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ANALYTIC REPRESENTATION OF THE DISTRIBUTIONAL FINITE HANKEL TRANSFORM

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ABSTRACT. Various representations of finite Hankel transforms of generalized functions are obtained. One of the representations is shown to be the limit of a certain family of regular generalized functions and this limit is interpreted as a process of truncation for the generalized functions (distributions). An inversion theorem for the generalized finite Hankel transform is established (in the distributional sense) which gives a Fourier-Bessel series representation of generalized functions.

KEY WORDS AND PHRASES. Hankel transforms of generalized functions, Test function spaces, Finite Hankel transforms.

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1. INTRODUCTION.

Zemanian [1] extended Hankel transformations to the distribution space H'_μ. H'_μ is the dual of the space of testing functions H_μ defined as follows: for each real number μ, let

$$H_\mu = \{\phi : (0,\infty) \to \mathbb{C} \mid \phi \text{ is smooth on } (0,\infty) \text{ and } \phi \text{ satisfies (1.1)} \}$$

$$\gamma^\mu_{m,k} (\phi) = \sup_{0<x<\infty} |x^m (x^{-1} D)^k [x^{-\mu-\frac{1}{2}} \phi(x)]| < \infty, \text{ for each } m,k = 0,1,2.... \quad (1.1)$$

H'_μ consists of certain distributions of slow growth. Then later [2] he obtained a more general result by removing the restriction on the slow growth of the distributions. He defined the Hankel transformation of a distribution of rapid growth in the space B'_μ. B'_μ is the dual of B_μ, the strict inductive limit of the testing function spaces $B_{\mu,b}$ (defined in section 2) as b tends to infinity through a monotonically increasing sequence of positive numbers.

We take advantage of the fact that functions in $B_{\mu,b}$ are identically zero after b, to define the finite Hankel transformation for the generalized functions in its dual $B'_{\mu,b}$. This is done by generalizing Parseval's equation. We find that for $\mu \geq -\frac{1}{2}$, the finite Hankel transform h_μ maps $B'_{\mu,b}$ isomorphically onto the generalized function space $Y'_{\mu,b}$ (defined in section 3). The aim of the present paper is to obtain various representations of the generalized functions in $Y'_{\mu,b}$ and to find an inversion formula for the generalized finite Hankel transform which also gives another representation of the members of $B'_{\mu,b}$ as a Fourier-Bessel series.

We follow the notation and terminology of Schwartz [3] and Zemanian [4,5]. Here I denotes the open interval $(0,\infty)$. The letters x,y,t and w are used as real variables on I. The k^{th} derivative of an ordinary or generalized function f(x) is usually denoted by $D^k f(x)$ (though the symbol $D_x^k f(x)$ is also used). D(I) denotes the space of smooth functions that have compact support on I. The topology of D(I) is that which makes its dual the space D'(I) of Schwartz's distributions [3, vol. I, p.65].

2. TESTING FUNCTION SPACES $B_{\mu,b}$ AND $Y_{\mu,b}$.

Let $b > 0$ be a fixed arbitrary real number. Then for $\mu \in R$, where R is the net of real numbers, we define

$$B_{\mu,b} = \{\phi : I \to \mathbb{C} \mid \phi(x) \text{ is smooth}, \phi(x) \equiv 0 \text{ for } x > b \text{ and } \phi \text{ satisfies (2.1)}\}$$

$$\gamma_k^\mu(\phi) = \sup_{0<x<\infty} |(x^{-1} D)^k [x^{-\mu-\frac{1}{2}} \phi(x)]| < \infty, \text{ for each } k = 0,1,2.... \tag{2.1}$$

$B_{\mu,b}$ is a linear space to which we assign the topology generated by the countable set of seminorms γ_k^μ. $B_{\mu,b}$ is a sequentially complete countably multi-normed space [2].

Classically, for $\mu + \frac{1}{2} \geq 0$, the finite Hankel transform of a testing function ϕ in $B_{\mu,b}$ is defined as

$$\Phi(\lambda_n) = \int_0^b \phi(x) \sqrt{\lambda_n x}\, J_\mu(\lambda_n x)dx, \; n=1,2,3,..., \tag{2.2a}$$

where as usual J_μ denotes the Bessel function of the first kind of order μ and λ_n $(n = 1,2,3,...)$ are positive roots of $J_\mu(bz) = 0$ (arranged in ascending order of magnitude). However, $\Phi(\lambda_n)$ can be extended to the analytic function of the complex variable $z = y+iw$ by

$$\Phi(z) = \int_0^b \phi(x) \sqrt{xz}\, J_\mu(xz)dx. \tag{2.2}$$

Note that $\Phi(z)$ is an analytic function on the finite z-plane except for a branch point at $z = 0$ [4, p. 145]. Henceforth, the finite Hankel transform of a testing function ϕ in $B_{\mu,b}$ shall be defined as the analytic function $\Phi(z)$ given in (2.2) and denoted by $h_\mu \phi = \Phi$.

For a given real number $b > 0$, $Y_{\mu,b}$ is the space of functions $\Phi(z)$ which satisfy:

$z^{-\mu-\frac{1}{2}}\Phi(z)$ is an even entire function of z and for each non-negative integer k, the quantity

$$\alpha^{\mu}_{b,k}(\Phi) = \sup_{z} |e^{-b|w|} z^{2k-(\mu+\frac{1}{2})} \Phi(z)| \tag{2.3}$$

is finite. The topology of $Y_{\mu,b}$ is the one generated by using the $\alpha^{\mu}_{b,k}$, $k = 0,1,2,\ldots,$ as seminorms. $Y_{\mu,b}$ is a sequentially complete countably normed space. For further properties of these spaces one can look into Zemanian [4, 2].

For a given testing function Φ in $Y_{\mu,b}$, consider the function

$$\phi(x) = \int_0^\infty \Phi(y) \sqrt{xy}\, J_\mu(xy)dy = h_\mu^{-1}[\Phi]. \tag{2.4}$$

Then for $\mu \geq -\frac{1}{2}$, Zemanian [2, Theorem 1] has proved:

<u>Theorem</u> 2.1. For $\mu \geq -\frac{1}{2}$, h_μ is an isomorphism from $B_{\mu,b}$ onto $Y_{\mu,b}$.

