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NORM-PRESERVING L-L INTEGRAL TRANSFORMATIONS

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ABSTRACT. In this paper we consider an L-L integral transformation G of the form $F(x) = \int_0^\infty G(x,y)f(y)dy$, where $G(x,y)$ is defined on $D = \{(x,y): x \geq 0, y \geq 0\}$ and $f(y)$ is defined on $[0,\infty)$. The following results are proved: For an L-L integral transformation G to be norm-preserving, $\int_0^\infty |G_*(x,t)|dx = 1$ for almost all $t \geq 0$ is only a necessary condition, where $G_*(x,t) = \lim_{h\to 0}\inf \frac{1}{h}\int_t^{t+h} G(x,y)dy$ for each $x \geq 0$. For certain G's. $\int_0^\infty |G_*(x,t)|dx = 1$ for almost all $t \geq 0$ is a necessary and sufficient condition for preserving the norm of certain $f \in L$. In this paper the analogous result for sum-preserving L-L integral transformation G is proved.

KEY WORDS AND PHRASES. *ℓ-ℓ method. L-L integral transformation. Absolutely continuity of integrals. Fubini-Tonelli Theorem.*
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1. INTRODUCTION.

The well-known summability method defined by a ℓ-ℓ matrix $A = (a_{nk})$, mapping from ℓ into ℓ, is sum-preserving if and only if for each k, $\Sigma_{n\geq 1} a_{nk} = 1$. In our present study we also discuss conditions under which G defined by $G(x,y)$, mappings L into L, is norm-preserving or sum-preserving.

2. NOTATION.

The notation and terms used are:

The statement that f is Lebesgue integrable on $[0,\infty)$ means that for every $u > 0$, if f is Lebesgue integrable on $[0,u]$ and that $\int_0^u f(x)dx$ tends to a finite limit as $u \to \infty$.

L - the space of functions that are Lebesgue integrable on $[0,\infty)$ with norm $\|f\| = \int_0^\infty |f(x)|dx$.

D - the first quadrant of the plane, i.e., $D = \{(x,y): x \geq 0, y \geq 0\}$.

G - an integral transformation, $G: f \to F$, of the form (*) $F(x) = \int_0^\infty G(x,y)f(y)dy$, for all $x \geq 0$, where f is defined on $[0,\infty)$ and $G(x,y)$ defined on D.

G - the collection of all G of the form (*).

*G*L - the subcollection of *G* such that $F \in L$ whenever $f \in L$.

L^∞ - the space of functions which are measurable and essentially bounded on $[0,\infty)$ with norm $\|f\|_\infty = \text{ess} - \sup_{x\geq 0}|f(x)|$

3. MAIN THEOREM

THEOREM 1. If $G \in GL$ and for every $f \in L$

$$\int_0^\infty |F(x)|dx = \int_0^\infty |f(y)|dy ,$$

then for almost all $y \geq 0$,

$$\int_0^\infty |G_*(x,y)|dx = 1 ,$$

where $G_*(x,y) = \liminf_{h\to 0} \frac{1}{h}\int_y^{y+h} G(x,t)dt$, for each $x \geq 0$.

PROOF. Suppose that there is a set $A \subseteq \{y \geq 0\}$ satisfying $0 < mA < \infty$ such that either $\int_0^\infty |G_*(x,y)|dx > 1$ for all $y \in A$ or $\int_0^\infty |G_*(x,y)|dx < 1$ for all $y \in A$. Since $G \in GL$, for each $x \geq 0$, it follows from Theorem (T.S.T.), see [1], that for every measurable set A of finite measure, $\int_A G(x,y)dy < \infty$, without loss of generality, we can assume that A is a bounded measurable set.

