BANACH SPACES OF FUNCTIONS ANALYTIC IN A POLYDISC

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ABSTRACT.   This paper is concerned with functions of several complex variables analytic
in the unit polydisc.   Certain Banach spaces to which these functions might belong are
defined and some relationships between them are developed.   The space of linear
functionals for the Banach space of functions analytic in the open unit polydisc and
continuous on the unit torus is then described in terms of analytic functions using
an extension of the Hadamard product.

1. INTRODUCTION.

In 1950 A. E. Taylor [1] studied Banach spaces of functions analytic in the unit
disc.   One of his principal results was a representation of linear functionals in
terms of functions analytic in the unit disc.   In this paper, the results of Taylor are
extended to functions analytic in the unit polydisc in n-dimensional complex space.
The goal is the representation theorem for linear functions, Theorem 4.5, in which the
functionals are expressed in terms of a Hadamard product.   Taylor's results have proved
to be very useful in work involving certain singular linear differential operators.
His representation theorem for linear functionals was a key part of the work of Grimm
and Hall in [2], and of subsequent work of Hall in [3] and [4].

The following notation will be used extensively:

U = open unit disc in the complex plane C, $\qquad$ (1.1)

T = unit circle in C, $\qquad$ (1.2)

$T_\varepsilon = \{z \in C: |z| = 1-\varepsilon\}, \ 0 < \varepsilon < 1$, $\qquad$ (1.3)

Z_+ = nonnegative integers, $\qquad$ (1.4)

$U^n = U \times U \times \ldots \times U$, (n copies of U) the unit polydisc in C^n, $\qquad$ (1.5)

$T^n = T \times T \times \ldots \times T$, (n copies of T) the unit torus in C^n, $\qquad$ (1.6)

$T_\varepsilon^n = T_\varepsilon \times T_\varepsilon \times \ldots \times T_\varepsilon$, (n copies of T_ε), $\qquad$ (1.7)

$Z_+^n = Z_+ \times Z_+ \times \ldots \times Z_+$ (n copies of Z_+). $\qquad$ (1.8)

If $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$ are in C^n, define zw by $zw = (z_1 w_1, z_2 w_2, \ldots, z_n w_n)$. Further, if $z \in C^n$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in Z_+^n$ define z^α by $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$ and $|\alpha|$ by $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$. Denote by A^n the class of functions on C^n which are analytic in U^n and define the function $u_\alpha \in A^n$, for each $\alpha \in Z_+^n$, by $u_\alpha(z) = z^\alpha$. Also define the operators $U_x : A^n \to A^n$ and $T_w : A^n \to A^n$ by

$$U_x f = g, \text{ where } x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \text{ and} \tag{1.9}$$

$$g(z) = f(ze^{ix}) \equiv f(z_1 e^{ix_1}, z_2 e^{ix_2}, \ldots, z_n e^{ix_n}), \tag{1.10}$$

$$T_w f = g, \text{ where } w \in \overline{U^n}, \text{ and } g(z) = f(wz). \tag{1.11}$$

The operators U_x and T_w are easily seen to be linear.

If $f \in A^n$ then f has the power series expansion

$$f(z) = \sum_{\alpha \in Z_+^n} f_\alpha z^\alpha. \tag{1.12}$$

Denote f_α by $\gamma_\alpha(f)$ where γ_α is the functional defined by

$$\gamma_\alpha(f) = (\alpha_1! \alpha_2! \ldots \alpha_n!)^{-1} \left. \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}} \right|_{z=(0,\ldots,0)}. \tag{1.13}$$

If f and g are in A^n with power series

$$f(z) = \sum_{\alpha \in Z_+^n} f_\alpha z^\alpha \text{ and } g(z) = \sum_{\alpha \in Z_+^n} g_\alpha z^\alpha, \tag{1.14}$$

define the Hadamard product of f and g by

$$H(f,g;z) = \sum_{\alpha \in Z_+^n} f_\alpha g_\alpha z^\alpha. \tag{1.15}$$

$H(f,g;z)$ is clearly in A^n, and the following also can be proved:

