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THIN CIRCULAR PLATE UNIFORMLY LOADED OVER A CONCENTRIC ELLIPTIC PATH AND SUPPORTED ON COLUMNS

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ABSTRACT. Within the limitations of the classical thin plate theory expressions are obtained for the small deflections of a thin isotropic circular plate uniformly loaded over a concentric ellipse and supported by four columns at the vertices of a rectangle whose sides are parallel to the axes of the ellipse. Formulae are given for the moments and shears at the centre of the plate and on the edge. Limiting cases are investigated.

KEY WORDS AND PHRASES. Deflections of circular plates, four point supports, free boundary, uniform loading on a concentric ellipse
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1. INTRODUCTION.

The technological importance of thin elastic plates is sufficiently well established to require no elaboration. Thin slabs of material are structures which are widely used in engineering work and their transverse flexure has been extensively studied by many authors both theoretically and experimentally when the boundary of the slab is clamped, simply supported or free. Support of circular discs at a discrete number of points is of interest to the designer of reflecting surfaces and receivers, particularly when the surfaces are parts of astronomical and aeronautical structures. The deflection surface of a thin circular plate subjected to a symmetrical loading and supported by equally spaced point columns along the periphery of the plate has been considered by Nadai [1] whose results are quoted by Timoshenko and Woinowsky-Krieger [2]. When the boundary of the circular plate is free and the plate is supported at interior points and acted upon by two types of normal loadings distributed over an eccentric circular patch the solutions have been obtained by the author [3,4], using the complex variable approach of Muskhelishvili, references to previous work are given at the ends of these papers. Thin circular plates on multipoint supports have also been discussed by Yu and Pan [5], Leissa and Wells [6], Kirstein, Pell, Woolley and Davis [7], Kirstein and Wooley [8,9], Vaughn [10], Chantaramungkorn, Karasudhi and Lee [11] and Williams and Brinson [12]. There is good agreement between the theoretical results of [3] and the experimental results from the tests reported in [7]. Most of the results in [5-10,12] are special cases of [3]. In a series of papers [13-15] complex variable methods were applied to study the bending of an elastically restrained circular plate subject to uniform, linearly varying and parabolic loadings over a concentric ellipse. Frishbeir and Lucht [16] used the //L// method of complex potentials to derive the solution for a clamped circular plate which is transversely and uniformly loaded over the area of a polygon. In this paper, expressions are obtained for the deflection at any point of a thin circular plate which is uniformly loaded over a concentric elliptic patch and supported by four equal concentrated forces located at the corners of a rectangle whose sides are parallel to the axes of the ellipse. Formulae are given for the boundary and central values of the moments and shears. The limiting cases in which the radius of the plate $\to \infty$, the eccentricity of the ellipse $\to 0$ or its minor axis $\to 0$ are investigated.

2. BASIC EQUATIONS AND BOUNDARY CONDITIONS.

Let C denote the boundary of a thin circular plate of centre 0 and radius c. If $2h$ is the constant thickness of the plate, then its flexural rigidity D is given by [2]

$$D = 2Eh^3/3(1-\nu^2) , \tag{2.1}$$

where E is the modulus of elasticity and ν is Poisson's ratio for the material of the plate. The mid-plane of the plate is chosen as the plane $Z = 0$ of a rectangular Cartesian frame $0(x,y,Z)$ and the notation used is that of [2]. According to the classical small bending theory of thin plates the deflection w of the mid-plane in the downward direction OZ at any point $z = x + iy = re^{i\theta}$ satisfies the biharmonic equation

$$D\nabla^4 w = p(z,\bar{z}), \tag{2.2}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} , \tag{2.3}$$

and $p(z,\bar{z})$ is the normal load intensity at the point z. The general solution of (2.2) may be written as

$$w = \bar{z}\Omega(z) + z\bar{\Omega}(\bar{z}) + \omega(z) + \bar{\omega}(\bar{z}) + W(z,\bar{z}), \tag{2.4}$$

where $\Omega(z)$, $\omega(z)$ are functions of z which are regular in the region occupied by the plate and W is a particular integral of (2.2). The moments and shears at any point (r,θ) of the mid-plane of the plate are given by [2]

$$M_r = -D(d^2 + \nu r^{-1}d + \nu r^{-2}d'^2)w = -D[\nu\nabla^2 + (1-\nu)d^2]w, \tag{2.5a}$$

$$M_\theta = -D(\nu d^2 + r^{-1}d + r^{-2}d'^2)w = -D[\nabla^2 + (\nu-1)d^2]w, \tag{2.5b}$$

$$M_{r\theta} = (1-\nu)Dr^{-1}(d-r^{-1})d'w , \tag{2.5c}$$

$$Q_r = -Dd(\nabla^2 w), \quad Q_\theta = -Dr^{-1}d'(\nabla^2 w), \tag{2.6}$$

where $d = \partial/\partial r$, $d' = \partial/\partial\theta$. In terms of the complex potentials $\Omega(z)$, $\omega(z)$ and particular integral $W(z,\bar{z})$ we have [3,p.730]

$$M_r + M_\theta = -4(1+\nu)D[2\mathrm{Re}\ \Omega' + \partial^2 W/\partial z\partial\bar{z}], \tag{2.7a}$$

$$M_r - M_\theta + 2iM_{r\theta} = -4(1-\nu)D[z\Omega'' + z^2(\omega'' + W'')/r^2], \tag{2.7b}$$

$$Q_r - iQ_\theta = -8Dz[\Omega'' + \partial^3 W/\partial z^2\partial\bar{z}]/r , \tag{2.8}$$

where accents denote differentiation with respect to z .
The conditions for the circular edge C to be free are [2]

$$(M_r)_{r=c} = 0,\ (V_r)_{r=c} = \left(Q_r - \frac{1}{r}\frac{\partial M_{r\theta}}{\partial\theta}\right)_{r=c} = 0 . \tag{2.9}$$

Substitution from (2.5a,c) and (2.6) in (2.9) leads to

$$[f_r(d,d')w]_{r=c} = 0,\ [F_r(d,d')w]_{r=c} = 0 , \tag{2.10}$$

where

$$f_r(d,d') = d^2 + \nu r^{-1}d + \nu r^{-2}d'^2 , \tag{2.11a}$$

$$F_r(d,d') = d^3 + r^{-1}d^2 - r^{-2}d + (2-\nu)r^{-1}dd'^2 + (\nu-3)r^{-3}d'^2. \tag{2.11b}$$

From (2.5a) and the first equation of (2.9) we see that

$$\nu(\nabla^2 w)_{r=c} = (\nu-1)(d^2 w)_{r=c} ,\ \nu[d'(\nabla^2 w)]_{r=c} = (\nu-1)[d'd^2 w]_{r=c}$$

Equation (2.5b) and the second equation of (2.6) then give

$$(M_\theta)_{r=c} = \frac{1-\nu^2}{\nu} D[d^2 w]_{r=c} ,\ (Q_\theta)_{r=c} = \frac{1-\nu}{\nu} . \frac{D}{c} [d'd^2 w]_{r=c} , \tag{2.12}$$

from which we deduce that

$$(1 + \nu)\ c\ (Q_\theta)_{r=c} = (d'M_\theta)_{r=c} . \tag{2.13}$$

This relation and the second equation of (2.9) serve to determine the periphery values of the shears in terms of the moment values.

