Internat. J. Math. & Math. Sci.
Vol. 8 No. 2 (1985) 359-365

A GENERALIZED MEIJER TRANSFORMATION

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(Received on May 13, 1983 and in revised form January 15, 1985)

ABSTRACT. In a series of papers [1-6], Kratzel studies a generalized version of the classical Meijer transformation with the Kernel function $(st)^{\nu}\ \eta\ (q,\ \nu + 1;\ (st)^{q})$. This transformation is referred to as GM transformation which reduces to the classical Meijer transform when q = 1. He also discussed a second generalization of the Meijer transform invoiving the Kernel function $\lambda_{\nu}^{(n)}(x)$ which reduces to the Meijer function when n = 2 and the Laplace transform when n = 1. This is called the Meijer-Laplace (or ML) transformation. This paper is concerned with a study of both GM and ML transforms in the distributional sense. Several properties of these transformations including inversion, uniqueness, and analyticity are discussed in some detail.

KEY WORDS AND PHRASES. Distributional GM and ML transforms, Meijer Transform.

1980 AMS MATHEMATICS SUBJECT CLASSIFICATION COD..S. 46F12, 44A20.

1. INTRODUCTION.

In Zemanian's book [7, p170] the Meijer transformation is defined by means of the integral

$$K_{\nu}\ [f(t)] = 2 \int_{0}^{\infty} (st)^{\nu/2}\ K_{\nu}\ (2\sqrt{st})\ f(t)\ dt, \tag{1.1}$$

where $K_{\nu}\ (z)$ is the modified Bessel function of third kind of order ν, and has the integral representation [7, p148]

$$K_{\nu}(z) = \frac{\sqrt{\pi}}{\Gamma(\nu + \frac{1}{2})}\ (\frac{z}{2})^{\nu} \int_{1}^{\infty} (t^{2} - 1)^{\nu-\frac{1}{2}}\ e^{-zt}\ dt, \tag{1.2}$$

for Re $\nu > -\frac{1}{2}$, Re $z > 0$.

An alternative form of (1.2) is

$$K_{\nu}(z) = \frac{1}{2}(\frac{z}{2})^{\nu} \int_{0}^{\infty} t^{-\nu-1}\ e^{-t - \frac{z^{2}}{4t}}\ dt \tag{1.3}$$

Kratzel [1, p149] has introduced a generalization of the Meijer trnasformation in the form

$$F(s) = K_\nu^{(q)} \{f(t)\} = \int_0^\infty (st)^\nu \eta(q, s+1; (st)^q) f(t)\, dt, \qquad (1.4)$$

where $q \geq 1$ and $|\arg s| < \frac{\pi}{2}(1 + \frac{1}{q})$.

In his other paper [3, p143], Kratzel considered an integral representation of $\eta(\rho, \beta; z)$ in the form

$$\eta(\rho, \beta; z) = \int_0^\infty t^{-\beta} e^{-t - zt^{-\rho}} dt \qquad (1.5)$$

where $\rho > o$ and $|\text{arc } z| < \frac{\pi}{2}$. When $\rho = 1$, $\beta = \nu + 1$,

$$\eta(1, \nu + 1; \frac{z^2}{4}) = 2(\frac{z^2}{2})^{\nu/2} K_\nu(z) \qquad (1.6)$$

Result (1.4) reduces to (1.1) when $q = 1$.

Also, Kratzel introduced a second generalization of the Meijer transformation ([1, p148], [2, p328], [3, p 369], [4, p 383] and [5, p105]) in the form

$$F(s) = L_\nu^{(n)} \{f(t)\} = \int_0^\infty \lambda_\nu^{(n)} \{n(st)^{1/n}\} f(t)\, dt \qquad (1.7)$$

where Re $\nu > \frac{1}{n} - 1$, Re $\{n(st)^{\frac{1}{n}}\} > 0$ and the Kernel $\lambda_\nu^{(n)}(z)$ is given by

$$\lambda_\nu^{(n)}(z) = \frac{(2\pi)^{\frac{n-1}{2}} \sqrt{n}\ (\frac{z}{n})}{\Gamma(\nu + 1 - \frac{1}{n})} \int_1^\infty (t^n - 1)^{\nu - \frac{1}{n}} e^{-zt} dt, \qquad (1.8)$$

with Re $\nu > \frac{1}{n} - 1$, Re $z > 0$ and $n = 1,2,3.....$

It is noted that (1.7) reduces to (1.1) when $n = 2$, and to the Laplace transform when $n = 1$. Also, (1.7) is a special case of a more general transformation studied by Dimovski [8, p23; 9, p141; 10, p156].