$$\gamma_\alpha(T_w f) = w^\alpha \gamma_\alpha(f), \tag{1.16}$$

$$H(f,g;z) = H(g,f;z), \tag{1.17}$$

$$H(af+bg,h;z) = aH(f,h;z) + bH(g,h;z), \tag{1.18}$$

$$H(T_w f,g;z) = H(f,g;wz), \tag{1.19}$$

$$H(f,g;z) = \left(\frac{1}{2\pi i}\right)^n \int_{T_{\varepsilon_1} \times \cdots \times T_{\varepsilon_n}} f(\xi_1, \ldots, \xi_n) g\left(\frac{z_1}{\xi_1}, \ldots, \frac{z_n}{\xi_n}\right) \frac{d\xi_1}{\xi_1} \cdots \frac{d\xi_n}{\xi_n}, \tag{1.20}$$

where $|z_i| < |\xi_i|$ for $i = 1, \ldots, n$.

Equation (1.20) can also take the form

$$H(f,g;z) = \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} \ldots \int_0^{2\pi} f(r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}) g\left(\frac{z_1}{r_1} e^{-i\theta_1}, \ldots, \frac{z_n}{r_n} e^{-i\theta_n}\right) d\theta_1 \ldots d\theta_n \tag{1.21}$$

where $r_i = 1 - \varepsilon_i$ and $z \in U^n$.

2   SPACES OF TYPE A_k.

Let B be a nontrivial complex Banach space, each element of which belongs to A^n. Such a space will be called a space of type A. B may also have one or more of the following properties.

P_1: There exists a constant A such that $|\gamma_\alpha(f)| \leq A||f||$ if $f \in B$ and $\alpha \in Z_+^n$.

The least such constant will be denoted $A_1(B)$.

P_2: $u_\alpha \in B$ for all $\alpha \in Z_+^n$. There exists a constant A such that $||u_\alpha|| \leq A$

for all α. The least such constant will be denoted $A_2(B)$.

P_3: $U_x f \in B$ if $f \in B$ and x is a real n-tuple. Also $||U_x f|| = ||f||$.

P_4: $T_r f \in B$ if $f \in B$ and $r = (r_1, r_2, \ldots, r_n)$ with $0 \leq r_i < 1$. There exists a

constant A such that $||T_r f|| \leq A||f||$. The least such constant will be

denoted $A_4(B)$.

B will be called a space of type A_k if B is of type A and also satisfies axioms $P_1, \ldots, P_k$.

Let B* denote the space of continuous linear functionals on B. Then by P_1, $\gamma_n \in B*$ and it can be shown that

$$A_1(B) = \sup_\alpha ||\gamma_\alpha||, \text{ and} \qquad (2.1)$$

$$A_2(B) = \sup_\alpha ||u_\alpha||. \qquad (2.2)$$

Other relations satisfied by the constants $A_k(B)$ are

$$A_4(B) = \sup_r ||T_r||, \qquad (2.3)$$

$$1 \leq A_1(B)A_2(B), \qquad (2.4)$$

$$1 \leq A_4(B) \qquad (2.5)$$

Let B be a space of type A_1. For any fixed $g \in A^n$ and $z \in U^n$ define

$$N(g;z) = \sup_{||f||=1} |H(f,g;z)|. \qquad (2.6)$$

The following are clear from the properties of the Hadamard product.

$$N(g+h;z) \leq N(g;z) + N(h;z) \qquad (2.7)$$

$$N(ag;z) = |a|N(g;z) \qquad (2.8)$$

$$N(T_w g;z) = N(g;wz), \quad |w| < 1. \qquad (2.9)$$

THEOREM 2.1. Let B be a space of type A_3. Then the function N(g;z) has the properties:

$$N(g,z) = N(g,\hat{z}), \text{ where } g \in A^n, z \in U^n \text{ and } \hat{z} = (|z_1|,|z_2|,\ldots,|z_n|), \qquad (2.10)$$

$N(g;(r_1, \ldots, r_n))$ is a nondecreasing function of each r_i, i=1,...,m,   (2.11)

where $0 \leq r_i < 1$,

$N(g;(r_1,\ldots,r_n)) = 0$ for all $(r_1,\ldots,r_n)$ if and only if $g \equiv 0$.   (2.12)