Case i). Suppose that for all $y \in A$, $\int_0^\infty |G_*(x,y)|dx < 1$. Without loss of generality we assume that for each $y \in A$

$$\int_0^\infty |G_*(x,y)|dx < 1 - \varepsilon,$$

where ε is a small positive number. Let $f(y) = \chi_A(y)$ then

$$\begin{aligned} F(x) &= \int_0^\infty G(x,y)\ \chi_A(y)dy \\ &= \int_A G(x,y)dy , \end{aligned}$$

and

$$\begin{aligned} \|F\| &= \int_0^\infty \left|\int_0^\infty G(x,y)\ \chi_A(y)dy\right|dx \\ &\leq \int_0^\infty \int_A |G(x,y)|dydx . \end{aligned}$$

Since for each $x \geq 0$, $G_*(x,y) = G(x,y)$ for almost all $y \geq 0$, see [2, Theorem 5. P. 255], so it follows from the Fubini-Tonelli Theorem that

$$\begin{aligned} \|F\| &\leq \int_0^\infty \int_A |G(x,y)|dydx \\ &= \int_0^\infty \int_A |G_*(x,y)|dydx \\ &= \int_A \int_0^\infty |G_*(x,y)|dxdy \\ &\leq \int_A (1 - \varepsilon)dy \\ &= (1 - \varepsilon)\ mA \\ &< mA = \|\chi_A\|. \end{aligned}$$

Hence, for case i), G is not norm-preserving.

Case ii). Suppose that for all $y \in A$, $\int_0^\infty |G_*(x,y)|dx > 1$. Without loss of generality, we assume that for each $y \in A$

$$\int_0^\infty |G_*(x,y)|dx > 1 + \varepsilon .$$

where ε is a small positive number. Let $f(y) = \chi_A(y)$; then $F(x) = \int_A G(x,y)dy$ for all $x \in [0,\infty)$. If $F(x) = 0$ for almost all $x \in [0,\infty)$, then

$$\|F\| = 0 < mA = \|\chi_A\| = \|f\| ,$$

and we're done.

Suppose that $F(x) \neq 0$ for all x in some set with positive measure. Since $G \in GL$, so it follows from Theorem (T. S. T) by author, see [1], $G_*(x,y)$ is measurable on D and $\int_0^\infty |G_*(x,y)|dx < M$ for almost all $y \geq 0$, where M is a constant. Thus

$$\int_A G_*(x,y)dy = \int_A G(x,y)dy \quad \text{and} \quad \int_A \int_0^\infty |G_*(x,y)|dxdy < MmA < \infty, \quad \text{and}$$
$$\int_0^\infty \int_A |G_*(x,y)|dydx = \int_A \int_0^\infty |G_*(x,y)|dxdy < \infty.$$

Given $1 > \varepsilon/2 > \eta > 0$, there is an $X_0 > 0$ such that

$$\int_{X_0}^\infty \int_A |G_*(x,y)|dydx < \eta \cdot mA/2 < \varepsilon \cdot mA/4 \quad .$$

It follows that there is at least a subset $A_0 \subseteq A$, having positive measure and for all $y \in A_0$, satisfying

$$\int_{X_0}^\infty |G_*(x,y)|dx < \varepsilon/8$$

and from $\int_0^\infty |G_*(x,y)|dx > 1 + \varepsilon$ for each $y \in A$ that

$$\int_0^{X_0} |G_*(x,y)|dx > 1 + 3\varepsilon/4$$

for all $y \in A_0$. Let $E = \{(x,y) \in [0,X_0] \times A_0 : |G_*(x,y)| < \varepsilon/2^4X_0\}$ and for any $y \in A_0$, let $E_y = \{x_1 \in [0,X_0]: (x_1,y) \in E\}$. Then $0 \leq mE_y \leq X_0$ for all $y \in A_0$.

Since

$$\int_{A_0} \int_0^{X_0} |G_*(x,y)|dxdy \leq \int_A \int_0^\infty |G_*(x,y)|dxdy \quad ,$$

so it follows from the absolute continuity of the integral that there is a $\delta > 0$ such that for every measurable set $H \subseteq [0,x_0] \times A_0$ satisfying $mH < \delta$, and

$$\iint_H |G_*(x,y)|dydx < \eta \cdot mA_0/4 \quad .$$

If $\int_{A_0} G(x,y)dy = 0$ for almost all $x \geq 0$, then

$$\begin{aligned} \|F\| &= \int_0^\infty \left| \int_0^\infty G(x,y)\, \chi_{A_0}(y)dy \right| dx \\ &= \int_0^\infty \left| \int_{A_0} G(x,y)dy \right| dx \\ &= 0 < mA_0 = \|\chi_{A_0}\| \quad , \end{aligned}$$