The purpose of this paper is to study both (1.4) and (1.7) in the distributional sense and establish theorems concerning complex inversion, uniqueness and analyticity.

2. DIFFERENTIAL OPERATORS.

we use the notation and the terminology similar to those of Kratzel [1 - 3] and Zemanian [7, pp170-200]. The following differential operators will be needed for this study:

$$S_{\nu,q}^k\ \phi(t) = [t^{\nu-q+1} D_t\{t^{q-\nu} D_t^q\ \phi(t)\}]^k,\ k = 0, 1, 2, \ldots. \qquad (2.1)$$

where $\phi(t)$ is a complex smooth function.

$$M_{\nu,n}[\lambda_\nu^{(n)}(t)] = t^{\nu n} D_t^{n-1} [t^{1-n\nu} D_t\ \lambda_\nu^{(n)}(t)],\ n = 1, 2, \ldots, \qquad (2.2)$$

where $\lambda_\nu^{(n)}(t)$ is defined in (1.8).

The operators (2.1) and (2.2) will be used to investigate (1.4) and (1.7) respectively.

3. FUNCTION SPACE $K_{\nu,a}$ AND ITS DUAL.

We define the following seminorms on certain complex smooth functions $\phi(t)$ (Zemanian [7, p176]):

$$\gamma^k_{\nu,a}(\phi) = \sup_{o < t < \infty} |e^{at}\, t^{\nu-\frac{1}{2}}\, S^k_{\nu,q}\, \phi(t)| \tag{3.1}$$

where a is a real number, ν is a complex number with Re $\nu > o$.

We next define $K_{\nu,a}$ as the linear space of all functions $\phi(t)$ on $o < t < \infty$ for which the seminorms $\gamma^k_{\nu,a}$ exist for each k = 0, 1, 2,...... Each $\gamma^k_{\nu,a}$ is a seminorm on $K_{\nu,a}$ which is complete and hence a Frechet space. We note that D(I) is subspace of $K_{\nu,a}$. The differential operator $S^k_{\nu,q}$ is a continuous linear mapping of $K_{\nu,a}$ into itself [7, p171]. It is noted that the differential operator is slightly different from that used in the book [7].

LEMMA 3.1: If

$$\phi(z) = z^{\nu}\, \eta\, (q,\ \nu + 1;\ z^q) \tag{3.2}$$

where Re $\nu > o$ and $q \geq 1$, $|\arg z| < \frac{\pi}{2}(1 + \frac{1}{q})$, then $\phi(st) \in K_{\nu,a}$ for every t in (o,∞) and for every fixed nonzero s.

PROOF: We have from (3.1)

$$\gamma^k_{\nu,a}\phi(st) = \sup_{o < t < \infty} |e^{at}\, t^{\nu-\frac{1}{2}}\, S^k_{\nu,q}\, \phi\,(st)|,\ \text{Re}\ \nu > 0.$$

Making reference to [1, p153], we use the fact

$$S^k_{\nu,q}\, \phi(st) = (-1)^{k(q+1)}\, s^k\, \phi(st) \tag{3.3}$$

combined with the asymptotic property of $\phi(t)$ [1, p 153] as $t \to 0$. We prove that, as $t \to 0$, the seminorms $\gamma^k_{\nu,a}\, \phi(st)$ are finite for $\nu > \frac{1}{2}$ and for every fixed $s \neq 0$. Also, as $t \to \infty$, it can be shown that $\gamma^k_{\nu,a}\, \phi$ are finite for $a < 0$ which follows from the asymptotic property of the function η [1, p 149].

DEFINITION 1: The distributional generalized Meijer transform of f(t) is defined by

$$F(s) = K^{(q)}_{\nu,a}\, f(t) = \langle f(t),\ (st)^q\, \eta\,(q,\ \nu + 1;\ (st)^q)\rangle, \tag{3.4}$$

for every s in $\Omega_f = \{s;\ s \neq 0,\ |\arg s| \leq \frac{\pi}{2}(1 + \frac{1}{q}|$ and $q \geq 1\}$, where $\langle f, \phi \rangle$ represents the number assigned to some ϕ in a testing function space by a member of the dual space.