3. STATEMENT OF THE PROBLEM.

The problem to be solved consists of determining the deflection surface of a thin circular plate of centre 0 and radius c subjected to the following conditions:

(1) The boundary C of the plate is free.

(2) The total normal load $L = \pi p_0 ab$ (3.14a)

is uniformly distributed over the area of the ellipse

$$x^2/a^2 + y^2/b^2 = 1 (0 \le b \le a \le c) . \tag{3.14b}$$

(3) The plate is supported by four equal concentrated forces, each of magnitude $L/4$, and located at the points $P_\lambda(z_\lambda = se^{i\gamma_\lambda}, \lambda = 1,2,3,4)$, where $0 \le s \le c$, $\gamma_1 = \gamma$, $\gamma_2 = \pi - \gamma$, $\gamma_3 = \gamma - \pi$, $\gamma_4 = -\gamma$, $0 \le \gamma \le \pi/2$. The four points of support lie in the loaded or unloaded region according as $s^2\cos^2\gamma/a^2 + s^2\sin^2\gamma/b^2$ is less or greater than 1. For $\gamma = 0$ we have two supports at $(\pm s, 0)$ while for $\gamma = \pi/2$ we have two supports at $(0, \pm s)$. See Figure 1. Symmetry with respect to both axes show that it is sufficient to find the deflection w at any point z in the positive quadrant. Deflections, moments and shears at the four points $\pm z$, $\pm\bar{z}$ are the same.

4. METHOD AND SOLUTION.

Let Γ denote the boundary of the ellipse (3.14b) and let the indices 1 and 2 refer to the loaded region inside Γ and the unloaded region between Γ and C, respectively. The particular integrals W_1 and W_2 of (2.2) corresponding to the uniform intensities of normal loading $p_1 = p_0$ and $p_2 = 0$ may be taken as

$$W_1(z,\bar{z}) = p_0 z^2\bar{z}^2/64D, \quad W_2(z,\bar{z}) = 0. \tag{4.15}$$

The continuity requirements for the deflections, slopes, moments and shears at any point on Γ lead to

$$[w]_1^2 = [\partial w/\partial z]_1^2 = [\partial^2 w/\partial z\partial\bar{z}]_1^2 = [\partial^3 w/\partial z^2\partial\bar{z}]_1^2 = 0 \tag{4.16}$$

along Γ. It was proved [13, p.105] that these transition conditions along Γ are satisfied by

$$[\Omega(z)]_1^2 = \frac{L}{96\pi D}\left[\frac{d^2z^3}{abf^2} - 2\left(2 + \frac{z^2}{f^2}\right)Z + 6z\ln\frac{z+Z}{a+b}\right], \tag{4.17}$$

$$[\omega(z)]_1^2 = -\frac{L}{96\pi D}\left[\left(\frac{2ab}{f^4} + \frac{1}{4ab}\right)z^4 - \frac{3}{2}ab\left(1 + \frac{4z^2}{f^2}\right) + d^2\left\{\frac{zZ}{f^2}\left(\frac{5}{2} - \frac{z^2}{f^2}\right) - \frac{3}{2}\ln\frac{z+Z}{a+b}\right\}\right], \tag{4.18}$$

where
$$Z = \sqrt{(z^2 - f^2)}, \quad f^2 = a^2 - b^2, \quad d^2 = a^2 + b^2. \tag{4.19}$$

Introducing (4.15), (4.17) and (4.18) in (2.4) we get, using (3.14a)

$$k[w]_1^2 = \mathrm{Re}\left[\left(r^2 + \tfrac{1}{4}d^2\right)\ln\frac{z+Z}{a+b} + \frac{1}{3}\left\{\frac{d^2 z}{2f^2}\left(\frac{z^2}{f^2} - \frac{5}{2}\right) - \left(2\bar{z} + \frac{r^2 z}{f^2}\right)\right\}Z - \frac{r^4}{8ab} + ab\left(\frac{1}{4} + \frac{z^2}{f^2}\right) + \frac{d^2r^2z^2}{6abf^2} - \frac{1}{3}\left(\frac{ab}{f^4} + \frac{1}{8ab}\right)z^4\right], \tag{4.20}$$

where
$$k = 8\pi D/L. \tag{4.21}$$

If ϕ is the eccentric angle of any point z on Γ then

$$z = a\cos\phi + ib\sin\phi, \quad Z = b\cos\phi + ia\sin\phi,$$

and it is checked that the expression between the square brackets in (4.20) vanishes along Γ as it should. It is to be noted that $Z = \sqrt{(z^2 - f^2)}$ is not uniform in region 1 while it is uniform in region 2. In fact, the two branches of Z interchange when z traces a closed path round any of the two foci $(\pm f, 0)$ of the ellipse. Thus, the terms containing Z in (4.20) should appear in w_2 and not in w_1. It is also known that the singular part of the deflection w at any point P near a downward concentrated force F is

$$w_{sin} = \frac{F}{8\pi D} R^2 \ln R \,, \tag{4.22}$$

where R is the distance between P and the point of application of the force. Guided by these remarks and using (4.20), we assume that

$$kw_1 = \frac{r^4}{8ab} - ab\left(\tfrac{1}{4} + \frac{r^2}{f^2}\cos 2\theta\right) - \frac{d^2 r^4}{6abf^2}\cos 2\theta + \frac{1}{3}\left(\frac{ab}{f^4} + \frac{1}{8ab}\right) r^4 \cos 4\theta$$

$$- S + \sum_0^{\infty} (A_n + C_n r^2) r^{2n} \cos 2n\theta \,, \tag{4.23}$$

$$kw_2 = \left(r^2 + \tfrac{1}{4} d^2\right) \ln \left|\frac{z+Z}{a+b}\right| + \frac{1}{3}\,\mathrm{Re}\left\{\frac{d^2 z}{2f^2}\left(\frac{z^2}{f^2} - \frac{5}{2}\right) - \left(2\bar{z} + \frac{r^2 z}{f^2}\right)\right\} Z$$

$$- S + \sum_0^{\infty} (A_n + C_n r^2) r^{2n} \cos 2n\theta \,, \tag{4.24}$$

where

$$S = \tfrac{1}{4} \sum_{\lambda=1}^{4} R_\lambda^2 \ln R_\lambda \,, \tag{4.25}$$