and we're done. So we suppose that $\int_{A_0} G(x,y)dy \neq 0$ for some set of $x \geq 0$ with positive measure. By the Generalization of Luzin's Theorem we can choose a closed set $F \subseteq [0,X_0] \times A_0$ such that if $H = [0,X_0] \times A_0 \sim F$ then $mH < \delta$ and

$$\iint_H |G_*(x,y)|dxdy < \eta \cdot mA_0/4 \quad ,$$

and $G_*(x,y)$ is continuous over F. It is clear that $G_*(x,y)$ is uniformly continuour on F. Thus we can have a finite number N of subsets A_i of $mA_i > 0$ of set A_0 such that $F = \cup_{i=1}^N [0,X_0] \times A_i$ and within each strip $[0,X_0] \times A_i$ for each $x \in [0,X_0]$ the value of $G_*(x,y)$ are close to one another. More precisely, for (x,y'), $(x,y'') \in [0,X_0] \times A_i$ and (x,y'), $(x,y'') \notin E$, $|G_*(x,y') - G_*(x,y'')| < \varepsilon/2^5X_0$. Then for each A_i there are three sets A_i^+, A_i^- and E_y of $x \in [0,X_0]$, such that

$$G_*(x,y) > 0, \quad \text{if} \quad (x,y) \in F \cap (A_i^+ \times A_i)$$
$$G_*(x,y) < 0, \quad \text{if} \quad (x,y) \in F \cap (A_i^- \times A_i) \quad ,$$

and

$$|G_*(x,y)| < \varepsilon/2^4X_0, \quad \text{if} \quad (x,y) \in E_y \times A_i \quad .$$

Hence, if $(x,y) \in F \cap [0,X_0] \times A_i$, then

$$\int_0^{X_0} \left| \int_0^\infty G(x,y)\, \chi_{A_i}(y)dy \right| dx = \int_0^{X_0} \left| \int_{A_i} G(x,y)dy \right| dx$$

$$= \int_0^{X_0} \left| \int_{A_i} G_*(x,y)dy \right| dx$$

$$= \int_{A_i^+} \left| \int_{A_i} G_*(x,y)dy \right| dx + \int_{A_i^-} \left| \int_{A_i} G_*(x,y)dy \right| dx$$

$$+ \int_{E_y} \left| \int_{A_i} G_*(x,y)dy \right| dx$$

$$= \int_{A_i^+} \int_{A_i} G_*(x,y)dydx + \int_{A_i^-} \left(- \int_{A_i} G_*(x,y)dy\right) dx$$

$$+ \int_{E_y} \left| \int_{A_i} G_*(x,y)dy \right| dx$$

$$= \int_{A_i} \int_{A_i^+} G_*(x,y)dxdy + \int_{A_i} \int_{A_i^-} (-G_*(x,y))dydx$$

$$+ \int_{E_y} \left| \int_{A_i} G_*(x,y)dy \right| dx$$

$$= \int_{A_i} \int_{A_i^+ \cup A_i^-} |G_*(x,y)|\, dydx + \int_{E_y} \left| \int_{A_i} G_*(x,y)dy \right| dx$$

$$= \int_{A_i^+ \cup A_i^-} \int_{A_i} |G_*(x,y)|\, dydx + \int_{E_y} \left| \int_{A_i} G_*(x,y)dy \right| dx$$

$$= \int_0^{X_0} \int_{A_i} |G(x,y)|dydx - \int_{E_y} \int_{A_i} |G(x,y)|dydx + \int_{E_y} \left| \int_{A_i} G_*(x,y)dy \right| dx$$

$$= \int_0^{X_0} \int_{A_i} |G_*(x,y)|dydx - \{ \int_{E_y} [\int_{A_i} |G_*(x,y)|dy - \left| \int_{A_i} G_*(x,y)dy \right|]dx$$

$$\geq \int_{A_i} \int_0^{X_0} |G_*(x,y)|dxdy - 2 \int_{E_y} \int_{A_i} |G_*(x,y)|dydx$$

$$> (1 + 3\varepsilon/4)mA_i - \varepsilon \cdot mA_i/8 \qquad (\text{since } mE_y < X_0) \ .$$

