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ABOUT THE DEFECTS OF CURVES HOLOMORPHIC IN THE HALF PLANE

MOQBUL HOSSAIN

Department of Mathematics
Jahangirnagar University
Savar, Dhaka, Bangladesh

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ABSTRACT. A study is made on the defects of curves holomorphic in the half plane. Several results are proved.

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1. INTRODUCTION.

Let $f(z)$ be a meromorphic function in the half plane $\{\mathrm{Im}z \geqslant 0\}$ and $f(z)$ is not equal to a constant identically. We denote by $\bar{n}(r,f), r \geqslant 1$, the number of poles of $f(z)$ lying inside the set $\{ z:|z-\frac{ir}{2}| < \frac{r}{2}, |z| > 1\}$ $1 < r < \infty$. We put [1,2,3]

$$\bar{N}(r,f) = \int_1^r \frac{\bar{n}(t,f)}{t^2}\,dt = \sum_{\rho_\ell e^{i\psi_\ell} \in D(r,1)} \left(\frac{\sin\psi_\ell}{\rho_\ell} - \frac{1}{r}\right),$$

where $\rho_\ell e^{i\psi_\ell}$ are the poles of the function $f(z)$.

$$\bar{N}(r,\frac{1}{f-a}) = \bar{N}(r,a),$$

$$\bar{m}(r,f) = \frac{1}{2\pi}\int_{\arcsin\frac{1}{r}}^{\pi-\arcsin\frac{1}{r}} \ln^+|f(re^{i\theta}\sin\theta)|\,\frac{d\theta}{r\sin^2\theta},$$

$$\bar{m}(r,a,f) = \bar{m}(r,a),$$

$$\bar{T}(r,f) = \bar{m}(r,f) + \bar{N}(r,f).$$

When the half circle $\{|z| = 1, \mathrm{Im}z > 0\}$ does not contain neither zeros nor poles of the function $f(z) - a$, we get the following equality [1]:

$$\bar{N}(r,\frac{1}{f-a}) - \bar{N}(r,f) =$$

$$\frac{1}{2\pi}\int_{\arcsin\frac{1}{r}}^{\pi-\arcsin\frac{1}{r}} \ln|f(re^{i\theta}\sin\theta) - a|\,\frac{d\theta}{r\sin^2\theta} + \tilde{\theta}(r,1,f-a) \tag{1.1}$$

where $$\tilde{\theta}(r,1,f-a) = \frac{1}{2\pi}\int_{\arcsin\frac{1}{r}}^{\pi-\arcsin\frac{1}{r}} \{\ln|f(e^{i\theta}\sin\theta) - a|(-\frac{\sin\theta}{r^2})\}\,d\theta.$$

The order $\rho_T(f)$ of the funciton $f(z)$ in the sense of TSUJI is called the number

$$\rho_T(f) = \overline{\lim_{r\to\infty}} \frac{\ln \overline{T}(r,f)}{\ln r} .$$

Similarly the lower order $\lambda_T(f)$ of $f(z)$ is defined by the quantity

$$\lambda_T(f) = \underline{\lim_{r\to\infty}} \frac{\ln \overline{T}(r,f)}{\ln r} .$$

The quantity

$$\delta_T(a,f) = \delta_T(a) = \underline{\lim_{r\to\infty}} \frac{\overline{m}(r,a)}{\overline{T}(r,f)} = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r,a)}{\overline{T}(r,f)}$$

is called the deficiency of $f(z)$ at the point a. The value a of $f(z)$, for which $\delta_T(a) > 0$, is called a deficiency of $f(z)$. For a meromorphic function of of order ρ_T and lower order λ_T in the half z-plane $\{\mathrm{Im} z \geqslant 0\}$, we put $E_T(f) = \{a: \delta_T(a,f) > 0\}$. The set $E_T(f)$ is called the set of deficient values of $f(z)$ in the sense of TSUJI.

2. RESULTS ANALOGOUS TO TSUJI'S WORK.

We will derive analogue of TSUJI'S work for curves holomorphic in the half plane.

Let C^p be a p-dimensional complex unitary space and $\overline{a}$ - vectors from C^p. The vector

$$\vec{G}(z) = \{g_1(z), g_2(z), \ldots, g_p(z)\}$$

is called p-dimensional entire curve of the complex parameter z, where the function $\{g_n(z)\}_{n=1}^{p}$ are linearly independent entire functions without common zeroes. The scalar product:

$$(\vec{G}(z).\vec{a}) = \sum_{k=1}^{p} g_k(z)a_k$$

is called an ordinary entire function of the parameter z.

Definition: The finite ($\geqslant p$) or infinite system A of vectors is called admissible, if any p-vectors of the system are linearly independent.

For $n = 2$, we shall get ordinary meromorphic functions. In this case the characteristics of T^* and $\overline{T}$ coincide.