$$\left.\begin{aligned} R_1^2 &= r^2 + s^2 - 2sr\cos(\theta-\gamma), \quad R_2^2 = r^2 + s^2 + 2sr\cos(\theta+\gamma)\,, \\ R_3^2 &= r^2 + s^2 + 2sr\cos(\theta-\gamma), \quad R_4^2 = r^2 + s^2 - 2sr\cos(\theta+\gamma)\,, \end{aligned}\right\} \tag{4.26}$$

and $A_n, C_n (n = 0,1,2,\ldots)$ are real constants to be determined. It is now easily seen that the expressions (4.23) and (4.24) for w_1 and w_2 satisfy the biharmonic equation (2.2) corresponding to the load intensities $p_1 = p_0$ and $p_2 = 0$, satisfy the required transition conditions along Γ and exhibit the appropriate singular behaviour at the four points of support. The unkown constants $A_n (n = 1,2,\ldots)$ and $C_n (n = 0,1,2,\ldots)$ will now be determined from the boundary conditions (2.10). The condition of zero deflection at any point of support serves to find A_0. To achieve this goal, all the terms in (4.24) will be explicitly expressed in terms of biharmonic functions of r and θ of the types

$$r^{\pm n} \begin{matrix}\cos\\ \sin\end{matrix}\, n\theta \quad \text{and} \quad r^{2\pm n} \begin{matrix}\cos\\ \sin\end{matrix}\, n\theta \,.$$

Assuming that $\psi(z,f) = \ln\left(z + \sqrt{(z^2 - f^2)}\right)$ we have

$$\psi'(z,f) = (z^2-f^2)^{-\frac{1}{2}} = \frac{1}{z}\left(1 - \frac{f^2}{z^2}\right)^{-\frac{1}{2}} = \frac{1}{z}\sum_0^{\infty} \binom{-\frac{1}{2}}{n} \left(-\frac{f^2}{z^2}\right)^n \left(\left|\frac{z}{f}\right| > 1\right), \tag{4.27a}$$

where, with the usual notation

$$\binom{-\frac{1}{2}}{n} = (-1)^n\, 2^{-2n} \binom{2n}{n} \,. \tag{4.27b}$$

Integrating (4.27a), using (4.27b) and noting that $\psi(z,0) = \ln(2z)$ we get

$$\psi(z,f) = \ln(2z) - \tfrac{1}{2} \sum_1^{\infty} \frac{1}{n}\binom{2n}{n}\left(\frac{f}{2z}\right)^{2n} \quad (|z| \geq f) \,.$$

Similarly, we obtain

$$\psi(z,f) = \ln(if) - 2i \sum_0^{\infty} \frac{1}{2n+1}\binom{2n}{n}\left(\frac{z}{2f}\right)^{2n+1} \quad (|z| \leq f) \,.$$

Thus we have

$$\ln\frac{|z+Z|}{|a+b|} = \ln\frac{2r}{a+b} - \frac{1}{2}\sum_{1}^{\infty}\frac{1}{n}\binom{2n}{n}\left(\frac{f}{2r}\right)^{2n}\cos 2n\theta \quad (r \geq f), \tag{4.28a}$$

$$\ln\frac{|z+Z|}{|a+b|} = \ln\frac{f}{a+b} + 2\sum_{0}^{\infty}\frac{1}{2n+1}\binom{2n}{n}\left(\frac{r}{2f}\right)^{2n+1}\sin(2n+1)\theta\,(r \leq f). \tag{4.28b}$$

Expanding $(1-f^2/z^2)^{\frac{1}{2}}$ by the binomial theorem we find for $|z| < f$:

$$\frac{1}{3}\operatorname{Re}\left\{\frac{d^2z}{2f^2}\left(\frac{z^2}{f^2}-\frac{3}{2}\right)-\left(2\bar{z}+\frac{r^2z}{f^2}\right)\right\}Z = \frac{3}{16}d^2 - \frac{1}{2}r^2 - \frac{r^2}{f^2}\left(\frac{1}{3}r^2+\frac{1}{2}d^2\right)\cos 2\theta$$

$$+\frac{d^2r^4}{6f^4}\cos 4\theta + \frac{1}{2}\sum_{1}^{\infty}\frac{(f/2r)^{2n}}{n+1}\binom{2n}{n}\left\{\frac{2n+1}{2n-1}r^2+\frac{n+3}{4(n+2)}d^2\right\}\cos 2n\theta \quad (r > f). \tag{4.29a}$$

Similarly, writing $Z = if\,(1-z^2/f^2)^{\frac{1}{2}}$ and expanding, we obtain the following expression for the left side of (4.29a) when $|z| < f$:

$$\frac{r}{3f}\left(\frac{5}{4}d^2-2f^2\right)\sin\theta - \sum_{1}^{\infty}\frac{(r/2f)^{2n+1}}{2n-1}\binom{2n}{n}\left\{\frac{2nr^2}{n+1}+\frac{2n-5}{2(2n-3)}d^2\right\}\sin(2n+1)\theta \quad (r < f). \tag{4.29b}$$

Substitution from (4.28a,b) and (4.29a,b) in (4.24) gives

$$kw_2 = \frac{3d^2}{16} - \frac{1}{2}r^2 + \left(r^2+\frac{1}{4}d^2\right)\ln\frac{2r}{a+b} - \frac{r^2}{f^2}\left(\frac{1}{3}r^2+\frac{1}{2}d^2\right)\cos 2\theta + \frac{d^2r^4}{6f^4}\cos 4\theta$$

$$- S + \sum_{1}^{\infty}\frac{(f/2r)^{2n}\cos 2n\theta}{4n(n+1)}\binom{2n}{n}\left(\frac{2r^2}{2n-1}-\frac{d^2}{n+2}\right) + \sum_{0}^{\infty}(A_n + C_nr^2)r^{2n}\cos 2n\theta \quad (r \geq f), \tag{4.30a}$$

$$kw_2 = \frac{r}{f}\left(r^2+\frac{4}{3}b^2\right)\sin\theta + \left(r^2+\frac{1}{4}d^2\right)\ln\frac{f}{a+b} - S$$

$$+ 2\sum_{1}^{\infty}\frac{(r/2f)^{2n+1}}{4n^2-1}\binom{2n}{n}\left(\frac{2d^2}{2n-3}-\frac{r^2}{n+1}\right)\sin(2n+1)\theta + \sum_{0}^{\infty}(A_n+C_nr^2)r^{2n}\cos 2n\theta \quad (r \leq f). \tag{4.30b}$$

It can be easily shown that for $r \geq s$

$$R_\lambda^2\ln R_\lambda = (r^2+s^2)\ln r + s^2 - sr\left(1+2\ln r+\frac{s^2}{2r^2}\right)\cos(\theta-\gamma_\lambda)$$

$$+\sum_{2}^{\infty}\frac{1}{n}\left(\frac{r^2}{n-1}-\frac{s^2}{n+1}\right)\left(\frac{s}{r}\right)^n\cos n(\theta-\gamma_\lambda) \tag{4.31a}$$

and

$$S = (r^2+s^2)\ln r + s^2 + \sum_{2}^{\infty}\frac{1}{2n}\left(\frac{r^2}{2n-1}-\frac{s^2}{2n+1}\right)\left(\frac{s}{r}\right)^{2n}\cos 2n\gamma\cos 2n\theta \quad (r \geq s). \tag{4.31b}$$

For $r \leq s$ we interchange r and s in (4.31a,b) .