If $m\{H \cap [0,X_0] \times A_i\} = 0$ for some $A_i \in \{A_i\}_1^N$, then for such an A_i,

$$\| F \| = \int_0^\infty \left| \int_0^\infty G(x,y)\, \chi_{A_i}(y)dy \right| dx$$

$$= \int_0^\infty \left| \int_{A_i} G(x,y)dy \right| dx$$

$$= \int_0^{X_0} \left| \int_{A_i} G_*(x,y)dy \right| dx + \int_{X_0}^\infty \left| \int_{A_i} G_*(x,y)dy \right| dx$$

$$\geq \underset{(x,y)\, \in\, F}{\int_0^{X_0} \left| \int_{A_i} G_*(x,y)dy \right| dx} - \underset{(x,y)\, \in\, H}{\int_0^{X_0} \left| \int_{A_i} G_*(x,y)dy \right| dx}$$

$$= \int_0^{X_0} \left| \int_{A_i} G_*(x,y)dy \right| dx$$

$$> (1 + 3\varepsilon/4)mA_i - \varepsilon mA_i/8$$

$$> mA_i = \| \chi_{A_i} \| ,$$ and we're done.

If $m\{H \cap [0,X_0] \times A_i\} \neq 0$ for all $A_i \in \{A_i\}_1^N$, then there is at least an A_i such that

$$\int\int_{H \cap [0,X_0] \times A_i} |G_*(x,y)|dydx < (\eta \cdot mA_0/4) \cdot mA_i/mA_0 \ ,$$

and for such an A_i,

$$\begin{aligned}
\| F \| &= \int_0^\infty \left| \int_0^\infty G(x,y)\, \chi_{A_i}(y)dy \right| dx \\
&= \int_0^{X_0} \left| \int_0^\infty G_*(x,y)\, \chi_{A_i}(y)dy \right| dx + \int_{X_0}^\infty \left| \int_0^\infty G_*(x,y)\, \chi_{A_i}(y)dy \right| dx \\
&\geq \int_0^{X_0} \left| \int_{\substack{A_i \\ (x,y)\,\varepsilon\, F}} G_*(x,y)dy \right| dx - \int_0^{X_0} \left| \int_{\substack{A_i \\ (x,y)\,\varepsilon\, H}} G_*(x,y)dy \right| dx \\
&\geq \int_0^{X_0} \left| \int_{\substack{A_i \\ (x,y)\,\varepsilon\, F}} G_*(x,y)dy \right| dx - \eta \cdot mA_i/4 \\
&\geq (1 + 3\varepsilon/4)mA_i - 2\ \varepsilon\ mA_i/8 \\
&= (1 + \varepsilon/2)mA_i \\
&> mA_i = \| \chi_{A_i} \| .
\end{aligned}$$

Hence, case (ii) we have proved that G is not norm-preserving and so the proof is complete.

Theorem 1 shows us that if $G \varepsilon GL$, for almost all $y \geq 0$, $\int_0^\infty |G_*(x,y)|dx = 1$ is a necessary condition for $\int_0^\infty |F(x)|dx = \int_0^\infty |f(y)|dy$ whenever $f \varepsilon L$. The next example will tell us that for almost all $y \geq 0$, $\int_0^\infty |G_*(x,y)|dx = 1$ is not a sufficient condition for $\int_0^\infty |F(x)|dx = \int_0^\infty |f(y)|dy$ for every $f \varepsilon L$.

But the following theorem will show that for certain G's, $\int_0^\infty |G_*(x,y)|dx = 1$ is a necessary and sufficient condition for preserving the norms of certain $f \varepsilon L$.

Example. Define

$$G(x,y) = \begin{cases} -1/4x^{1/2}, & \text{if } x \varepsilon (0,1), \\ 1/2x^2, & \text{if } x \varepsilon [1,), \end{cases} \quad \text{for all } y \geq 0;$$

and

$$f(y) = \begin{cases} -2/(y+1)^2, & \text{if } y \varepsilon [0,1), \\ 10/(y+1)^2, & \text{if } y \varepsilon [1,\infty). \end{cases}$$

Then

$$\begin{aligned}
\int_0^\infty |f(y)|dy &= \int_0^1 2/(y+1)^2 dy + \int_1^\infty 10/(y+1)^2 dy \\
&= -2(y+1)^{-1}\Big|_0^1 - 10(y+1)^{-1}\Big|_1^\infty \\
&= -1 + 2 + 5 = 6,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^\infty |G_*(x,y)|dx &= \int_0^1 1/4x^{1/2}dx + \int_1^\infty 1/2x^2\, dx \\
&= 2x^{1/2}/4\Big|_0^1 - 1/2x\Big|_1^\infty \\
&= 1/2 + 1/2 = 1 .
\end{aligned}$$

