ob(u_1) = Ob(u_2)$. Hence $\overline{\varepsilon(D)} = \theta_\varepsilon(\bar{D})$ for every elementary homomorphism and thus for every homomorphism of D. Conversely, partition V(D) according to the relation: u_1 and u_2 are in V_i if and only if $Ib(u_1) = Ib(u_2)$ and $Ob(u_1) = Ob(u_2)$. We need only show that if u_1 is in V_i and u_2 is in V_j, $i \neq j$, then either u_1u_2 or u_2u_1 is in E(D). Suppose u_1 and u_2 are mutually non-adjacent and let ε be the elementary homomorphism identifying them. Since $\overline{\varepsilon(D)} = \theta_\varepsilon(\bar{D})$, $Ob(u_1) = Ob(u_2)$ and $Ib(u_1) = Ib(u_2)$ by Theorem 1 and hence u_1 and u_2 are in the same partition set. Thus if u_1 is in V_i and u_2 is in V_j, $i \neq j$, there is an arc between them and D must be pseudo-complete n-partite.

If, for every vertex u of D, $Ib(u) = Ob(u)$, D is a symmetric digraph and can be represented by a graph G. This leads to the following corollaries to Theorems 1 and 2.

COROLLARY 2. An elementary homomorphism ε identifying vertices u and v of a graph G satisfies $\overline{\varepsilon(G)} = \theta_\varepsilon(\overline{G})$ if and only if $A(u_1) = A(u_2)$.

COROLLARY 3. A homomorphism ϕ of G satisfies $\overline{\phi(G)} = \theta_\phi(\overline{G})$ if and only if ϕ is a sequence of elementary homomorphisms, each satisfying Corollary 2.

COROLLARY 4. $\overline{\phi(G)} = \theta_\phi(\overline{G})$ for every homomorphism ϕ of G if and only if G is complete n-partite.

A study of the equation $\phi(D) = \theta_\phi(\overline{D})$ would be interesting, yet is apparently difficult considering the work done in [2] for graphs. We conjecture that if $D = \overline{D}$ and $\phi(D) = \theta_\phi(\overline{D})$, ϕ nontrivial, then D is a symmetric digraph.

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