The quantity $n^*(r,\vec{a},\vec{G})$ will denote the number of zeroes of the holomorphic function $(\vec{G}(z).\vec{a})$ in the region $\{z: |z - \frac{ir}{2}| < \frac{r}{2}, |z| \geqslant 1\}$, where $\vec{a}\epsilon A$, the admissible system of vectors.

We put

$$N^*(r,\vec{a},\vec{G}) = \int_1^r \frac{n^*(t,\vec{a},\vec{G})}{t^2} dt , \quad 1 < r < \infty.$$

We define the function $m^*(r,\vec{a},\vec{G})$ and $T^*(r,\vec{G})$ by the quantities:

$$m^*(r,\vec{a},\vec{G}) = \frac{1}{2\pi} \int_{\arcsin\frac{1}{r}}^{\pi-\arcsin\frac{1}{r}} \ln^+ \frac{||\vec{G}(re^{i\theta}\sin\theta)||\,||\vec{a}||}{|(\vec{G}(re^{i\theta}\sin\theta)\vec{a})|} \cdot \frac{d\theta}{r\sin^2\theta} ,$$

$$T^*(r,\vec{G}) = \frac{1}{2\pi} \int_{\arcsin \frac{1}{r}}^{\pi - \arcsin \frac{1}{r}} \ln || \vec{G}(re^{i\theta}\sin\theta)|| \frac{d\theta}{r\sin^2\theta} ,$$

where

$$||\vec{G}(z)|| = \sqrt{\sum_{i=1}^{p} |g_i(z)|^2} .$$

For all admissable system $A \subset C^p$ of vectors we get the equality:

$$T^*(r,\vec{G}) = m^*(r,\vec{a},\vec{G}) + N^*(r,\vec{a},\vec{G}) + O(1),$$

when $r \to \infty$.

The order ρ_T and the lower order λ_T of the entire curve $\vec{G}(z)$ may be defined as before.

For a meromorphic function in the half plane the following assertions are true:

THEOREM A. [2] Let $0 < \rho < \infty$ and M be an arbitrary, not more than countable, set of points in the extended complex plane. Then there exists a meromorphic function in the half plane $\{\mathrm{Im}\, z \geq 0\}$ of order ρ_T , for which the set $E_T(f)$ of dificient values coincides with M.

THEOREM B. [2] Let $\{\eta_\nu\}$ be a sequence of positive number which satisfies the condition $\sum_{\nu=1}^{\infty} \eta_\nu = 1$ and let $\{a_k\}$ be an arbitrary sequence of different complex number. For any λ , $0 < \lambda < \infty$, there exists a meromorphic function in the half plane $\{\mathrm{Im}\, z \geq 0\}$ of lower order λ such that $(q = [\lambda] + 1)$

$$\delta_T(a_k,f) \geq K_1(\rho,\theta_1)\eta_k^3 , \text{ for } \lambda \leq 1 ,$$

$$\delta_T(a_k,f) \geq K_1(\rho,\theta_1)\eta_k^3 , \text{ for } \lambda > 1 ,$$

where $K_1(\rho,\theta_1) = \min \{ \frac{\rho^3}{36(\rho+1)}, \frac{\rho^3}{18(\rho+1)} \sin^{\rho-1}\theta_1 \}$.

THEOREM C. [2] For any λ , $0 < \lambda < \infty$, there exists a meromorphic function in the half plane $\{\mathrm{Im}\, z > 0\}$ of finite lower order λ such that the series $\Sigma\, \delta_T^\alpha(a_k,f)$ converges for each $\alpha < \frac{1}{3}$.

THEOREM D. [3] Let $\{w_\nu\}$ be an arbitrary sequence of complex number and $\{\delta_\nu\}_1^N$ be a sequence of positive number $(N \leq \infty)$ satisfying the relationship $\sum_{\nu=1}^{\infty} \delta_\nu \leq 1$. Then tnere exists a meromorphic function in the half plane $\{\mathrm{Im}\, z \geq 0\}$ of finite order λ , $0 < \lambda < \infty$, such that

$$\delta_T(w_\nu,f) = \delta_\nu,\ w_\nu \in \{w_\nu\} \text{ and } \delta_T(w,f) = 0, \text{ if } w \overline{\in} \{w_\nu\} .$$

MAIN RESULTS.

In this article we shall consider the following assertions:

We denote by A(W) the fixed admissible system A of vectors, if the coordinates of the vector $\vec{a}$ of the system depend on a complex parameter w . We shall denote the the vectors of the admissible system A(w) of vectors by $\vec{a}(w)$.