Introducing (4.31b) in (4.30a) yields

$$kw_2 = \sum_{0}^{\infty} L_n(r)\cos 2n\theta \quad , \tag{4.32}$$

where $r \geq$ the greater of f and s ,

$$L_0(r) = A_0' + B_0'\ln r + C_0'r^2,\quad L_n(r) = A_n'r^{2n} + B_n'r^{-2n} + C_n'r^{2+2n} + D_n'r^{2-2n}\,(n \geq 1), \tag{4.33}$$

$$A_0' = A_0 - s^2 + \frac{d^2}{4}\left(\frac{3}{4}+\ln\frac{2}{a+b}\right),\quad B_0' = \frac{1}{4}d^2 - s^2,\quad C_0' = C_0 - \frac{1}{2} + \ln\frac{2}{a+b}\,, \tag{4.34}$$

$$A_1' = A_1 - \frac{d^2}{2f^2}\ ,\quad C_1' = C_1 - \frac{1}{3f^2}\ ,\quad A_2' = A_2 + \frac{d^2}{6f4}\ , \tag{4.35}$$

$$A_n' = A_n (n \geq 3),\quad B_n' = \frac{s^{2n+2}\cos 2n\gamma}{2n(2n+1)} - \frac{d^2(f/2)^{2n}}{4n(n+1)(n+2)}\binom{2n}{n}\ (n \geq 1), \tag{4.36}$$

$$C_n' = C_n (n \geq 2),\quad D_n' = -\frac{s^{2n}\cos 2n\gamma}{2n(2n-1)} + \frac{(f/2)^{2n}}{2n(n+1)(2n-1)}\binom{2n}{n}\ (n \geq 1). \tag{4.37}$$

Applying the differential operators (2.11) and (2.12) to the functions in (4.32) gives

$$f_r(d,d')\{L_0(r)\} = \sigma\left[(\kappa-1)C_0' - r^{-2}B_0'\right],\quad F_r(d,d')\{L_0(r)\} = 0\ , \tag{4.38}$$

$$f_r(d,d')\{L_n(r)\cos 2n\theta\} = 2n\sigma\left[(2n-1)A_n' r^{2n-2} + (2n+1)B_n' r^{-2n-2}\right.$$
$$\left. + \left(1 + \frac{1}{2n}\right)(2n+\kappa-1)\ C_n' r^{2n} + \left(1 - \frac{1}{2n}\right)(2n-\kappa+1)\ D_n' r^{-2n}\right]\cos 2n\theta\ , \tag{4.39}$$

$$rF_r(d,d')\{L_n(r)\cos 2n\theta\} = -4n^2\sigma\left[(2n-1)\ A_n' r^{2n-2} - (2n+1)\ B_n' r^{-2n-2}\right.$$
$$\left. + \left(1 + \frac{1}{2n}\right)(2n-\kappa-1)\ C_n' r^{2n} - \left(1 - \frac{1}{2n}\right)(2n+\kappa+1)\ D_n' r^{-2n}\right]\cos 2n\theta\ , \tag{4.40}$$

where

$$\sigma = 1 - \nu,\quad \kappa = (3 + \nu)/(1 - \nu)\ . \tag{4.41}$$

Inserting (4.32) in the boundary conditions (2.10), using (4.33)-(4.40), equating the coefficients of $\cos 2n\theta$ $(n = 0,1,2,\ldots)$ in the resulting identities to zero and solving the obtained systems of linear equations we find

$$C_0' = \frac{1}{8}\beta(v^2 - 4u^2),\quad C_0 = \frac{1}{2} + \frac{1}{8}\beta(v^2 - 4u^2) + \ln\frac{a+b}{2}\ , \tag{4.42a}$$

$$A_n' = \frac{c^{2-2n}}{2n\kappa}\left[\left\{u^2 - \frac{4n^2+\kappa^2-1}{2n(2n-1)}\right\}u^{2n}\cos 2n\gamma + \frac{(f/2c)^{2n}}{2n+2}\binom{2n}{n}\left\{\frac{4n^2+\kappa^2-1}{n(2n-1)} - \frac{2n+1}{n+2}v^2\right\}\right]\ (n \geq 1), \tag{4.42b}$$

$$C_n' = \frac{c^{-2n}}{2n\kappa}\left[\left(1 - \frac{2nu^2}{2n+1}\right)u^{2n}\cos 2n\gamma + \frac{(f/2c)^{2n}}{n+1}\binom{2n}{n}\left(\frac{nv^2}{n+2} - 1\right)\right]\ (n \geq 1), \tag{4.42c}$$

where

$$\beta = (1-\nu)/(1+\nu),\quad u = s/c\ ,\quad v = d/c\ . \tag{4.43}$$

When the values of A_1, A_2 and C_1 are replaced by their values in terms of A_1', A_2' and C_1' by means of (4.35) it is found that (4.23) and (4.30a) take the forms

$$kw_1 = \frac{a-b}{2(a+b)}\left(1 - \frac{r^2}{3ab}\right)r^2\cos 2\theta + \frac{r^4}{8ab}\left\{1 + \frac{1}{3}\left(\frac{a-b}{a+b}\right)^2\cos 4\theta\right\} - \frac{1}{4}ab - S$$
$$+ A_0 + C_0 r^2 + \sum_1^\infty (A_n' + C_n' r^2)\ r^{2n}\cos 2n\theta\ , \tag{4.44}$$

$$kw_2 = \frac{3}{16}d^2 - \frac{1}{2}r^2 + \left(r^2 + \frac{1}{4}d^2\right)\ln\frac{2r}{a+b} + \frac{1}{4}\sum_1^\infty\binom{2n}{n}\frac{(f/2r)^{2n}}{n(n+1)}\left(\frac{2r^2}{2n-1} - \frac{d^2}{n+2}\right)\cos 2n\theta - S$$
$$+ A_0 + C_0 r^2 + \sum_1^\infty (A_n' + C_n' r^2)\ r^{2n}\cos 2n\theta\ (r \geq f), \tag{4.45}$$

where C_0 is given by (4.42a) and A_n', C_n' are given by (4.42b,c). For points of region 2 at which $r \leq f$ the deflection w_2 is furnished by (4.30b). It is easily seen that such points exist only if $f \geq b$, i.e., if the eccentricity of the ellipse $\geq \sqrt{2}/2$. In any case, w_2 is given by (4.24). The constant A_0 can always be determined from the condition that the deflection vanishes at any of the four points of support. If all these points lie in the loaded region then (4.44) gives

$$A_0 = \frac{1}{4}ab + s^2\Big[\sin^2\gamma \ln \sin\gamma + \cos^2\gamma \ln\cos\gamma - \frac{1}{2} + \ln\frac{8s^2}{a+b} + \frac{1}{8}\beta(4u^2 - v^2) + \frac{a-b}{2(a+b)}\left(\frac{s^2}{3ab} - 1\right)\cos 2\gamma - \frac{s^2}{8ab}\left\{1 + \frac{1}{3}\left(\frac{a-b}{a+b}\right)^2\cos 4\gamma\right\}\Big] - \sum_1^\infty (A'_n + C'_n s^2)s^{2n}\cos 2n\gamma. \tag{4.46}$$