Here isomorphism means topological isomorphism. Henceforth, the symbol Φ shall be used to denote a testing function in $Y_{\mu,b}$ whose pre-image is a testing function ϕ in $B_{\mu,b}$.

For a given Φ in $Y_{\mu,b}$, the classical inverse of the finite Hankel transform (2.2a) is a Fourier-Bessel series of the form, [6,7],

$$\frac{2}{b^2} \sum_{n=1}^{\infty} (x/\lambda_n)^{\frac{1}{2}} \frac{J_\mu(x\lambda_n)}{J^2_{\mu+1}(b\lambda_n)} \Phi(\lambda_n). \tag{2.5}$$

Since Φ satisfies (2.3), we have

$$|\Phi(\lambda_n)| \leq \frac{A_{k\mu}}{\lambda_n^{2k-(\mu+\frac{1}{2})}}, \; k = 0,1,2,\ldots, \; n = 1,2,3,\ldots,$$

where $A_{k\mu}$ is constant. Also $(x/\lambda_n)^{\frac{1}{2}} [J_\mu(x\lambda_n)/J^2_{\mu+1}(b\lambda_n)]$ is smooth and bounded on $0 < \lambda_n x < \infty$, for $\mu \geq -\frac{1}{2}$. Consequently, the Fourier-Bessel series (2.5) converges absolutely and uniformly in x for all $x > 0$. Let us write

$$\Psi(x) = \frac{2}{b^2} \sum_{n=1}^{\infty} (x/\lambda_n)^{\frac{1}{2}} \frac{J_\mu(x\lambda_n)}{J^2_{\mu+1}(b\lambda_n)} \Phi(\lambda_n),$$

then

$$|(x^{-1}D)^k x^{-\mu-\frac{1}{2}} \Psi(x)| = \frac{2}{b^2} \left| \sum_{n=1}^{\infty} (x\lambda_n)^{-\mu-k} \frac{J_{\mu+k}(x\lambda_n)}{J^2_{\mu+1}(b\lambda_n)} \lambda_n^{2k+\mu-\frac{1}{2}} \Phi(\lambda_n) \right|. \tag{2.6}$$

Since $\Phi(\lambda_n)$ is of rapid descent as $\lambda_n \to \infty$ and $(x\lambda_n)^{-\mu-k} J_{\mu+k}(x\lambda_n)$ is smooth and bounded on $(0,\infty)$ for $\mu \geq -\frac{1}{2}$, it follows that the right-hand side of (2.6) converges absolutely and uniformly for all $x > 0$ and for every $k = 0,1,2\ldots.$ Hence the left-hand side is continuous and bounded on $0 < x < \infty$ for each $k = 0,1,2,\ldots.$ Hence

$$\gamma^{\mu}_{k}(\Psi) < \infty, \; k = 0,1,2,\ldots.$$

Moreover, $(x^{-1}D)^k (x^{-\mu-\frac{1}{2}}\Psi(x)) = x^{-\mu-\frac{1}{2}}\left[a_{k0} \frac{\Psi}{x^{2k}} + a_{k1} \frac{D\Psi}{x^{2k-1}} + \ldots + a_{kk} \frac{D^k\Psi}{x^k}\right]$,

where the a_{ki} denote constants and $D = \frac{d}{dx}$. So we see that the Fourier-Bessel series defines an infinitely differentiable function $\Psi(x)$ satisfying $\gamma^{\mu}_{k}(\Psi) < \infty$ for each $k = 0,1,2,\ldots.$ But Ψ may not be in $B_{\mu,b}$ as Ψ may not be zero for $x > b$. But

$$\phi(x) = \lim_{\varepsilon \to 0+} \lambda_\varepsilon(x)\Psi(x) \in B_{\mu,b},$$

where $\lambda_\varepsilon(x)$ is defined as:

$$\lambda_\varepsilon(x) = \begin{cases} E(x/2\varepsilon), & 0 < x < 2\varepsilon \\ 1, & 2\varepsilon \le x \le b-2\varepsilon \\ 1-E(\frac{x-b+2\varepsilon}{2\varepsilon}), & b-2\varepsilon < x < b, \\ 0, & x \ge b, \end{cases} \tag{2.7}$$

for $0 < \varepsilon < b/4$, and

$$E(u) = \frac{\int_0^u \exp(1/x(x-1))dx}{\int_0^1 \exp(1/x(x-1))dx} . \tag{2.8}$$

Note that $\lambda_\varepsilon(x)$ is a multiplier in $B_{\mu,b}$ for each $0 < \varepsilon < b/4$ since, for any $\phi \in B_{\mu,b}$, we have

$$\gamma_k^\mu(\lambda_\varepsilon\phi) \le \sum_{n=0}^{k} \binom{k}{n} \gamma_n^\mu(\phi) \sup_{0<x<b} |(x^{-1}D)^{k-n}\lambda_e(x)|.$$

Now pick X such that $0 < X < 2\varepsilon$. Then

$$\sup_{X<x<b} |(x^{-1}D)^m \lambda_\varepsilon(x)| < \infty,$$

and

$$\sup_{0<x<X} |(x^{-1}D)^m\lambda_\varepsilon(x)| \le A \sup_{0<x<X} |(x^{-1}D)^{m-1}[x^{-1}\exp(\frac{4\varepsilon^2}{x(2\varepsilon-x)})]| ,$$

where A is a constant.