PROOF. Let $\hat{M}$ denote the set of all nonzero elements of B and define, for $f \in \hat{M}$

$$M(f;z) = \frac{H(f,g;z)}{||f||} \tag{2.13}$$

where g is fixed in A^m. Also define

$$M(z) = \sup_{f \in \hat{M}} |M(f;z)|. \tag{2.14}$$

Observe that $M(z) = N(g;z)$ and also, for $x \in R^n$ and $z \in U^n$

$$M(U_x f;z) = M(f;ze^{ix}). \tag{2.15}$$

Let $z \in U^n$ be fixed. Then

$$|M(f;z)| = \left|H(\frac{f}{||f||},g;z)\right|$$

$$= \frac{1}{||f||}|H(f,g;z)| \tag{2.16}$$

$$\leq \frac{1}{||f||} \sum_\alpha |f_\alpha||g_\alpha| |z^\alpha|$$

$$\leq \frac{1}{||f||} \sum_\alpha A_1(B)||f|| |g_\alpha| |z^\alpha|$$

$$= A_1(B) \sum_\alpha |g_\alpha| |z^\alpha|.$$

Hence, for fixed $z \in U^n$, $|M(f;z)|$ is bounded as f varies over $\hat{M}$.

Now for $x \in R^n$, $U_x f$ varies over all of $\hat{M}$ as f varies over all of $\hat{M}$ since $f = U_x(U_x f)$, and so by (2.15) we get $M(z) = \hat{M}(z)$ if x is chosen carefully, and (2.10) is proved.

Next, let $R = (R_1,...,R_i,...,R_n)$ and $r = (r_1,...,r_i,...,r_n)$ with $0 \leq R_k = r_k < 1$ for $k=1,...,n$, $k \neq i$ and $0 \leq r_i < R_i < 1$. Also suppose $f \in \hat{M}$. Then let z_R and z_r be the points on the tori $\{z: |z_k| = R_k\}$ and $\{z: |z_k| = r_k\}$, respectively, at which $|M(f;z)|$ assumes its maximum value. The maximum principle for polydisc functions now yields

$$|M(f;z_r)| \leq |M(f;z_R)|. \tag{2.17}$$

But $|M(f;z_R)| \leq M(z_R) = \hat{M}(z_R) = M(R)$ and so

$$|M(f;r)| \leq |M(f;z_r)| \leq M(R), \tag{2.18}$$

which implies $M(r) \leq M(R)$, and (2.11) is proved.

Finally, if $g \equiv 0$, $N(g;r) \equiv 0$ for all r. If $g \not\equiv 0$, then $\gamma_\alpha(g) \neq 0$ for some α. Let $f = u_\alpha$ for this α. Then $H(f,g;r) = \gamma_\alpha(g)r_1^{\alpha_1}...r_n^{\alpha_n}$ which is nonzero if $r_k \neq 0$, $k=1,...,n$, and the proof is complete.

3. THE SPACES B' AND B°.

Suppose B is a space of type A_3. Two related spaces which will be used later in characterizing B* will be defined as follows:

$$B' = \{F \in A^n: N(F;r) \text{ is uniformly bounded in } r, 0 \leq r_k < 1, k=1,...,n\}, \tag{3.1}$$

$$B^0 = \{F \epsilon A^n: \lim_{r \to (1, \ldots, 1)} H(f, F; r) \text{ exists for each } f \epsilon B\}. \tag{3.2}$$

For $F \epsilon B^1$ define $N(F) = \sup\{N(f; r): 0 \le r_k < 1, k=1, \ldots, n\}$. By (2.11) we have

$$N(F) = \lim_{r (1, \ldots, 1)} N(F; r) \tag{3.3}$$

THEOREM 3.1. If B is a space of type A_3 then B' is a Banach space with norm $N(F)$.

PROOF. From the properties of the Hadamard product and $N(F; r)$, and from theorem 2.1, B' is clearly a normed linear space with norm $N(F)$. To prove completeness proceed as follows.