In short, we call it as the distributional GM - transform of f.

Since by Lemma 3.1, $\phi(st) \in K_{\nu,a}$ for every fixed nonzero s, and for $\nu > \frac{1}{2}$; definition (3.4) has a sense as the application of $f(t) \in K'_{\nu,a}$ to $\phi(st) \in K_{\nu,a}$ where a is any negative real number and $K'_{\nu,a}$ is the dual space of $K_{\nu,a}$.

DEFINITION 2: A distribution f is called a GM-transformable distribution if $f \in K'_{\nu,a}$ for some real number a.

NOTE: Lemma 3.1 is not true for (i) Re $\nu = 0$, $\nu \neq 0$; (ii) $\nu = 0$; and (iii) Re $\nu < 0$.

4. ANALYTICITY OF F (s)

The analyticity of F(s) can be expressed in the following theorem:

THEOREM 4.1: If

$$F(s) = \langle f(t),\ (st)^q\, \eta\,(q,\ \nu + 1;\ (st)^q)\rangle \tag{4.1}$$

for $s \in \Omega_f$, then F(s) is analytic on Ω_f; and

$$D_s\, F(s) = \langle f(t),\ D_s\, (st)^q\, \eta\,(q,\ \nu + 1;\ (st)^q)\rangle \tag{4.2}$$

PROOF: A fairly standard procedure can be used to prove this theorem. However, we state some initial steps for the proof.

$$\frac{F(s + \Delta s) - F(s)}{\Delta s} - < f(t), D_s (st)^q \eta (q,\nu + 1; (st)^q) > = < f(t), \Psi_{\Delta s}(t) > \tag{4.3}$$

where

$$\Psi_{\Delta s}(t) = \frac{1}{\Delta s} [(st + \Delta st)^\nu \eta (q,\nu + 1; (st + \Delta st)^q - (st)^q \eta (q,\nu + 1; (st)^q)] \tag{4.4}$$

We use the series expansion of η from [6, p 142] as

$$\eta (q,\alpha; z) = \sum_{n=o}^{\infty} \frac{1}{n} (1 - \alpha - nq) (-z)^n + \sum_{m=o}^{\infty} \frac{1}{m} (\frac{\alpha-1-m}{q}) (-1)^m z^{\frac{m-\alpha+1}{q}} \tag{4.5}$$

and then asymptotic behavior of η as given in [6, p 142]. After some calculation, it can be shown that

$$D_s (st)^q \eta (q,\nu + 1; (st)^q) \in K_{\nu,a} \tag{4.6}$$

so that (4.2) and (4.3) have a sense. We next follow the arguments given in [7, pp 185-186] combined with the use of Cauchy's integral formula to complete the proof of the theorem.

5. FUNCTION SPACE $G_{\nu,a}$ AND ITS DUAL

We now define $G_{\nu,a}$ as the linear space of all complex-valued smooth functions $\phi(t)$ on $o < t < \infty$. The topology of this space is generated by a set of seminorms $\sigma^n_{\nu,a,n}$ as

$$\sigma^n_{\nu,a,n} \lambda^{(n)}_\nu (t) = \sup_{o < t < \infty} |e^{at} t^{\nu-\frac{1}{2}} M^n_{\nu,n} \lambda^{(n)}_\nu (t)|, \tag{5.1}$$

where $M^n_{\nu,n}$ is the differential operator defined by (2.2). It is noted that (5.1) exists. We denote the dual space of $G_{\nu,a}$ by $G'_{\nu,a}$.

LEMMA 5.1: If

$$\phi(st) = \lambda^{(n)}_\nu \{\nu(st)^{\frac{1}{n}}\} \tag{5.2}$$

for Re $\nu > o$, then $\phi(st) \in G_{\nu,a}$ for t in $(o,\infty)$and for every fixed s such that $s \neq o$ provided $\nu > \frac{1}{2} - \frac{1}{n}$.