So we see that $\gamma_k^\mu(\lambda_\varepsilon\phi) < \infty$. Also $\lambda_\varepsilon(x)$ is smooth on $(0,\infty)$. Hence $\lambda_\varepsilon\phi \in B_{\mu,b}$. It is easy to see that

$$\lim_{\varepsilon\to 0^+} \lambda_\varepsilon(x)\phi(x) = \phi(x), \quad \text{for any } \phi \text{ in } B_{\mu,b}.$$

3. GENERALIZED FUNCTION SPACES $B'_{\mu,b}$ AND $Y'_{\mu,b}$.

The spaces $B'_{\mu,b}$ and $Y'_{\mu,b}$ are the dual spaces of $B_{\mu,b}$ and $Y_{\mu,b}$ respectively. We shall use only the weak topology of $B'_{\mu,b}$, that is, the topology assigned to it by the seminorms

$$\rho_\varphi(f) = |\langle f,\phi\rangle|, \ \phi \in B_{\mu,b}, \ f \in B'_{\mu,b}.$$

Since $B_{\mu,b}$ is a sequentially complete countably normed space, $B'_{\mu,b}$ is also sequentially complete [4, Theorem 1.83]. Similarly, we equip $Y'_{\mu,b}$ with the weak topology generated by the seminorms $\zeta_\Phi(F) = |\langle F,\Phi\rangle|$, $\Phi \in Y_{\mu,b}$, $F \in Y'_{\mu,b}$. $Y'_{\mu,b}$ is a sequentially complete space.

We now construct a generalized function in $B'_{\mu,b}$ which is not in $D'(I)$. Let $\{\tau_n\}$ be a monotone increasing sequence of positive numbers with limit $b+1$. For every $\phi \in B_{\mu,b}$, the formula

$$\langle f,\phi\rangle = \sum_{n=1}^{\infty} \phi(\tau_n) \tag{3.1}$$

is easily seen to define a generalized function f in $B'_{\mu,b}$. On the other hand, if ϕ is an arbitrary testing function in $D(I)$, then $\Sigma_n \phi(\tau_n)$ is in general an infinite sum and it need not be convergent.

Note that

(i) $B'_{\mu,b}$ contains every regular distribution that corresponds to a function which is Lebesgue integrable on $0 < x < b$. In this case we have

$$< f ,\phi > = {}_0\int^b f(x)\ \phi(x)\ dx,\ \phi \in B_{\mu,b}.$$

(ii) If f is a tempered distribution whose support is contained in $[X, \infty)$ for some $X > 0$, then $f \in B'_{\mu,b}$.

(iii) Similarly, every regular distribution F, which can be defined by a locally integrable function $F(y)$ through the equation

$$< F, \Phi > = {}_0\int^\infty F(y)\ \Phi(y)\ dy$$

for every Φ in $Y_{\mu,b}$, belongs to $Y'_{\mu,b}$. Note that $F(y)$ need not be integrable over $0 < y < \infty$, though typically it would be of slow growth, i.e., for some integer $N > 0$, $y^{-N} F(y) \to 0$ as $y \to \infty$.

4. FINITE HANKEL TRANSFORMATION OF $B'_{\mu,b}$.

Henceforth, we assume that $\mu \geq -\frac{1}{2}$. For $f \in B'_{\mu,b}$, $\phi \in B_{\mu,b}$ and $\Phi = h_\mu \phi \in Y_{\mu,b}$, we define the finite Hankel transform $F = h_\mu f$ by

$$< F,\Phi > = < f,\phi >. \tag{4.1}$$

The above equation also defines the inverse Hankel transform $f = h_\mu^{-1}F$. From Theorem 2.1 one readily obtains:

Theorem 4.1. h_μ is an isomorphism from $B'_{\mu,b}$ onto $Y'_{\mu,b}$.

Example 1. The finite Hankel transform of the delta function $\delta(x-k)$ is given by the equation (4.1):

$$\begin{aligned} < h_\mu \delta(x-k),\ \Phi(z) > &= < \delta(x-k),\ \phi(x) >,\ \text{for } 0 < k < b, \\ &= < \delta(x-k),\ {}_0\int^\infty \Phi\ \sqrt{xy}\ J_\mu(xy)dy >,\ (\text{using } (2.4)) \\ &= {}_0\int^\infty \Phi(y)\ \sqrt{ky}\ J_\mu(ky)dy < \infty. \end{aligned}$$

This defines a regular distribution $F(z) = (kz)^{\frac{1}{2}}\ J_\mu(kz)$ in $Y'_{\mu,b}$. Consequently, $h_\mu \delta(x-k) = (kz)^{\frac{1}{2}} J_\mu(kz)$, for $0 < k < b$.

Example 2. The finite Hankel transform of $\delta(x-k)$ for $k > b$ is the "Zero" generalized function in $Y'_{\mu,b}$, since $<\delta(x-k),\ \phi(x)> = \phi(k) = 0$ for all ϕ in $B_{\mu,b}$.

Example 3. The finite Hankel transform of the generalized function in $B'_{\mu,b}$ defined by (3.1) is the generalized function defined by the series

$$F = \sum_{n=1}^{\infty} \sqrt{y\tau_n}\ J_\mu(y\tau_n)$$

since, for $\Phi \in Y_{\mu,b}$,

$$\begin{aligned} < F,\Phi > &= \sum_{n=1}^{\infty} {}_0\int^\infty \Phi(y)\ \sqrt{\tau_n y}\ J_\mu(\tau_n y)dy \\ &= \sum_{n=1}^{\infty} \phi(\tau_n) = < f,\phi > \end{aligned}$$

Example 4. Suppose f is a regular generalized function, corresponding to a Lebesgue integrable function over $(0,b)$, in $B'_{\mu,b}$. Than the ordinary finite Hankel transform of f is given by

$${}_0\int^b f(x)\ \sqrt{\lambda_n x}\ J_\mu(\lambda_n x)dx;\ n = 1,2,3\dots .$$

Since f is integrable over (0,b), its finite Hankel transform may be extended to the analytic function

$$F(z) = \int_0^b f(x) \sqrt{zx}\, J_\mu(zx)dx.$$

We show that $F = h_\mu(f)$. Since f is a regular generalized function,

$$< h_\mu f,\Phi > = < f,\phi>$$

$$= \int_0^b f(x)\ \phi(x)dx$$

$$= \int_0^b f(x)\ (\int_0^\infty \Phi(y)\ \sqrt{xy}\ J_\mu(xy)dy)dx \quad \text{(using (2.4))}.$$