But

$$\begin{aligned}
F(x) &= \int_0^\infty G(x,y)f(y)dy \\
&= \begin{cases} \int_0^\infty (-1/4x^{1/2})\, f(y)dy, & \text{if } x \varepsilon (0,1), \\ \int_0^\infty (1/2x^2)f(y)dy, & \text{if } x \varepsilon [1,\infty), \end{cases}
\end{aligned}$$

where

$$- \int_0^\infty (1/4x^{1/2})\ f(y)dy = -1/4x^{1/2}[\int_0^1 -2/(y+1)^2dy + \int_1^\infty 10/(y+1)^2dy]$$
$$= -1/4x^{1/2}[-2(-1)(y+1)^{-1}|_0^1 + 10(-1(y+1)^{-1}|_1^\infty]$$
$$= -1/4x^{1/2}[1 - 2 + 5]$$
$$= -1/x^{1/2}\ , \quad \text{if} \quad x \in (0,1) \quad ,$$

and

$$\int_0^\infty (1/2x^2)f(y)dy = (1/2x^2)[\int_0^1 -2/(y+1)^2dy + \int_1^\infty 10/(y+1)^2dy]$$
$$= (1/2x^2)(1 - 2 + 5) = 2/x^2\ , \quad \text{if} \quad x \in [1,\infty) \quad .$$

Therefore

$$\int_0^\infty |F(x)|dx = \int_0^1 1/x^{1/2}dx + \int_1^\infty 2/x^2\ dx$$
$$= 2x^{1/2}|_0^1 + 2(-1)x^{-1}|_1^\infty$$
$$= 2 + 2$$
$$= 4 \neq 6 = \int_0^\infty |f(y)|dy \quad .$$

THEOREM 2. Suppose that $G(x,y)$ is a nonnegative function on D and $G \in GL$; then the folowing are equivalent;

i) $\| F \| = \| f \|$ whenever $f \in L$ and $f(y) \geq 0$ on $[0,\infty)$;

ii) $\| F \| = \| f \|$ whenever $f \in L$ and $f(y) \leq 0$ on $[0,\infty)$;

iii) $\| F \| = \| f \|$ whenever $f \in L$, if $F(x) = \int_0^\infty G(x,y)|f(y)|dy$;

iv) $\int_0^\infty G_*(x,y)dx = 1$, for almost all $y \geq 0$.

PROOF. Since $\| f \| = \int_0^\infty |f(y)|dy$, $F(x) = \int_0^\infty G(x,y)\ f(y)dy$ and

$$\| F \| = \int_0^\infty |F(x)|dx \quad ,$$

it is clear that i) is equivalent to i). We now prove that i) is equivalent to iv). Assuming that $G(x,y) \geq 0$ on D and $f(y) \geq 0$ for all $y \in [0,\infty)$, we have $f(y)G(x,y) \geq 0$ on D. Hence

$$F(x) = \int_0^\infty G(x,y)f(y)dy \geq 0 \quad ,$$

so

$$|F(x)| = F(x) \quad .$$

Therefore

$$\| F \| = \int_0^\infty |F(x)|dx = \int_0^\infty F(x)dx \quad ,$$

and

$$\| F \| = \int_0^\infty \int_0^\infty G(x,y)f(y)dydx \quad .$$

By the Fubini-Tonelli Theorem and for each $x \geq 0$, $G_*(x,y) = G(x,y)$ for almost all $y \geq 0$,

$$\| F \| = \int_0^\infty f(y) \int_0^\infty G_*(x,y)dxdy \quad .$$

Hence,

$$\| F \| = \| f \| \text{ if and only if}$$
$$\int_0^\infty G_*(x,y)dx = 1 \text{ for almost all } y \geq 0 \quad .$$

Next we prove that iii) is equivalent to iv). Let

$$f^+ = \begin{cases} f(y), & \text{if } f(y) \geq 0 \\ 0, & \text{if } f(y) < 0 \end{cases} ;$$

$$f^- = \begin{cases} -f(y), & \text{if } f(y) < 0 \\ 0, & \text{if } f(y) \geq 0 \end{cases} .$$