For a single support at the centre we have $s = 0$, $u = 0$, $A_0 = \frac{1}{4}ab$ and $S = r^2\ln r$. If all the support points lie in the unloaded region, then either (4.45) or (4.30b) can be used to determine A_0 according as $s \geq f$ or $s \leq f$. If $s \geq f$ we have

$$A_0 = s^2\Big[\sin^2\gamma \ln(4s\sin\gamma) + \cos^2\gamma\ln(4s\cos\gamma) + \frac{1}{8}\beta(4u^2 - v^2)\Big] - \frac{d^2}{4}\left(\frac{3}{4} + \ln\frac{2s}{a+b}\right) + \sum_1^\infty\left\{\frac{(f/2s)^{2n}}{4n(n+1)}\binom{2n}{n}\left(\frac{d^2}{n+2} - \frac{2s^2}{2n-1}\right) - (A'_n + C'_n s^2)s^{2n}\right\}\cos 2n\gamma, \tag{4.47a}$$

but if $s \leq f$ then (4.30b) gives

$$A_0 = s^2\Big[\sin^2\gamma\ln\sin\gamma + \cos^2\gamma\ln\cos\gamma - \frac{1}{2} + \ln\frac{8s^2}{f} + \frac{1}{8}\beta(4u^2 - v^2)\Big] - \frac{d^2}{4}\ln\frac{f}{a+b} - \frac{s}{f}\left(s^2 + \frac{4}{3}b^2\right)\sin\gamma + 2\sum_1^\infty\frac{(s/2f)^{2n+1}}{4n^2-1}\binom{2n}{n}\left(\frac{s^2}{n+1} - \frac{2d^2}{2n-3}\right)\sin(2n+1)\gamma - \sum_1^\infty (A_n + C_n s^2)s^{2n}\cos 2n\gamma\,. \tag{4.47b}$$

In any case, the deflection at the centre of the plate is

$$w_0 = \frac{L}{8\pi D}\left(A_0 - \frac{1}{4}ab - s^2\ln s\right), \tag{4.48}$$

where the appropriate value of A_0 is taken.

5. BOUNDARY AND CENTRAL VALUES OF MOMENTS AND SHEARS.

It can be easily shown that the deflections (4.23) and (4.24) may be written in the forms

$$w_1 = 2\mathrm{Re}\,[\bar{z}\Omega_1(z) + \omega_1(z)] + W_1(z,\bar{z}), \tag{5.49}$$

$$w_2 = 2\mathrm{Re}\,[\bar{z}\Omega_2(z) + \omega_2(z)], \tag{5.50}$$

where

$$2k\Omega_1(z) = -\frac{d^2z^3}{6abf^2} - \frac{1}{4}\sum_1^4 Z_\lambda\ln Z_\lambda + \sum_0^\infty C_n z^{2n+1}, \tag{5.51a}$$

$$2k\omega_1(z) = \frac{1}{3}\left(\frac{ab}{f^4} + \frac{1}{8ab}\right)z^4 - ab\left(\frac{1}{4} + \frac{z^2}{f^2}\right) + \frac{1}{4}\sum_1^4 \bar{z}_\lambda Z_\lambda\ln Z_\lambda + \sum_0^\infty A_n z^{2n}, \tag{5.51b}$$

$$2k\Omega_2(z) = z\ln\frac{z+Z}{a+b} - \frac{1}{3}\left(2 + \frac{z^2}{f^2}\right)Z - \frac{1}{4}\sum_1^4 Z_\lambda\ln Z_\lambda + \sum_0^\infty C_n z^{2n+1}, \tag{5.52a}$$

$$2k\omega_2(z) = \frac{d^2}{4}\left\{\frac{2zZ}{3f^2}\left(\frac{z^2}{f^2} - \frac{5}{2}\right) + \ln\frac{z+Z}{a+b}\right\} + \frac{1}{4}\sum_1^4 \bar{z}_\lambda Z_\lambda\ln Z_\lambda + \sum_0^\infty A_n z^{2n}, \tag{5.52b}$$

$Z_\lambda = z - z_\lambda$ and the real constants A_n, C_n have been determined in the previous section. The moments and shears at any point of the plate can be obtained either by substitution from (4.15), (5.51a,b) and (5.52a,b) in (2.7a,b) and (2.8) or by introducing (4.44), (4.45), (4.30b) in (2.5a,b,c) and (2.6), noting that S is defined by (4.25), (4.26) and its expansion is (4.31b) if $r \geq s$ and interchanging r,s in (4.31b) if $r \leq s$. After extensive algebraic manipulation, it is found that $(M_r)_{r=c}=0$ as expected and

$$(M_\theta)_{r=c} = -\frac{(1+\nu)L}{4\pi\kappa}\left[\beta\kappa\left(\frac{1}{4}v^2-u^2\right)+\sum_1^\infty\left(2-2u^2+\frac{\kappa+1}{n}\right)u^{2n}\cos 2n\gamma\cos 2n\theta\right.$$
$$\left.+\sum_1^\infty\binom{2n}{n}\frac{(f/2c)^{2n}}{n+1}\left(\frac{2n+1}{n+2}v^2-\frac{2n+\kappa+1}{n}\right)\cos 2n\theta\right], \tag{5.53a}$$

$$(M_{r\theta})_{r=c} = \frac{L}{4\pi\kappa}\left[\sum_1^\infty\left(2u^2-2+\frac{\kappa-1}{n}\right)u^{2n}\cos 2n\gamma\sin 2n\theta\right.$$
$$\left.+\sum_1^\infty\binom{2n}{n}\frac{(f/2c)^{2n}}{n+1}\left(2-\frac{\kappa-1}{n}-\frac{2n+1}{n+2}v^2\right)\sin 2n\theta\right], \tag{5.53b}$$

$$(Q_r)_{r=c} = \frac{L}{2\pi\kappa c}\left[\sum_1^\infty(2nu^2-2n+\kappa-1)u^{2n}\cos 2n\gamma\cos 2n\theta\right.$$
$$\left.+\sum_1^\infty\binom{2n}{n}\frac{(f/2c)^{2n}}{n+1}\left\{2n-\kappa+1-\frac{n(2n+1)}{n+2}v^2\right\}\cos 2n\theta\right], \tag{5.54a}$$

$$(Q_\theta)_{r=c} = \frac{L}{2\pi\kappa c}\left[\sum_1^\infty(2n+\kappa+1-2nu^2)u^{2n}\cos 2n\gamma\sin 2n\theta\right.$$
$$\left.-\sum_1^\infty\binom{2n}{n}\frac{(f/2c)^{2n}}{n+1}\left\{2n+\kappa+1-\frac{n(2n+1)}{n+2}v^2\right\}\sin 2n\theta\right]. \tag{5.54b}$$

It is easily seen that (5.53b), (5.54a) satisfy the second boundary condition in (2.9) and (5.53a), (5.54b) satisfy (2.13) .