Since the integrand $f(x)\Phi(y)\ (xy)^{\frac{1}{2}}J_\mu(xy)$ is absolutely integrable over the domain $0 < x < b,\ 0 < y < \infty$, the order of integration may be changed, and we obtain

$$< h_\mu f,\Phi > = \int_0^\infty dy\ \Phi(y) \int_0^b dx\ f(x)\ (xy)^{\frac{1}{2}}\ J_\mu(xy)$$

$$= < F,\Phi >.$$

Note that $F(\lambda_n) = \int_0^b f(x)\ (\lambda_n x)^{\frac{1}{2}}\ J_\mu\ (\lambda_n x)dx$, gives that $|F(\lambda_n)|$ is bounded. Hence, for any Φ in $Y_{\mu,b}$, equation (2.3) ensures that the series $\sum\limits_{n=1}^{\mu} F(\lambda_n)\Phi(\lambda_n)$ converges. Furthermore, if a sequence $\{\Phi_m\}$ converges in $Y_{\mu,b}$ then the sequence of numbers $\{\sum\limits_{2}^{\infty} F(\lambda_n)\Phi_m(\lambda_n)\}$ also converges. Hence, the sum $\sum\limits_{1}^{\infty} F(\lambda_n)\Phi(\lambda_n)$ defines a continuous linear functional on $Y'_{\mu,b}$.

Next we investigate a representation for the finite Hankel transform of a generalized function in $B'_{\mu,b}$. Let D(0,b) be the space of infinitely differentiable functions on (0,b) with compact support contained in (0,b). The topology of D(0,b) is that which makes its dual D'(0,b) of Schwartz's distribution. Then $D(0,b) \subset B_{\mu,b}$ and h_μ maps D(0,b) into a subspace of $Y_{\mu,b}$. Let W be the subspace of $Y_{\mu,b}$ onto which D(0,b) is mapped. Then we have

<u>Theorem 4.2</u>. For any generalized function f in $B'_{\mu,b}$, there exists a continuous function F(y) of slow growth such that the finite Hankel transform $h_\mu f$ of f restricted to W is equivalent, in the functional sense, to the regular generalized function F in $Y'_{\mu,b}$.

<u>Proof</u>. For a given generalized function f, there exists an integer $r \geq 0$ and a continuous function h(x) [5, Theorem 3.4.2] such that

$$< f,\phi > = < D^r h,\phi >, \quad \text{for every } \phi \text{ in } D(0,b).$$

We take h = 0 outside (0,b). Then using (2.4), we have, for $\phi \in D(0,b)$,

$$< F,\Phi > = < h_\mu f,\ h_\mu\phi > = < f,\phi > = < D^r h,\phi >$$

$$= (-1)^r < h(x),\ D^r\phi(x) >$$

$$= (-1)^r \int_0^b dx\, h(x)\, D^r\phi(x)$$

$$= (-1)^r \int_0^b dx\, h(x) \int_0^\infty dy\, \Phi(y)\frac{\partial^r}{\partial x^r}(\sqrt{xy}\, J_\mu(xy)) \tag{4.2}$$

$$= (-1)^r \int_0^b dx\, h(x) \int_0^\infty dy\, \Phi(y) \sum_{i=0}^{r} [(-1)^i a_i(\mu) y^i x^{i-r}\sqrt{xy}\, J_{\mu+i}(xy)]$$

$$= \sum_{i=0}^{r} (-1)^{i+r} a_i(\mu) \int_0^b dx\, x^{i-r}h(x)[\int_0^\infty dy\, \Phi(y)y^i \sqrt{xy}\, J_{\mu+i}(xy)] \tag{4.3}$$

where $a_i(\mu)$ is a constant depending on μ, for each i. If $g_{i-r}(x) = x^{i-r}h(x)$, then $g_{i-r}(x)$ is continuous on $(0,\infty)$ and $g_{i-r}(x) = 0$ outside $(0,b)$. Since $\Phi(y)$ is of rapid descent and $(xy)^{\frac{1}{2}} J_{\mu+i}(xy)$ is bounded on $0 < xy < \infty$, the order of integration in (4.3) may be interchanged. Therefore, (4.3) becomes

$$< F,\Phi > = \sum_{i=0}^{r} (-1)^{i+r} a_i(\mu) \int_0^\infty dy\, \Phi(y)y^i h_{\mu+i}[g_{i-r}(x)].$$

Denote $h_{\mu+i}[g_{i-r}(x)]$ by $G_{i-r}^{\mu+i}(y)$, then for $\Phi \varepsilon W$,

$$< F,\Phi > = < \sum_{i=0}^{r} (-1)^{i+r} a_i(\mu)y^i G_{i-r}^{i+\mu}(y),\ \Phi(y)> . \tag{4.4}$$

Clearly, the continuous function

$$G_{i-r}^{i+\mu}(y) = \int_0^b g_{i-r}(x)(xy)^{\frac{1}{2}}J_{\mu+1}(xy)dx,$$

may be extended to an analytic function. Equation (4.4) gives

$$h_\mu f \mid W = \sum_{i=0}^{r} (-1)^{i+r} a_i(\mu)y^i G_{i-r}^{i+\mu}(y) = F(y).$$

Finally since, $| G_{i-r}^{i+\mu}(y) | < \infty$, it is obvious that $F(y)$ is of slow growth.