Let $\{F_j\}$ be a Cauchy sequence in B'. Then there is a $J > 0$ such that if $j > J$, $N(F_j) - N(F_J) < 1$, and $N(F_j)$ is bounded by, say, E. Let $0 \le r_k < 1$ for $k=1, \ldots, n$ and consider, for $\alpha \epsilon Z_+^m$,

$$\left| \gamma_\alpha (F_j - F_m) r^\alpha \right| = H(u_\alpha, F_j - F_m; r) \Big| \tag{3.4}$$

$$\le ||u_\alpha|| N(F_j - F_m) \le A_2(B) N(F_j - F_m).$$

Thus, $\{\gamma_\alpha(F_j)\}$ is also a Cauchy sequence for each $\alpha \epsilon Z_+^n$. Now define $a_\alpha = \lim_{j \to \infty} \gamma_\alpha(F_j)$ and note that convergence is uniform in α and that the sequence $\{a_\alpha\}$, $\alpha \epsilon Z_+^n$, is bounded. Using the above argument with $F_m = 0$ we obtain $|\gamma_\alpha(F_j) r^\alpha| \le A_2(B) E$ which yields $|a_\alpha| \le A_2(B) E$. Next define

$$F(z) = \sum_{\alpha \epsilon Z_+^n} a_\alpha z^\alpha. \tag{3.5}$$

Clearly $F \epsilon A$, and it will be shown that $F \epsilon B'$ and $\lim_{j \to \infty} F_j = F$.

Consider, for fixed r, and $f \epsilon A$,

$$\left| H(f, F_j; r) - H(f, F; r) \right| \le \sum_\alpha |\gamma_\alpha(f) r^\alpha \sup_\alpha |\gamma_\alpha(F_j) - a_\alpha|. \tag{3.6}$$

Since $\lim_{j \to \infty} [\gamma_\alpha(F_j) - a_\alpha] = 0$ uniformly in α,

$$\lim_{j \to \infty} H(f, F_j; r) = H(f, F; r). \tag{3.7}$$

Further, for $f \epsilon B$

$$\left| H(f, G_j; r) \right| \le ||f|| N(F_j) \le E||f|| \tag{3.8}$$

which implies $\left| H(f, F; r) \right| \le E||f||$ or $N(F; r) \le E$. Hence, $F \epsilon B'$.

Finally, for $\epsilon > 0$, choose $j_0(\epsilon)$ such that if $j, m \ge j_0(\epsilon)$ then $N(F_j - F_m) < \epsilon$. For $f \epsilon B$ (3.5) yields

$$\left| H(f, F_j - F_m; r) \right| \le N(F_j - F_m) ||f|| < \epsilon ||f||. \tag{3.9}$$

As $m \to \infty$ (3.4) yields $\left| H(f, F_j - F; r) \right| \le \epsilon ||f||$, if $j \ge j_0(\epsilon)$, which means $N(F_j - F) \le \epsilon$ and the proof is complete.

THEOREM 3.2. If B is a Banach space of type A_3 then B^0 is a linear subset of B' and is a Banach space of type A_4 with norm $N(F)$.

PROOF: Clearly B^0 is a linear subset of A. For $f \in B$ and $F \in B^0$ let $\psi_r(f) =$ $H(f,F;r)$ define a functional of f over B. Then

$$\sup_{||f||=1} |\psi_r(f)| = N(F;r), \tag{3.10}$$

and since $F \in B^0$ then $N(f;r)$ is bounded as a function of r by the uniform boundedness principle. Thus, B^0 is a linear subset of B'.

In order to show that B^0 is closed in B', let $\{F_j\} \subset B^0$ and $F \in B'$ be such that $\lim_{j\to\infty} N(F_j-F) = 0$ For $f \in B$

$$|H(f,F;r)-H(f,F_j;\rho)| \leq |H(f,F-F_j;r)| + |H(f,F_j;r)-H(f,F_j;\rho)| + |H(f,F_j-F;\rho)|$$

$$\leq 2N(F_j-F)||f|| + |H(f,F_j;r)-H(f,F_j;\rho)|. \tag{3.11}$$

For $f \in B$ fixed and $\varepsilon > 0$ choose j large enough so that $2N(F_j-F)||f|| < \varepsilon/2$ and choose r and ρ close enough to $(1,1,\ldots,1)$ so that $|H(f,F_j;r)-H(f,F_j;\rho)|<\varepsilon/2$. Therefore $\lim_{r,\rho\to(1,\ldots,1)} |H(f,F;r)-H(f,F;\rho)| = 0$ and $F \in B^0$.