All the infinite series appearing in this section and in section 4 are convergent in the intervals mentioned and some of them will be summed in section 8.

The following formulae are obtained for the moments and shears at the centre:

$$\left.\begin{matrix}(M_r)_0\\(M_\theta)_0\end{matrix}\right\} = \frac{L}{8\pi}\left[(1+\nu)\left\{\beta\left(u^2-\frac{1}{4}v^2\right)+1+2\ln\frac{2s}{a+b}\right\}\pm(1-\nu)\left\{\cos 2\gamma\right.\right.$$
$$\left.\left.-\frac{a-b}{a+b}+\frac{u^2}{2\kappa}(\kappa^2+3-2u^2)\cos 2\gamma+\frac{f^2}{8\kappa c^2}(v^2-\kappa^2-3)\right\}\cos 2\theta\right], \tag{5.55a}$$

$$(M_{r\theta})_0 = \frac{(1-\nu)L}{8\Pi}\left[\cos 2\gamma-\frac{a-b}{a+b}+\frac{u^2}{2\kappa}(\kappa^2+3-2u^2)\cos 2\gamma+\frac{f^2}{8\kappa c^2}(v^2-\kappa^2-3)\right]\sin 2\theta. \tag{5.55b}$$

$$(Q_r)_0 = (Q_\theta)_0 = 0\ . \tag{5.56}$$

6. INFINITE PLATE UNIFORMLY LOADED OVER AN ELLIPTIC PATCH AND SUPPORTED ON COLUMNS.

Letting $c \to \infty$ in (4.44), (4.45) and (4.30b) leads to

$$kw_1 = A_0 - \frac{1}{4}ab - S + \left(\frac{1}{2}+\ln\frac{a+b}{2}\right)r^2 + \frac{a-b}{2(a+b)}\left(1-\frac{r^2}{3ab}\right)r^2\cos 2\theta$$
$$+\frac{r^4}{3ab}\left[1+\frac{1}{3}\left(\frac{a-b}{a+b}\right)^2\cos 4\theta\right], \tag{6.57}$$

$$kw_2 = A_0 + \frac{d^2}{4}\left(\frac{3}{4} + \ln\frac{2r}{a+b}\right) + r^2 \ln r - S$$
$$+ \sum_1^{\infty} \frac{(f/2r)^{2n}}{4n(n+1)} \binom{2n}{n} \left(\frac{2r^2}{2n-1} - \frac{d^2}{n+2}\right) \cos 2n\theta \quad (r \geq f), \tag{6.58a}$$

$$kw_2 = A_0 + \left(\frac{1}{2} + \ln\frac{f}{2}\right) r^2 + \frac{1}{4} d^2 \ln\frac{f}{a+b} + \frac{r}{f}\left(r^2 + \frac{4}{3} b^2\right) \sin\theta + \frac{r^2}{f^2}\left(\frac{1}{3} r^2 + \frac{1}{2} d^2\right) \cos 2\theta$$
$$- \frac{d^2 r^4}{6f^4} \cos 4\theta - S + 2\sum_1^{\infty} \frac{(r/2f)^{2n+1}}{4n^2-1} \binom{2n}{n} \left(\frac{2d^2}{2n-3} - \frac{r^2}{n+1}\right) \sin(2n+1)\theta \quad (r \leq f), \tag{6.58b}$$

where

$$A_0 = \frac{1}{4} ab + s^2 \left[\sin^2\gamma \ln\sin\gamma + \cos^2\gamma \ln\cos\gamma - \frac{1}{2} + \ln\frac{8s^2}{a+b}\right.$$
$$\left. + \frac{a-b}{2(a+b)}\left(\frac{s^2}{3ab} - 1\right) \cos 2\gamma - \frac{s^2}{8ab}\left\{1 + \frac{1}{3}\left(\frac{a-b}{a+b}\right)^2 \cos 4\gamma\right\}\right], \tag{6.59}$$

if the supports lie in the loaded region;

$$A_0 = s^2 \sin^2\gamma \ln(4s \sin\gamma) + s^2\cos^2\gamma \ln(4s\cos\gamma) - \frac{d^2}{4}\left(\frac{3}{4} + \ln\frac{2s}{a+b}\right)$$
$$+ \sum_1^{\infty} \frac{(f/2s)^{2n}}{4n(n+1)} \binom{2n}{n} \left(\frac{d^2}{n+2} - \frac{2s^2}{2n-1}\right) \cos 2n\gamma, \tag{6.60a}$$

if the supports lie in the unloaded region and $s \geq f$;

$$A_0 = s^2\left\{\sin^2\gamma \ln\sin\gamma + \cos^2\gamma \ln\cos\gamma + \ln\frac{3s^2}{f} - \frac{1}{2}\right\} + \frac{d^2}{4}\ln\frac{a+b}{f}$$
$$- \frac{s}{f}\left(s^2 + \frac{4}{3}b^2\right)\sin\gamma - \frac{s^2}{f^2}\left(\frac{s^2}{3} + \frac{d^2}{2}\right)\cos 2\gamma + \frac{d^2 s^4}{6f^4}\cos 4\gamma$$
$$+ 2\sum_1^{\infty} \frac{(f/2s)^{2n+1}}{4n^2-1} \binom{2n}{n} \left(\frac{s^2}{n+1} - \frac{2d^2}{2n-3}\right)\sin(2n+1)\gamma, \tag{6.60b}$$

if the supports lie in the unloaded region and $s \leq f$.