Example 5. From Example 1, we know that $h_\mu\delta(x-k) = (kz)^{\frac{1}{2}}J_\mu(kz)$, for $0 < k < b$. On the other hand, if we define

$$h(x) = \begin{cases} 0, & \text{for } x \geq k, \\ x-k, & \text{for } x > k, \end{cases}$$

we obtain from Theorem 4.2

$$h_\mu\delta(x-k) = F(y) = a_0(\mu)G_{-2}^{\mu}(y) - a_1(\mu)y\, G_{-1}^{\mu+1}(y) + a_2(\mu)y^2G_0^{\mu+2}(y).$$

It is easily seen that

$$a_0(\mu) = \mu^2 - \tfrac{1}{4},\ a_1(\mu) = 2(\mu+1), \text{ and } a_2(\mu) = 1.$$

Also

$$G_{-2}^{\mu}(Y) = \int_k^b(\frac{1}{x} - k/x^2)(xy)^{\frac{1}{2}}J_\mu(xy)dx,$$

$$G_{-1}^{\mu+1}(y) = \int_k^b(1 - \frac{k}{x})(xy)^{\frac{1}{2}}J_{\mu+1}(xy)dx,$$

and

$$G_0^{\mu+2}(y) = \int_k^b(x-k)(xy)^{\frac{1}{2}}J_{\mu+2}(xy)dx.$$

Hence (4.4) gives

$$F(y) = (yk)^{\frac{1}{2}}J_\mu(yk) - \tfrac{1}{2}(1 + \frac{k}{b})(yb)^{\frac{1}{2}} J_\mu(yb) + (1 - \frac{k}{b})y(yb)^{\frac{1}{2}}J_\mu'(yb).$$

This is another representation of $h_\mu\delta(x-k)$. It can be shown that this representation is equivalent to the one given in example 1, since

$$< -\tfrac{1}{2}\ (1 + \frac{k}{b})(yb)^{\frac{1}{2}}J_\mu(yb),\ \Phi > = -\tfrac{1}{2}(1 + \frac{k}{b})\ {}_0\!\int^\infty\Phi(y)\ (yb)^{\frac{1}{2}}J_\mu(yb)dy,\ \Phi \in Y_{\mu,b}$$

$$= -\tfrac{1}{2}(1 + \frac{k}{b})\ \phi(b) = 0.$$

Note. $\phi(b) = 0$ follows from the continuity property.

Thus $(1 - \frac{k}{b})\ <y(yb)^{\frac{1}{2}}\ J_\mu'\ (yb),\ \Phi > = ((1 - \frac{k}{b})\ {}_0\!\int^\infty\ y\ \Phi(y)\ (yb)^{\frac{1}{2}}J_\mu'(yb)dy.$

From (2.4) we have

$$\phi'(x) = \frac{1}{2x}\ {}_0\!\int^\infty\ \Phi(y)\ (xy)^{\frac{1}{2}}J_\mu(xy)dy + {}_0\!\int^\infty\ y\ \Phi(y)(xy)^{\frac{1}{2}}J_\mu'\ (xy)(dy).$$

Therefore,

$$\phi'(b) = \frac{1}{2b}\ {}_0\!\int^\infty\ \Phi(y)\ (by)^{\frac{1}{2}}J_\mu(by)dy + {}_0\!\int^\infty\ y\ \Phi(y)(by)^{\frac{1}{2}}J_\mu'\ (by)(dy).$$

So we have

$$(1 - \frac{k}{b})\ <\ y(by)^{\frac{1}{2}}J'_\mu(by),\ \Phi > = (1 - \frac{k}{b})\ [\phi'(b) - \frac{1}{2b}\ \phi(b)] = 0.$$

Hence

$$< F,\ \Phi > = <\ (yk)^{\frac{1}{2}}\ J_\mu(yk),\ \Phi(y) >.$$

Thus we get the same result as derived in Example 1.

Example 6. We have shown in Example 2 that $h_\mu\delta(x-k) = 0$ for $k > b$. This also follows from Theorem 4.2. Take $r = 2$ and define $h(x) = 0$ for $x \le k$ and $h(x) = x-k$ for $x > k$. It is easily seen that $G^\mu_{-2}(y) = G^{\mu+1}_{-1}(y) = G^{\mu+2}_0(y) = 0$, hence $F(y) = 0$.

Corollary 4.3. For any generalized function f in $B'_{\mu,b}$ with r, $h(x)$ and $F(y)$ defined as in the proof of Theorem 4.2, we have

$$F(y) = <\ D^r\ h(x),\ \lambda_\varepsilon(x)\ (xy)^{\frac{1}{2}}\ J_\mu(xy) >,\ \text{as}\ \varepsilon \to 0^+. \tag{4.5}$$

Proof. Theorem 4.2 gives

$$h_\mu f|_W = F(y) = \sum_{i=0}^{r}\ (-1)^{i+r}a_i(\mu)y^iG^{i+\mu}_{i-r}(y),\quad \text{for}\quad f \in B'_{\mu,b}$$

$$= (-1)^r\ {}_0\!\int^b\ dx\ \sum_0^r\ (-1)^i a_i(\mu)x^{-r}(xy)^i(xy)^{\frac{1}{2}}J_{\mu+i}(xy)$$

$$= (-1)^r\ {}_0\!\int^b\ \frac{\partial^r}{\partial x^r}\ [\sqrt{xy}\ J_\mu(xy)]\ h(x)\ dx$$

$$= (-1)^r\ {}_0\!\int^b\ \lambda_\varepsilon(x)\ \frac{\partial^r}{\partial x^r}\ [\sqrt{xy}\ J_\mu(xy)]\ h(x)\ dx\ \text{as}\ \varepsilon \to 0^+$$

$$= (-1)^r< h(x),\ \lambda_\varepsilon\ \frac{\partial^r}{\partial x^r}\ \ [\sqrt{xy}\ J_\mu\ (xy)] > \qquad \text{as}\quad \varepsilon \to 0^+ .$$

But since $\lambda_\varepsilon(x) = 1$ on $(0,b)$ as $\varepsilon \to 0^+$ (and $h(x) = 0$ outside $(0,b)$), the order of differentiation may be interchanged in the preceding equation to give

$$F(y) = (-1)^r\ <\ h(x),\ D^r_x\ [\lambda_\varepsilon(x)(xy)^{\frac{1}{2}}\ J_\mu(xy)] > \text{as}\ \varepsilon \to 0^+,$$

$$= <\ D^r_x\ h(x),\ \lambda_\varepsilon(x)\ (xy)^{\frac{1}{2}}J_\mu(xy) >,\qquad \text{as}\ \varepsilon \to 0^+,$$

(from the definition of distributional differentiation).