4. REPRESENTATION OF LINEAR FUNCTIONALS.

In this section a series of results which culminate in a representation of the space B* and its relationship to the spaces B' and B^0 will be proved.

THEOREM 4.1. Let B be a Banach space of type A_4. If $\gamma \in B*$ let

$$G(z) = \sum_{\alpha \in Z_+^n} \gamma(u_\alpha) z^\alpha, \; z \in U^n. \tag{4.1}$$

Then $G \in B'$ and $||G|| = N(G) \leq A_4(B)||\gamma||$.

PROOF: $G \in A^n$ since $|\gamma(u_\alpha)| \leq ||\gamma||A_2(B)$. Also, $\gamma(u) = \gamma_\alpha(G) = G_\alpha$. Let $f \in A^n$ and $w \in U^n$. Then $T_w f \in B$ and

$$\gamma(T_w f) = \sum_\alpha f_\alpha \gamma(z^\alpha)w^\alpha = \sum_\alpha f_\alpha G_\alpha w^\alpha = H(f,G;w). \tag{4.2}$$

If $f \in B$ then

$$|H(f,G;r)| = |\gamma(T_r f)| \leq A_4(B)||\gamma|| \; ||f||. \tag{4.3}$$

Hence $N(G;r) = \sup_{||f||=1} |H(f,G;r)| \leq A_4(B)||\gamma||$ and $G \in B'$. Also,

$$||G|| = N(G) = \sup_r N(G;r) \leq A_4(B) ||\gamma||.$$

The passage from γ to G given by (4.1) defines an operator $\Gamma: B* \to B'$ where $\Gamma(\gamma) = G$.

THEOREM 4.2. Let B be a Banach space of type A_4. Then Γ is a linear operator and $||\Gamma|| = A_4(B)$. Γ has an inverse if and only if the linear subspace of B spanned by $\{u_\alpha\}$, $\alpha \in Z_+^n$, is dense in B.

PROOF. By Theorem 4.1 Γ is linear and $||\Gamma|| \leq A_4(B)$. By (4.2)

$$|\gamma(T_r f)| \leq ||f|| \; ||G|| \leq ||f|| \; ||\Gamma|| \; ||\gamma||. \tag{4.4}$$

Now choose γ so that $||\gamma|| = 1$ and $|\gamma(T_r f)| = ||T_r f||$ so that $||T_r f|| \leq ||f|| \; ||\Gamma||$.

Then $||T_r|| \leq ||\Gamma||$ and $A_4(B) \leq ||\Gamma||$. Hence $||\Gamma|| = A_4(B)$.

Suppose $\Gamma(\gamma) = 0$. Then $\gamma(u_\alpha) = 0$ for all $\alpha \in Z_+^n$ and hence saying that Γ^{-1} exists if and only if $\gamma(u_\alpha) = 0$ for all $\alpha \in Z_+^n$ is equivalent to saying $\gamma = 0$. Thus Γ^{-1} exists if and only if $\{u_\alpha\}$ is total in B which is equivalent to the subspace spanned by $\{u_\alpha\}$ being dense in B.

For $f \in B$, $H(f,F;r)$ defines a linear functional with norm $N(F;r)$, so if $F \in B^0$

$$\gamma(f) = \lim_{r \to (1,\ldots,1)} H(f,F;r) \tag{4.5}$$

defines an element of B* corresponding to F. The passage from F to γ given by (4.5) defines an operator $\Lambda : B^0 \to B*$ where $\Lambda(F) = \gamma$.

THEOREM 4.3. Let B be a Banach space of type A_4. Then Λ is a linear operator,

$$\Gamma\Lambda(F) = F, \text{ for } F \in B^0, \text{ and} \tag{4.6}$$

$$||\Lambda(F)|| \leq N(F) \leq A_4(B)||\Lambda(F)||, \text{ for } F \in B^0. \tag{4.7}$$

Thus Λ defines an injective mapping, with bounded inverse, of B^0 onto a subspace of B*.