PROOF: It follows from [3, p 371] that

$$M^k_{\nu,n} \lambda^{(n)}(z) = z^n \frac{d^{n-1}}{dz^{n-1}} [z^{1-n\nu} \frac{d}{dz} \lambda^{(n)}_\nu (z)] = (-1)^n z \lambda^{(n)}_\nu (z), \ (k=o).$$

Using the following asymptotic property of $\lambda^{(n)}_\nu (z)$ given in [3, p 371] in the form

$$\lambda^{(n)}_\nu (z) = \prod_{r=o}^{n-1} \Gamma(\nu + \frac{r}{n}) + O(1) \text{ as } z \to o, \text{ Re } \nu > o, \tag{5.3}$$

we obtain

$$\sigma^n_{\nu,n,a} \lambda_\nu \{n(st)^{1/n}\} = \sup_{o<t<\infty} |e^{at} t^{\nu-1} n(st)^{1/n} \lambda^{(n)}_\nu \{n(st)^{1/n}\}|$$

which are finite for each n=1,2,.... as $t \to o$ if

$$\sup_{o < t < \infty} |e^{at} t^{\nu-\frac{1}{2}-\frac{1}{n}} n s^{\frac{1}{n}} \lambda^{(n)}_\nu \{n(st)^{1/n}\}|$$

are finite provided $\nu > \frac{1}{2} - \frac{1}{n}$.

We next consider the case for $t \to \infty$. For $-\frac{\pi}{2} < \arg z < \frac{\pi}{2}$, $z = n(st)^{1/n}$, we use equation (7) of [3, p 372] to obtain

$$e^{at}\, t^{\nu-\frac{1}{2}+\frac{1}{n}}\, n\, s^{\frac{1}{n}}\, \lambda_\nu^{(n)}\{n(st)^{1/n}\}$$

$$= e^{at}\, t^{\nu-\frac{1}{2}-\frac{1}{n}}\, n\, s^{\frac{1}{n}}\, (2n)^{\frac{n-1}{2}}\, n^{-\frac{1}{2}}\, \{(st)^{1/n}\}^{\nu(n-1)+\frac{1}{n}-1}$$

$$\times e^{-n(st)^{\frac{1}{n}}}\{1 + 0\,(\frac{1}{n(st)^{1/n}})\},\ t \to \infty,\ s \neq o$$

This expression is asymptotically equal to

$$e^{at}\, n^{\frac{1}{2}}\, s^{\frac{1}{n}}\, t^{\nu-\frac{1}{2}+\frac{1}{n}}\, (2\pi)^{\frac{n-1}{2}}\, s^{\{(n-1)\nu+\frac{1}{n}-1\}\frac{1}{n}}$$

$$\times\, t^{\frac{1}{n}\{(n-1)\nu+\frac{1}{n}-1\}}\, e^{-n(st)^{\frac{1}{n}}},\ s \neq o$$

which is finite if $a < o$.

REMARK: Even if we take a more general differential operator (that is, of a greater order, say k) it must involve terms $\exp[-n(st)^{1/n}]$ asymptotically as $t \to \infty$, which tends to zero as $t \to \infty$.

DEFINITION 3: A distribution $f(t)$ is called an M-L transformable distribution if $f(t) \in G'_{\nu,a}$ for some real number a and $\text{Re}\,\nu > \frac{1}{n} - 1$.

DEFINITION 4: The M-L transform of a M-L transformable distribution $g \in G'_{\nu,a}$ is defined by

$$G(s) = \langle g(t),\ \lambda_\nu^{(n)}\{n(st)^{\frac{1}{n}}\} \rangle \tag{5.4}$$

where $s \in \Omega'_f = \{s,\ \text{Re}\, s > o;\ -\frac{\pi}{2} < \arg s < \frac{\pi}{2}\}$ which is given in [3, p 372].

6. COMPLEX INVERSION THEOREM FOR THE TRANSFORM (1.4).

Kratzel [1, p 151] proved an inversion theorem for (1, p 151) in the classical sense.