Example 7. While calculating the finite Hankel transform of $\delta(x-k)$, $0 < k < b$ (Example 5) using the method of Theorem 4.2, it was necessary to evaluate certain integrals to find $F(y)$. This may be avoided by using the above Corollary. From the definition of $h(x)$, we see that

$$D\,h(x) = \begin{cases} 0, & 0 < x \le k, \\ 1, & x > k, \end{cases}$$

and

$$D^2\,h(x) = \delta(x-k).$$

Hence, (4.7) gives

$$\begin{aligned} F(y) &= \lim_{\varepsilon\to 0^+} < \delta(x-k),\ \lambda_\varepsilon(x)\ (xy)^{\frac{1}{2}} J_\mu(xy) > \\ &= \lim_{\varepsilon\to 0^+} \lambda_\varepsilon(k)(ky)^{\frac{1}{2}} J_\mu(ky) \\ &= (ky)^{\frac{1}{2}} J_\mu(ky), \text{ since } \lambda_\varepsilon(k) = 1 \text{ as } \varepsilon\to 0^+ . \end{aligned}$$

5. SOME STRUCTURE THEOREMS

In this section we shall obtain representations for members of $B_{\mu,b}$, $B'_{\mu,b}$ and $Y'_{\mu,b}$ under suitable conditions. Note that the structure formula given by Theorem 4.2 is valid only when $F \in Y'_{\mu,b}$ is restricted to W, a subspace of $Y_{\mu,b}$. Here we will obtain a more general result, viz., a structure formula shall be established for $F \in Y'_{\mu,b}$ restricted to a larger subspace than W of $Y_{\mu,b}$. This section is very similar to section 3.4 of Zemanian [5, p. 86-93], consequently the corresponding results will be stated without proof or, perhaps, with only an indication of the proof. To begin with, we define certain spaces associated with $B_{\mu,b}$.

Definition 5.1. We define the spaces $B^o_{\mu,b}$, $C^o_{\mu,b}$ and $B^{(1)}_{\mu,b}$ by

$$B^o_{\mu,b} = \{ \phi \in B_{\mu,b} : \phi = o(x^{\mu+\frac{1}{2}}) \text{ as } x \to 0^+ \}, \tag{5.1}$$

$$C^o_{\mu,b} = \{ g : (0,\infty) \to \mathbb{C} \mid g \text{ is continuous on } (0,\infty),\ g = 0 \text{ for } x > b \quad g(x) = o(x^{\mu+\frac{1}{2}}) \text{ as } x \to 0^+ \}, \tag{5.2}$$

and

$$B^{(1)}_{\mu,b} = \{ \phi \in B_{\mu,b} : \phi' = o(x^{\mu+3/2}) \text{ as } x \to 0^+\}. \tag{5.3}$$

$B^o_{\mu,b}$ carries the natural topology induced on it by $B_{\mu,b}$.

Note that

$$\phi \in B_{\mu,b} \to \phi = o(x^{\mu+\frac{1}{2}}), \text{ as } x \to 0^+.$$

This is true because $\gamma^\mu_o(\phi) = \sup_{0<x<b} |x^{-\mu-\frac{1}{2}}\, \phi(x)\, | < \infty$.

We prove an interesting property of functions in $B^o_{\mu,b}$ in

Lemma 5.2. Let $\phi \in B^o_{\mu,b}$. Then

$$\phi = o(x^{\mu+5/2}) \text{ as } x \to 0^+. \tag{5.4}$$

Proof. Let $\phi \in B^o_{\mu,b}$. Now

$$(t^{-1}\, D_t)[t^{-\mu-\frac{1}{2}}\phi(t)] = t^{-\mu-\frac{1}{2}}\, [\frac{\phi'}{t} - (\mu + \tfrac{1}{2})\, \frac{\phi}{t^2}]. \tag{5.5}$$

Write $\eta(t) = \frac{\phi'}{t} - (\mu + \frac{1}{2}) \frac{\phi}{t^2}$. Clearly $\eta(t)$ is a smooth function on $(0,\infty)$ and $\eta(t) = 0$, for $t > b$. Also, $\gamma_k^\mu(\eta) = \gamma_{k+1}^\mu(\phi) < \infty$, for each $k = 0,1,2,\ldots$.

Hence, $\eta(t) \in B_{\mu,b}$. Therefore, $\eta(t) = o(t^{\mu+\frac{1}{2}})$, as $t \to 0^+$.
Hence

$$\frac{d}{dt}(t^{-\mu-\frac{1}{2}}\phi(t)) = O(t), \quad \text{as } t \to 0^+. \tag{5.6}$$

Integrating (5.6), we obtain

$$t^{-\mu-\frac{1}{2}}\phi(t) = O(t^2), \quad \text{as } t \to 0^+, \tag{5.7}$$

proving Lemma 5.2.

We assign a norm to the space $C^0_{\mu,b}$ by

$$||j||_0 = \sup_{x\in(0,b)} |x^{-\mu-\frac{1}{2}}g(x)|, \quad \text{for each } g \in C^0_{\mu,b}. \tag{5.8}$$

Thus $C^0_{\mu,b}$ becomes a topological vector space. We need the following lemma which is stated without proof since the proof is identical to the proof of [5, Lemma 1, p. 88].

Lemma 5.3. $B^0_{\mu,b}$ is a dense subset of $C^0_{\mu,b}$.

The following proposition gives an integral representation for the functions in $B^0_{\mu,b}$.