At the centre of the ellipse the moments (5.55a,b) reduce to

$$\begin{matrix}(M_r)_0 \\ (M_\theta)_0\end{matrix} = \frac{L}{8\pi}\left[(1+\nu)\left\{1 + 2\ln\frac{2s}{a+b}\right\} \pm (1-\nu)\left\{\cos 2\gamma - \frac{a-b}{a+b}\right\}\cos 2\theta\right], \tag{6.61a}$$

$$(M_{r\theta})_0 = \frac{(1-\nu)L}{8\pi}\left[\cos 2\gamma - \frac{a-b}{a+b}\right]\sin 2\theta. \tag{6.61b}$$

7. THIN CIRCULAR PLATE UNDER A VARIABLE LINE LOADING ALONG A DIAMETER AND SUPPORTED ON COLUMNS.

When the minor axis of the loaded elliptic patch $\to 0$ we have the case of a variable line loading extending along the x-axis from $x = -a$ to $x = a$. If $b \to 0$ and $p_0 \to \infty$ such that $2bp_0 \to p_1$ then the intensity of this line loading at a distance x from the centre equals $p_1 \surd(1-x^2/a^2)$. Deflections, moments and shears corresponding to this case can be dedcued from those for region 2 in sections 4, 5 by setting $b = 0$, $d = f = a$, $L = \frac{1}{2}\pi a p_1$ and noting that separate expressions are obtained for w at any point (r,θ) according as r is greater or less than a and the columns lie outside or inside the circle $r = a$.

8. THIN CIRCULAR PLATE UNIFORMLY LOADED OVER A CONCENTRIC CIRCULAR PATCH AND SUPPORTED ON COLUMNS.

Setting $b = a$, $f = 0$ in (4.44), (4.45) and (4.42a,b,c) we get

$$kw_1 = A_0 - \tfrac{1}{4}a^2 + \left\{\tfrac{1}{2} + \ln a + \tfrac{1}{4}\beta(t^2 - 2u^2)\right\} r^2 + \frac{r^4}{8a^2} - S$$

$$+ \sum_1^\infty (A'_n + C'_n r^2) r^{2n} \cos 2n\theta \quad , \tag{8.62}$$

$$kw_2 = A_0 + \frac{3}{8} a^2 + \left\{\ln a + \tfrac{1}{4}\beta(t^2 - 2u^2)\right\} r^2 + \left(r^2 + \tfrac{1}{2}a^2\right) \ln \frac{r}{a} - S$$

$$+ \sum_1^\infty (A'_n + C'_n r^2) r^{2n} \cos 2n\theta \quad , \tag{8.63}$$

where $t = a/c$ and

$$A'_n = \frac{c^{2-2n} u^{2n}}{2n\kappa} \left\{u^2 - \frac{4n^2+\kappa^2-1}{2n(2n-1)}\right\} \cos 2n\gamma, \quad C'_n = \frac{c^{-2n} u^{2n}}{2n\kappa} \left(1 - \frac{2nu^2}{2n+1}\right) \cos 2n\gamma \ , \tag{8.64}$$

$$A_0 = \tfrac{1}{4}a^2 + s^2 \left[\sin^2\gamma \ln \sin\gamma + \cos^2\gamma \ln \cos\gamma - \tfrac{1}{2} + \ln \frac{4s^2}{a} + \tfrac{1}{4}\beta(2u^2-t^2) - \frac{s^2}{8a^2}\right]$$

$$- \sum_1^\infty (A'_n + C'_n s^2) s^{2n} \cos 2n\gamma \, (s \le a) \ , \tag{8.65a}$$

$$A_0 = s^2\left[\sin^2\gamma \ln \sin\gamma + \cos^2\gamma \ln \cos\gamma + \ln(4s) + \tfrac{1}{4}\beta(2u^2-t^2)\right] - \frac{a^2}{2} \left(\frac{3}{4} + \ln \frac{s}{a}\right)$$

$$- \sum_1^\infty (A'_n + C'_n s^2) s^{2n} \cos 2n\gamma \, (s \ge a) \ . \tag{8.65b}$$

The infinite series in (8.62) and (8.63) equals

$$\frac{r^2+s^2+(\kappa^2-1)c^2}{2\kappa} \{\mathcal{J}_0(\xi, \phi_1) + \mathcal{J}_0(\xi,\phi_2)\} - \frac{r^2u^2}{2\kappa} \{\mathcal{J}_1(\xi,\phi_1) + \mathcal{J}_1(\xi,\phi_2)\}$$

$$- \tfrac{1}{2} \kappa c^2 \{\mathcal{J}_{-1}(\xi,\phi_1) + \mathcal{J}_{-1}(\xi,\phi_2)\} + \frac{(\kappa^2-1)c^2}{8\kappa} \{ \mathcal{J}(\xi,\phi_1) + \mathcal{J}(\xi,\phi_2)\} \quad , \tag{8.66}$$

where

$$\xi = rs/c^2 \ , \ \phi_1 = \theta - \gamma \ , \quad \phi_2 = \theta + \gamma \ ,$$

$$\mathcal{J}_0(\xi,\phi) = \sum_1^\infty \frac{\xi^{2n}\cos 2n\phi}{2n} = - \tfrac{1}{4} \ln (1 + \xi^4 - 2\xi^2\cos 2\phi) \ , \tag{8.67a}$$

$$\mathcal{J}_1(\xi,\phi) = \sum_1^\infty \frac{\xi^{2n}\cos 2n\phi}{2n+1} = \frac{1}{4\xi} \left[\cos \phi \ln \frac{1+\xi^2+2\xi\cos\phi}{1+\xi^2-2\xi\cos\phi} + 2 \sin\phi \tan^{-1} \frac{2\xi\sin\phi}{1-\xi^2}\right] - 1, \tag{8.67b}$$

$$\mathcal{J}_{-1}(\xi,\phi) = \sum_1^\infty \frac{\xi^{2n}\cos 2n\phi}{2n-1} = \frac{1}{4} \xi \left[\cos \phi \ln \frac{1+\xi^2+2\xi\cos\phi}{1+\xi^2-2\xi\cos\phi} - 2 \sin\phi \tan^{-1} \frac{2\xi\sin\phi}{1-\xi^2}\right] \ , \tag{8.67c}$$

$$\mathcal{J}(\xi,\phi) = \sum_1^\infty \frac{\xi^{2n}\cos 2n\phi}{n^2} = \mathrm{Re} \sum_1^\infty \frac{\zeta^{2n}}{n^2} = \mathrm{Re} \left\{ - \int_0^\zeta \frac{\ln(1-\lambda)}{\lambda} d\lambda\right\} \quad (\zeta = \xi e^{i\phi}). \tag{8.68}$$

The last function is the dilogarithm studied in the last three references of [7].