Since $G(x,y) \geq 0$ on D, so whenever $f \in L$

$$F^+ = \int_0^\infty G(x,y) f^+(y)\,dy \geq 0 , \quad \text{for all } x \geq 0 ;$$

$$F^- = \int_0^\infty G(x,y) f^-(y)\,dy \geq 0 , \quad \text{for all } x \geq 0 ;$$

and

$$f(y) = f^+ - f^- ,$$

if $F(x) = \int_0^\infty G(x,y)|f(y)|dy$, then

$$|F(x)| = F^+ + F^- .$$

It follows from i) that

$$\|F(x)\| = \int_0^\infty |F(x)|dx = \int_0^\infty |f(y)|dy$$

if and only if

$$\int_0^\infty G_*(x,y)\,dx = 1 \quad \text{for almost all } y \geq 0 .$$

We are also interested in the analogous sum-preserving question for L-L integral transformations, viz., when is $\int_0^\infty F(x)dx = \int_0^\infty f(y)dy$ whenever $f \in L$?

Next we give the definition of sum-preserving for L-L integral transformations and a result concerning it.

DEFINITION. The integral transformation $G \in \mathcal{G}L$ is said to be sum-preserving if and only if

$$\int_0^\infty F(x)\,dx = \int_0^\infty f(y)\,dy$$

for all $f(y) \in L$, where $F(x) = \int_0^\infty G(x,y)f(y)dy$.

COROLLARY. Suppose that $G(x,y)$ is a nonnegative function on D and $G \in \mathcal{G}L$; then G is a sum-preserving transformation whenever $f \in L$ if and only if $\int_0^\infty G_*(x,y)dx = 1$ for almost all $y \geq 0$.

PROOF. Since $f \in L$, $f = f^+ - f^-$, where

$$f^+ = \begin{cases} f(y), & \text{if } f(y) \geq 0 \\ 0, & \text{if } f(y) < 0 \end{cases} ,$$

$$f^- = \begin{cases} -f(y), & \text{if } f(y) < 0 \\ 0, & \text{if } f(y) \geq 0 \end{cases} ,$$

and

$$\int_0^\infty f(y)\,dy = \int_0^\infty f^+ dy - \int_0^\infty f^- dy .$$

Then

$$F(x) = \int_0^\infty G(x,y)f(y)dy$$
$$= \int_0^\infty G(x,y)[f^+ - f^-]dy$$
$$= \int_0^\infty G(x,y)f^+(y)dy - \int_0^\infty G(x,y)f^-(y)dy$$

and

$$\int_0^\infty F(x)dx = \int_0^\infty [\int_0^\infty G(x,y)f^+(y)dy - \int_0^\infty G(x,y)f^-(y)dy]dx$$
$$= \int_0^\infty \int_0^\infty G(x,y)f^+(y)dydx - \int_0^\infty \int_0^\infty G(x,y)f^-(y)dydx \quad .$$

By the Fubini-Tonelli Theorem

$$\int_0^\infty \int_0^\infty G(x,y)f^+(y)dydx = \int_0^\infty f^+(y) \int_0 G_*(x,y)dxdy \; ;$$

and

$$\int_0^\infty \int_0^\infty G(x,y)f^-(y)dydx = \int_0^\infty f^-(y) \int_0^\infty G_*(x,y)dxdy \; .$$

Thus

$$\int_0^\infty \int_0^\infty G(x,y) \; f^+(y)dydx = \int_0^\infty f^+(y)dy \; ;$$

and

$$\int_0^\infty \int_0^\infty G(x,y) \; f^-(y)dydx = \int_0^\infty f^-(y)dy$$

if and only if

$$\int_0^\infty G_*(x,y)dx = 1 \quad \text{for almost all} \quad y \geq 0 \quad .$$

Therefore

$$\int_0^\infty F(x)dx = \int_0^\infty f^+dy - \int_0^\infty f^-dy$$

if and only if

$$\int_0^\infty G_*(x,y)dx = 1 \quad \text{for almost all} \quad y \geq 0 \; ,$$

i.e.,

$$\int_0^\infty F(x)dx = \int_0^\infty f(y)dy$$

if and only if

$$\int_0^\infty G_*(x,y)dx = 1 \quad \text{for almost all} \quad y \geq 0 \quad .$$

The proof is completed.

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