Proposition 5.4. Let $\phi \in B^0_{\mu,b}$. Then ϕ satisifies the integral equation

$$\phi(x) = \int_0^b u(x,t)(t^{-1}D_t^2)(t^{-\mu-\frac{1}{2}}\phi(t))dt, \tag{5.9}$$

where

$$u(x,t) = x^{\mu-\frac{1}{2}}u^*(x,t), \tag{5.10}$$

and

$$u^*(x,t) = \begin{cases} \frac{x^3}{2b^2}\, t(t^2-b^2), & \text{for } 0 < x \le t \le b, \\ \frac{t^3}{2b^2}\, x(x^2-b^2), & \text{for } 0 < t \le x \le b, \\ 0 & \text{, elsewhere,} \end{cases} \tag{5.11}$$

for $0 < x < \infty$, $0 < t < \infty$.

Proof. Trivial.

Next we prove that generalized functions in $B'_{\mu,b}$ are distributional derivatives of certain continuous functions.

We start with the following boundedness property of $f \in B'_{\mu,b}$.

For each $f \in B'_{\mu,b}$, there exists a non-negative integer r and a positive constant A such that for all $\phi \in B_{\mu,b}$,

$$|\langle f,\phi\rangle| \le A \max_{0\le k\le r} \gamma_k^\mu(\phi) = \rho_r^\mu(\phi) \quad \text{(say)}. \tag{5.12}$$

Suppose $f \in B'_{\mu,b}$ is such that (5.12) is satisfied with $r = 0$. Then

$$|\langle f,\phi\rangle| \le A \sup_{0<x<b} |x^{-\mu-\frac{1}{2}}\phi(x)|. \tag{5.13}$$

We now extend f, satisfying the inequality (5.13), continuously and uniquely onto the space $C^0_{\mu,b}$. Let g in $C^0_{\mu,b}$ be arbitrary. Then by Lemma 5.3, there exists a sequence $\{\phi_n\}$ of testing functions in $B^0_{\mu,b}$ such that ϕ_n converges to g in

$C^0_{\mu,b}$. We define $<f,g>$ by

$$<f,g> = \lim_{n\to\infty} <f,\phi_n>. \tag{5.14}$$

This defines a continuous linear functional on $C^0_{\mu,b}$ satisfying the inequality (5.13).

Clearly $u(x,t) \in C^0_{\mu,b}$, hence the following definition makes sense for $f \in B'_{\mu,b}$ satisfying (5.13).

Definition 5.5. For $f \in B'_{\mu,b}$ satisfying the inequality (5.13), define

$$h(t) = < f(x), u(x,t) >. \tag{5.15}$$

Note: (i) $h(t) = 0$ for $t \geq b$, as $u(x,t) = 0$ for $t \geq b$

(ii) $|h(t) - h(\tau)| \leq 3Ab^2|t-\tau|$, $0 < t \leq b$, $0 < \tau \leq b$. (5.16)

Lemma 5.6. For $\phi \in B_{\mu,b}$, $(\frac{1}{t} D_t)^2[t^{-\mu-\frac{1}{2}}\phi(t)]$ is uniformly continuous on $(0,b]$.

Proof. Let

$$\eta(t) = (\frac{1}{t}\frac{d}{dt})^2[t^{-\mu-\frac{1}{2}}\phi(t)],\ \phi \in B_{\mu,b}.$$

Then, $\eta(t) = O(1)$ as $t \to 0^+$, and $|\eta'(t)| < \infty$, proving the Lemma 5.6.

Theorem 5.7. For $f \in B'_{\mu,b}$ satisfying the condition

$$|<f,\phi>| \leq A \sup_{0<x<b} |x^{-\mu-\frac{1}{2}}\ \phi(x)|,\ \forall\ \phi \in B^0_{\mu,b},$$

we have

$$<f,\phi> = \int_0^b h(t)(t^{-1}D_t)^2[t^{-\mu-\frac{1}{2}}\phi(t)]dt \tag{5.17}$$

for every $\phi \in B^0_{\mu,b}$. Here, $h(t)$ is the continuous function defined by equation (5.15). If $D_t[t^{-1}D_t(t^{-1}h(t))]$ is Lebesgue integrable over $(0,b)$, (5.17) can be written as

$$<f,g> = <t^{-\mu-\frac{1}{2}}D_t(t^{-1}D_t(t^{-1}h(t))),\ \phi(t)>, \tag{5.18}$$

for every $\phi \in B^0_{\mu,b}$.

Proof. The proof of (5.17) is very similar to the proof of [5, equation (9), p. 90-91] and (5.18) follows easily from (5.17).

We now generalize Theorem 5.7 for the case when $|<f,\phi>| \leq \rho^\mu_r(\phi)$, $r > 0$. For this we need the following:

Definition 5.8. For each non-negative integer n, we define the spaces $B^{(n)}_{\mu,b}$, $H^{(n)}_{\mu,b}$, $B^\infty_{\mu,b}$, and $H^\infty_{\mu,b}$ by

$$B^{(n)}_{\mu,b} = \{\phi \in B_{\mu,b} : \phi^{(n)}(x) = o(x^{\mu+3/2}) \text{ as } x \to 0+\}, \tag{5.19}$$

$$H^{(n)}_{\mu,b} = \{\eta(x) = \frac{\phi'(x)}{(\mu+\frac{1}{2})x} - \frac{\phi(x)}{x^2} : \phi \in B^{(n)}_{\mu,b}\}, \tag{5.20}$$

where $\phi^{(n)}(x) = D^n\phi(x)$ for $n \neq 0$ and $\phi^{(0)}(x) = \phi(x)$,

$$B^\infty_{\mu,b} = \{ \phi \in B_{\mu,b} : \phi^{(k)}(x) = o(x^{\mu+\frac{1}{2}}), \text{ as } x \to 0+, \text{ for each } k = 0,1,2,\ldots.\}, \tag{5.21}$$

and

$$H^\infty_{\mu,b} = \{\eta(x) = \frac{\phi'(x)}{(\mu+\frac{1}{2})x} - \frac{\phi(x)}{x^2} : \phi \in B^\infty_{\mu,b}\}. \tag{5.22}$$

Note that $H^{(1)}_{\mu,b} \subset B^{o}_{\mu,b}$ and $B^{o}_{\mu,b} = B^{(o)}_{\mu,b}$. Let $\phi_o \in B^{o}_{\mu,b}$ be such that

$$\int_0^b t^{-\mu+\frac{1}{2}}\phi_o(t)dt = 1. \tag{5.23}$$

In the subsequent development we shall need the following lemma, whose proof is omitted since it is similar to that of [5, Lemma 1, p.68].