THEOREM 1. Let f(z) be an arbitrary meromorphic function in the half plane. It is possible to find a p-dimensional holomorphic curve $\vec{G}(z)$ $(p \geq 2)$ and an admissible system $A(w)$ of vectors such that $(r > 1)$

$$T^*(r,\vec{G}) = (p-1)\overline{T}(r,f) + O(1) ,$$

$$m^*(r,\vec{a}(w),\vec{G}) = (p-1)\overline{m}(r,w,f) + O(1) ,$$

$$\delta_T(\vec{a}(w),\vec{G}) = \delta_T(w,f), \ \Delta_T(\vec{a}(w),\vec{G}) = \Delta_T(w,f) .$$

COROLLARY 1. Let $0 < \rho_T < \infty$ and M be an arbitrary, not more than countable, set of vectors of the admissible system of vectors. Then there exists a p-dimensional $(p \geq 2)$ curve $\vec{G}(z)$ holomorphic in the half plane $\{\operatorname{Im} z \geq 0\}$ of order ρ_T, whose set of deficient values $E_T(f)$ coincides with M.

COROLLARY 2. For any λ, $0 < \lambda < \infty$ and $p > 2$, there exists a p-dimensional curve $\vec{G}_\lambda(z)$ holomorphic in the half plane $\{\operatorname{Im} z \geq 0\}$ of lower order λ and an admissible system $A(w)$ of vectors containing the sequence of distinct vectors :

$$\vec{a}_k = \vec{a}(w_k) \in A(w) \quad , \ k = 1,2, \ldots\ldots$$

such that $(q = [\lambda] + 1)$

$$\delta_T(\vec{a}_k,\vec{G}) \geq K_1(\rho,\theta_1)\eta_k^3 , \text{ for } \lambda \leq 1 ,$$

$$\delta_T(\vec{a}_k,\vec{G}) \geq K_1(\rho,\theta_1)\eta_k^3, \text{ for } \lambda > 1,$$

where

$$K_i(\rho,\theta_1) = \min \left\{ \frac{\rho^3}{36(\rho+1)} , \frac{\rho^3}{18(\rho+1)} \sin^{\rho-1}\theta_1 \right\} .$$

COROLLARY 3. For any λ, $0 < \lambda < \infty$, $p \geq 2$, there exists a p-dimensional curve $\vec{G}_\lambda(z)$ holomorphic in the half plane $\{\operatorname{Im} z > 0\}$ of finite lower order λ and an admissible system $A(w)$ of vectors such that the series $\sum_{\vec{a} \in A(w)} \delta_T^\alpha(\vec{a},\vec{G}_\lambda)$ diverges for $\alpha < \frac{1}{3}$.

COROLLARY 4. Let $\{w_\nu\}$ be an arbitrary sequence of complex number and $\{\delta_\nu\}_1^N$ be a sequence of positive number $(N \leq \infty)$ satisfying the condition $\sum_{\nu=1}^N \delta_\nu \leq 1$. Then for any λ, $0 < \lambda < \infty$, $p \geq 2$, there exists a p-dimensional curve $\vec{G}_\lambda(z)$ holomorphic in the half plane $\{\operatorname{Im} z \geq 0\}$ such that $\delta_T(\vec{a}(w_\nu),\vec{G}) = \delta_\nu$ and

$$\delta_T(\vec{a}(w_\nu),\vec{G}) = 0 , \text{ if } w \,\overline{\in}\, \{w_\nu\} .$$

PROOF OF THEOREM 1. Let

$$f(z) = \frac{g_1(z)}{g_2(z)} \tag{2.1}$$

be a meromorphic function in the half plane $\{\operatorname{Im} z \geq 0\}$, where $g_1(z)$ and $g_2(z)$ are entire functions having no zeroes in common.

We note that the entire functions:

$$h_k(z) = g_1^{p-k-1}(z)\, g_2^k(z) , \quad k = 0,1, \ldots, p-1 \tag{2.2}$$

are linearly independent in the field of complex number [5] .

We consider the curve

$$\vec{G}(z) = \{h_0(z), h_1(z), \ldots, h_{p-1}(z)\} \tag{2.3}$$

holomorphic in the half plane $\{\mathrm{Im}\ z \geqslant 0\}$ and the system A, $A = A(w) = \{\vec{a}(w)\}$ where for any complex $w \neq \infty$,

$$\vec{a}(w) = (1, \ldots, (-1)^k C_k^{p-1} \bar{w}^{k}, \ldots, (-1)^{p-1} \bar{w}^{p-1}) .$$

If $\{\vec{a}(w_k)\}_{k=1}^{p-1}$, p-different vectors from the system A, then for the determinant Δ of the system we have

$$|\Delta| = (-1)^{\frac{p(p-1)}{2}} \prod_{k=1}^{p-1} C_k^{p-1} \prod_{i\neq k} (\bar{w}_k - \bar{w}_i) \neq 0 .$$

Thus $A(w)$ is an admissible system of vectors. We shall investigate the defects of the curve $\vec{G}(z)$ holomorphic in the half plane $\{\mathrm{Im}\ z \geqslant 0\}$ for the vectors $\vec{a}(w)$ contained in this admissible system $A(w)$.