PROOF: Let $F \in B^0$, $\Lambda(F) = \gamma$ and $\Gamma(\gamma) = G$. Then

$$\gamma_\alpha(G) = \gamma(u_\alpha) = \lim_{r \to (1,\ldots,1)} H(u_\alpha,F;r) = \lim_{r \to (1,\ldots,1)} \gamma_\alpha(F)r^\alpha = \gamma_\alpha(F), \text{ and so } \Gamma(\Lambda(F)) = F.$$

Also, since $|H(f,F;r)| \leq ||f|| \, ||F||$ it follows that $|\gamma(f)| \leq ||f|| \, ||F||$ where $\gamma = \Lambda(F)$. Thus, from (4.6)

$$||F|| = ||\Gamma(\Lambda(F))|| \leq ||\Gamma|| \, ||\Lambda(F)|| = A_4(B)||\Lambda(F)||. \tag{4.8}$$

If the space B satisfies an additional axiom, viz.,

P_5 : If $f \in B$ then $T_r f \in B$ and $\lim\limits_{r \to (1,\ldots,1)} ||T_r f - f|| = 0$,

then B*, B' and B^0 turn out to be isometrically isomorphic.

THEOREM 4.4. Let B be a Banach space of type A_4 also satisfying P_5. Then

$$B^0 = B' \tag{4.9}$$

$$\Lambda \text{ is bijective and isometric, and} \tag{4.10}$$

$$\Lambda^{-1} = \Gamma. \tag{4.11}$$

PROOF: By (1.19) $H(T_r f,F;\rho) = H(T_\rho f,F;r)$ and so

$$H(f,F;r) - H(f,F;\rho) = H(f-T_\rho f,F;r) + H(T_r f - T_\rho f,F;\rho) + H(T_\rho f - f,F;\rho). \tag{4.12}$$

Therefore, if $f \in B$ and $F \in B'$

$$|H(f,F;r) - H(f,F;\rho)| \leq [2||f-T_\rho f|| + ||T_r f - T_\rho f||]N(F), \tag{4.13}$$

and by P_5

$$\lim_{r,\rho \to (1,\ldots,1)} |H(f,F;r) - H(f,F;\rho)| = 0, \tag{4.14}$$

so $F \in B^0$. Since $B^0 \subset B'$ already, this means $B' = B^0$.

From P_5, $\lim\limits_{r\to(1,\ldots,1)} ||T_r f|| = ||f||$. If $|w_i| = r_i < 1$, $i=1,\ldots,n$, and $T_w f$ is

analytic in w then $||T_w f|| = ||T_r f||$ by P_3. Hence from the maximum modulus theorem

for functions with values in a Banach space (see [3], p. 211) $||T_r f||$ is a nondecreasing

continuous function of r. This yields $A_4(B) = 1$ and by Theorem 4.3 Λ is isometric.

Now suppose $f \varepsilon B$ and $\gamma \varepsilon B*$. Then

$$[\Lambda(\Gamma(\gamma))](f) = \lim_{r\to(1,\ldots,1)} H(f,\Gamma(\gamma);r)$$

$$= \lim_{r\to(1,\ldots,1)} \gamma(T_r f) = \gamma(f). \tag{4.15}$$

As a corollary of Theorem 4.4 the following theorem provides a representation

of B* in terms of the Hadamard product.

THEOREM 4.5. Under the hypotheses of Theorem 4.4 every linear functional $\gamma \varepsilon B*$

is representable in the form

$$\gamma(f) = \lim_{r\to(1,\ldots,1)} H(f,F;r) =$$

$$\lim_{r\to(1,\ldots,1)} (\frac{1}{2\pi i})^n \int_{T_{\varepsilon_1} \times \ldots \times T_{\varepsilon_n}} f(\xi_1,\ldots,\xi_n) F(\frac{r_1}{\xi_1},\ldots,\frac{r_n}{\xi_n}) \frac{d\xi_1}{\xi_1} \cdots \frac{d\xi_n}{\xi_n}, \tag{4.16}$$

where, $0 < \varepsilon_i < 1$, $0 < r_i < \xi_i < 1$, $i=1,\ldots,n$ and $F \varepsilon B^0$. F uniquely determines and

is uniquely determined by γ and $||\gamma|| = N(F)$.

Equation (4.16) can be expressed in terms of real integrals using (1.21).

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