The deflection at the centre is given by

$$kw_0 = s^2 \left[\sin^2\gamma \ln \sin \gamma + \cos^2\gamma \ln \cos \gamma + \ln \frac{4s}{a} - \tfrac{1}{2} - \frac{s^2}{8a^2}\right.$$

$$\left. + \frac{1}{4} \beta(2u^2-t^2)\right] - \frac{c^2\delta}{4\kappa} \ (s \le a) \ , \tag{8.69a}$$

$$kw_0 = s^2 \quad \sin^2\gamma \ln \sin \gamma + \cos^2\gamma \ln \cos \gamma + \ln 4 + \tfrac{1}{4}\beta(2u^2-t^2)$$

where

$$- \frac{a^2}{2} \left(\frac{5}{4} + \ln \frac{s}{a}\right) - \frac{c^2\delta}{4\kappa} \quad (s \geq a) \qquad (8.69b)$$

$$\delta = (\kappa^2 + 2u^2 - 1)\{ 2\mathcal{J}_0(u^2,2\gamma) - \ln(1-u^4)\} - 2u^4 \left\{\mathcal{J}_1(u^2,2\gamma) + \frac{1}{2u^2} \ln \frac{1+u^2}{1-u^2} - 1\right\}$$

$$- 2\kappa^2 \left\{\mathcal{J}_{-1}(u^2,2\gamma) + \tfrac{1}{2} u^2 \ln \frac{1+u^2}{1-u^2}\right\} + \tfrac{1}{2} (\kappa^2-1) \{\mathcal{J}(u^2,0) + \mathcal{J}(u^2,2\gamma)\} . \qquad (8.70)$$

Setting $\gamma = \pi/4$ we obtain

$$kw_0 = s^2 \left[\tfrac{1}{2}(3 \ln 2-1) + \ln \frac{s}{a} - \frac{s^2}{8a^2} + \tfrac{1}{4}\beta(2u^2-t^2)\right] + \frac{c^2\delta'}{\kappa} \quad (s \leq a), \qquad (8.71a)$$

$$kw_0 = s^2 \left[\frac{3}{2} \ln 2 + \tfrac{1}{4}\beta(2u^2-t^2)\right] - \frac{a^2}{2} \left(\frac{5}{4} + \ln \frac{s}{a}\right) + \frac{c^2\delta'}{\kappa} \quad (s \geq a) , \qquad (8.71b)$$

where

$$\delta' = \sum_1^\infty \left[\frac{16n^2+\kappa^2-1}{16n^2(4n-1)} - \frac{u^2}{2n} + \frac{u^4}{4n+1} \right]$$

$$= \frac{1-\kappa^2}{16} \mathcal{J}(u^4,0) - u^4 + \tfrac{1}{4} \left\{\kappa^2 - 1 + (\kappa^2+3)u^2\right\} \ln (1+u^2)$$

$$+ \tfrac{1}{4}(\kappa^2 + 2u^2 - 1)\ln(1+u^4) + \tfrac{1}{2}(1-\kappa^2) \left\{u^2\tan^{-1}u^2 - \tfrac{1}{2}(1-u^2)\ln(1-u^2)\right\} . \qquad (8.72)$$

It is verified that (8.71a,b) agree with (3.14) and (3.6a) of [3], noting the difference in notation. There are misprints in equation (3.5a), p.738 of [3] and $\cos \alpha$ which appears twice in this equation must be replaced by $\cos s\alpha$.

Putting $b = a$, $f = 0$ in (5.53a,b), (5.54a,b) and summing the infinite series obtained we get the closed formulae

$$(M_\theta)_{r=c} = - \frac{(1+\nu)L}{4\pi\kappa} \left[u^2 - 1 + \beta\kappa(\tfrac{1}{2} t^2-u^2) - \tfrac{1}{4}(\kappa+1)\ln(I_1 I_2)\right.$$

$$\left. + \tfrac{1}{2}(1-u^2)(1-u^4)(I_1^{-1} + I_2^{-1})\right] . \qquad (8.73a)$$

$$(M_{r\theta})_{r=c} = \frac{L}{4\pi\kappa} \left[\frac{1}{\beta} \left(\tan^{-1} \frac{u^2\sin 2\varphi_1}{1-u^2\cos 2\varphi_1} + \tan^{-1} \frac{u^2\sin 2\varphi_2}{1-u^2\cos 2\varphi_2} \right)\right.$$

$$\left. - u^2(1-u^2) \left(\frac{\sin 2\varphi_1}{I_1} + \frac{\sin 2\varphi_2}{I_2} \right)\right] , \qquad (8.73b)$$

$$(Q_r)_{r=c} = \frac{L}{4\pi\kappa c} \left[1-\kappa+ (1-u^2)\left(1 + u^4 + \frac{1+u^2}{\beta}\right) (I_1^{-1} + I_2^{-1})\right.$$

$$\left. - (1-u^2)(1-u^4)^2(I_1^{-2} + I_2^{-2})\right] , \qquad (8.74a)$$

$$(Q_\theta)_{r=c} = \frac{L}{4\pi\kappa c} \left[(1+\kappa)u^2 \left(\frac{\sin 2\varphi_1}{I_1} + \frac{\sin 2\varphi_2}{I_2}\right)\right.$$

$$\left. + 2u^2(1-u^2)(1-u^4)\left(\frac{\sin 2\varphi_1}{I_1^2} + \frac{\sin 2\varphi_2}{I_2^2}\right)\right] , \qquad (8.74b)$$

where

$$I_j = 1 + u^4 - 2u^2 \cos 2\gamma_j \quad (j = 1,2) \ . \qquad (8.75)$$

It is easily verified that (8.73a), (8.74b) satisfy (2.13) and (8.73b),(8.74a) satisfy the second equation in (2.9) .

Formulae (5.55a,b) and (5.56) for the moments at the centre reduce to

$$\begin{matrix}(M_r)_0 \\ (M_\vartheta)_0\end{matrix} = \frac{L}{8\pi}\left[(1+\nu)\left\{1 + 2\ln\frac{s}{a} + \frac{1}{2}\beta(2u^2 - t^2)\right\}\right.$$

$$\left. \mp (1-\nu)\left\{1 + \frac{u^2}{2\kappa}(\kappa^2 + 3 - 2u^2)\right\}\cos 2\gamma \cos 2\theta\right] , \qquad (8.76a)$$

$$(M_{r\vartheta})_0 = \frac{(1-\nu)L}{8\pi}\left[1 + \frac{u^2}{2\kappa}(\kappa^2 + 3 - 2u^2)\right]\cos 2\gamma \sin 2\vartheta \ . \qquad (8.76b)$$

For $\gamma = \pi/4$ it is checked that the formulae (8.73)-(8.76) are in agreement with those obtained by applying equations (2.44)-(2.46) of [3], which were derived by a different method.

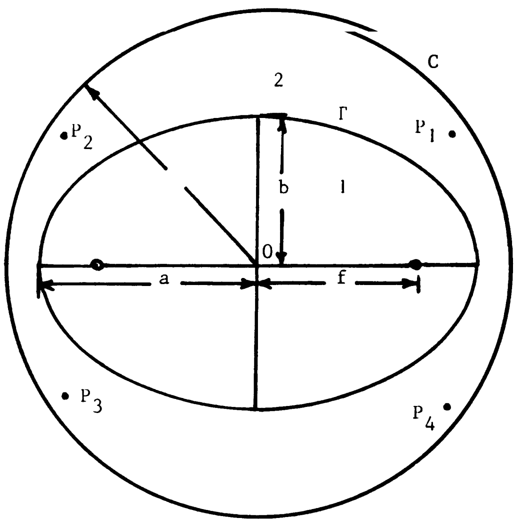

FIG.1

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