In order to discuss a complex inversion theorem, we need the Wright function $\Phi(q, \alpha; z)$ defined in [11] in the form

$$\Phi(q, \alpha; z) = \sum_{n=o}^{\infty} \frac{z^n}{n!\ \Gamma(qn+\alpha)} \tag{6.1}$$

This reduces to Bessel function for $q = 1$, that is,

$$\Phi(1, \nu+1; -\frac{z^2}{4}) = (\frac{z}{2})^{-\nu} J_\nu(z) \tag{6.2}$$

One of the properties of the Wright function [1, p 151] can be expressed as

$$K_\nu^{(n)}\{\Phi\,(\frac{1}{q}, \frac{\nu+1}{q}; t)\} = \frac{1}{q}\,\frac{1}{s-1} \tag{6.3}$$

which is needed in proving the following theorem:

THEOREM 6.1: If

(i) $G(s)$ is holomorphic in Ω where

$$\Omega = \{s;\ \text{Re}\, s^{\frac{q}{q+1}} \geq c,\ |\arg s| \leq \frac{\pi}{2}(1+\frac{1}{q}),\ q \geq 1, \tag{6.4}$$

$$\text{(ii)}\ g(t) = \frac{q}{2\pi i}\int_L \sigma^{-\nu}\, G(\sigma)\, \Phi\,(\frac{1}{q}, \frac{\nu+1}{q}; \sigma(t))\, d\sigma, \tag{6.5}$$

where the path of integration L is given by

$$L:\ \text{Re}\, s^{q/q+1} = c,\ |\arg s| \to \frac{\pi}{2}(1+\frac{1}{q})\ \text{as}\ s \to \infty;$$

then

$$G(s) = K_\nu^{(q)}\{g(t)\} \tag{6.6}$$

In other words, we prove that, for any $\Phi(s) \varepsilon D(I)$ in the sense of convergence in D'(I):

$$< K_\nu^{(q)}\{g(t)\}, \phi(s) > = < G(s), \Phi(s) > \qquad (6.7)$$

where $K_\nu^{(q)}$ is given by (1.4)

PROOF: In view of condition (ii) of the theorem, the left hand side of (6.7) can be written as

$$< < \frac{q}{2\pi i} \int_L \sigma^{-\nu} G(\sigma)\, \Phi(\frac{1}{q}, \frac{\nu+1}{q}; \sigma(t))\, d\sigma, (st)^\nu\, \eta\, (q, \nu + 1; (st)^q >, \Phi(s) >$$

$$= < \frac{q}{2\pi i} \int_L [\int_0^\infty (\frac{st}{\sigma})^\nu \eta\, (q, \nu + 1; (st)^q\, \Phi\, (\frac{1}{q}, \frac{\nu+1}{q}; \sigma(t))\, dt\, G\, (\sigma)\, d\sigma], \Phi(s) >$$

$$= I \text{ (say)}$$

In view of (6.3), this expression yields

$$I = < \frac{1}{2\pi i} \int_L \frac{G(\sigma)}{s-\sigma} d\sigma, \Phi(s) > \qquad (6.8)$$

which is equal to, using a relation in [1, p 152]

$$= < G(s), \Phi(s) >$$

This completes the proof.

We shall give here a weaker version of a uniqueness theorem.

THEOREM 6.2: If

$$F(s) = K_\nu^{(q)} f(t) \text{ on } \Omega_f$$

$$G(s) = K_\nu^{(q)} g(t) \text{ on } \Omega_g$$

and

$$F(s) = G(s) \text{ on } \Omega_f \cap \Omega_g,$$

then f(t) = g(t) in the sense of equality in D'(I).

PROOF: By Inversion Theorem (6.1), we have

$$F(s) - G(s) = K_\nu^{(q)} [f(t)] - K_\nu^{(q)} [g(t)]$$

$$= K_\nu^{(q)} [f(t) - g(t)] = 0 \text{ in } \Omega_f \cap \Omega_g.$$

This implies that f(t) = g(t) in $\Omega_f \cap \Omega_g$ in the sense of equality in D'(I).

7. CLOSING REMARKS: A transform more general than (1.4) and (1.7) was introduced by Oberchkoff in 1958. A modified version of that transform was studied by Dimorski [9 - 10] who proved both real and complex inversion theorems. We would like to discuss some of these thoerems in the sense of distribution in a subsequent paper.

ACKNOWLEDGEMENT: The first author expresses his grateful thanks to Professors E. Kratzel and I. H. Dimovski for their help. Thanks are due to Professor H. J. Glaeske for his kind invitation and hospitality to the first author at Jena. Authors would live to express their thanks to Professors Kratzel and Glaeske for useful discussions on the subject of matter of this paper.

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