We have

$$(\vec{G}(z)\ \vec{a}(w)) = \sum_{k=0}^{p-1} g_1^{p-k-1}(z)\ (-1)^k\ C_k^{p-1}\ g_2^k(z)\ w^k =$$

$$= (g_1(z) - g_2(z)w)^{p-1} = g_2^{p-1}(z)(f(z) - w)^{p-1} . \tag{2.4}$$

$$||\vec{G}(z)|| = \sqrt{\sum_{k=0}^{p-1} |g_1(z)|^{2(p-k-1)}|g_2(z)|^{2k}} =$$

$$= |g_1(z)|^{p-1}\sqrt{\sum_{k=0}^{p-1} \frac{1}{|f(z)|^{2k}}} = |g_2(z)|^{p-1}\sqrt{\sum_{k=0}^{p-1} |f(z)|^{2(p-k-1)}} \tag{2.5}$$

For $z=re^{i\theta}\sin\theta$, we get

$$\frac{||\vec{G}(re^{i\theta}\sin\theta)||}{|(\vec{G}(re^{i\theta}\sin\theta)\vec{a}(w))|} = \frac{\sqrt{\sum_{k=0}^{p-1} |f(z)|^{2(p-k-1)}}}{|f(re^{i\theta}\sin\theta) - w|^{p-1}} \geqslant \frac{\max(|f(z)|^{p-1},1)}{|f(re^{i\theta}\sin\theta) - w|^{p-1}} \tag{2.6}$$

Then

$$(p-1)\ \ln^+ \frac{1}{|f(re^{i\theta}\sin\theta) - w|} = \ln^+ \frac{1}{|f(re^{i\theta}\sin\theta) - w|^{p-1}}$$

$$\leq \ln^+ \frac{||\vec{a}(w)||\max(1,|f(z)|^{p-1})}{|f(re^{i\theta}\sin\theta) - w|^{p-1}} \leq \ln \frac{||\vec{G}(re^{i\theta}\sin\theta)||\,||\vec{a}(w)||}{|(\vec{G}(re^{i\theta}\sin\theta)\ \vec{a}(w))|} \tag{2.7}$$

$$[\ ||\vec{a}(w)|| \geqslant 1\ ] \quad .$$

From (2.6) and (2.7) we get

$$\frac{||\vec{G}(re^{i\theta}\sin\theta)||\,||\vec{a}(w)||}{|(\vec{G}(re^{i\theta}\sin\theta)\vec{a}(w))|} \leq \frac{\sqrt{p}(|f(re^{i\theta}\sin\theta)| + 1)^{p-1}}{|f(re^{i\theta}\sin\theta) - w|^{p-1}}\ ||\vec{a}(w)||. \tag{2.8}$$

For the curve $\vec{G}(z)$ holomorphic in the half plane defined by the relation (2.3) we get by (2.7) and (2.8)

$$(p - 1)\bar{m}(r,w,f) \leq m^*(r,\vec{a}(w),\vec{G}) \leq (p - 1)\bar{m}(r,w,f) + C . \tag{2.9}$$

Since the characteristics $\bar{n}(r,f)$ consider the arguments of the poles, the characteristics $\bar{n}(r,f)$ may decrease, if we simply change its argument for fixed value of its modules. Then from (2.4), (2.5), and (1.1), we get

$$(p - 1)\overline{N}(r,\infty,f) + (p - 1)\overline{N}(r,w,f) - (p - 1)\overline{N}(r,\infty,f) = N^{*}(r,\vec{a}(w),\vec{G}) .$$
$$(p - 1)\overline{N}(r,w,f) = N^{*}(r,\vec{a}(w),\vec{G}) \le T^{*}(r,\vec{G}) + C. \tag{2.10}$$

From (2.9) and (2.10) we get

$$T^{*}(r,\vec{G}) = m^{*}(r,\vec{a}(w),\vec{G}) + N^{*}(r,\vec{a}(w),\vec{G}) \le (p - 1)\overline{m}(r,w,f) + C + (p - 1)\overline{N}(r,w,f)$$
$$\le (p - 1)\overline{T}(r,f) + C .$$

If in (2.9), $f(z) = g(z)$ is an analytic function, then we shall proceed in the same way considering $g_1(z) = g(z)$ and $g_2(z) = 1$ in (2.1), (2.2) and (2.3) with

$$\vec{G}(z) = \{ g^{p-1}(z), g^{p-2}(z), \dots , 1 \} .$$

Corollary 1, 2, 3 follow from Theorem 1, and Theorem A, B, C, for meromorphic functions. Corollary 4 follows from Theorem 1 and Theorem D for analytic function defined in ([3], 131-135).

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