Lemma 5.9. Let ϕ_o be a fixed testing function in $B^{o}_{\mu,b}$ satisfying (5.23). Then any testing function ϕ in $B^{o}_{\mu,b}$ may be decomposed uniquely according to

$$\phi = K\phi_o + \eta \tag{5.24}$$

where η is in $H^{(1)}_{\mu,b}$ and the constant K is given by

$$K = \int_0^b t^{-\mu+\frac{1}{2}}\phi(t)dt. \tag{5.25}$$

Suppose f is a regular generalized function in $B'_{\mu,b}$ generated by a differentiable function f such that f is Lebesgue integrable over $(0,b)$ and f' is bounded on $(0,b]$. Then for $\eta \in H^{(1)}_{\mu,b}$ we have

$$\begin{aligned}\langle f,\eta\rangle &= \int_0^b f(x)\,\eta(x)dx\\ &= \frac{1}{\mu+\frac{1}{2}}\int_0^b x^{\mu-\frac{1}{2}}f(x)\frac{d}{dx}[x^{-\mu-\frac{1}{2}}\phi(x)]dx \quad (\text{for some } \phi \in B^{(1)}_{\mu,b})\\ &= -\frac{1}{\mu+\frac{1}{2}}\int_0^b \frac{(x^{\mu-\frac{1}{2}}f(x))'}{x^{\mu+\frac{1}{2}}}\phi(x)dx,\end{aligned}$$

since $\phi(b) = 0$ and $\frac{\phi(x)}{x} = o(x^{\mu+3/2})$ as $x \to 0+$.

Therefore

$$\langle f,\eta\rangle = \frac{1}{\mu+\frac{1}{2}}\langle x^{-\mu-\frac{1}{2}}Dx^{\mu-\frac{1}{2}}f(x), -\phi(x)\rangle. \tag{5.26}$$

But

$$\eta = \frac{x^{\mu-\frac{1}{2}}}{\mu+\frac{1}{2}}Dx^{-\mu-\frac{1}{2}}\phi(x),$$

therefore,

$$\langle f(x), x^{\mu-\frac{1}{2}}Dx^{-\mu-\frac{1}{2}}\phi(x)\rangle = \langle x^{-\mu-\frac{1}{2}}Dx^{\mu-\frac{1}{2}}f(x), -\phi(x)\rangle. \tag{5.27}$$

Let

$$L_\mu = x^{\mu-\frac{1}{2}}Dx^{-\mu-\frac{1}{2}}, \tag{5.28}$$

and

$$T_\mu = x^{-\mu-\frac{1}{2}}Dx^{\mu-\frac{1}{2}} = I_\mu. \tag{5.29}$$

Then for any $\phi \in B^{(1)}_{\mu,b}$,

$$L_\mu\phi = x^{\mu+\frac{1}{2}}(x^{-1}D)(x^{-\mu-\frac{1}{2}}\phi(x)) \in B^{o}_{\mu,b},$$

and

$$\gamma^{\mu}_{k}(L_\mu\phi) = \gamma^{\mu}_{k+1}(\phi).$$

We write the above as

Lemma 5.10. The operation $\phi \to L_\mu\phi$ is a continuous linear mapping of $B^{(1)}_{\mu,b}$ into $B^{o}_{\mu,b}$.

For an arbitrary f in $B'_{\mu,b}$ and any ϕ in $B^{(1)}_{\mu,b}$, set

$$< T_\mu f,\phi > = < x^{-\mu-\frac{1}{2}} Dx^{\mu-\frac{1}{2}} f, \phi > = < f, -L_\mu\phi >. \tag{5.30}$$

From (5.30) we see that $T_\mu f$ is a linear functional on $B^{(1)}_{\mu,b}$. Write

$$x^{-\mu-\frac{1}{2}} Dx^{\mu-\frac{1}{2}} f = g.$$

We then have formally,

$$Dx^{\mu-\frac{1}{2}} f = x^{\mu+\frac{1}{2}} g.$$

Now define,

$$x^{-\mu+\frac{1}{2}} [x^{\mu+\frac{1}{2}} g]^{(-1)} = f. \tag{5.31}$$

With this notation (5.26) suggests that

$$< x^{-\mu+\frac{1}{2}} [x^{\mu+\frac{1}{2}} f]^{(-1)}, \eta > = \frac{1}{\mu+\frac{1}{2}} < f,-\phi >. \tag{5.32}$$

Equation (5.32) defines a linear functional on $H^{(1)}_{\mu,b}$. This can be extended to all of $B^{o}_{\mu,b}$ by using Lemma 5.9. Assign

$$< x^{-\mu+\frac{1}{2}} [x^{\mu+\frac{1}{2}} f]^{(-1)}, \phi_o > = K_o \text{ (arbitrary)}. \tag{5.33}$$

Then for any $\phi \in B^{o}_{\mu,b}$, we have from Lemma 5.9,

$$\phi = K\phi_o + \eta.$$

Hence the operator $x^{-\mu+\frac{1}{2}} [x^{\mu+\frac{1}{2}} f]^{(-1)}$ is defined for all of $B^{o}_{\mu,b}$.

<u>Proposition 5.11</u>. Suppose $f \in B'_{\mu,b}$ satisfies

$$|< f,\phi >| \le A \sup_{o\le k\le r} \gamma^{\mu}_{k} (\phi), \quad r \ge 1 \tag{5.34}$$

for all ϕ in $B^{o}_{\mu,b}$. Then

$$|< x^{-\mu+\frac{1}{2}} [x^{\mu+\frac{1}{2}}] f(x)]^{(-1)}, \phi(x) > | \le A \sup_{o\le k\le r-1} \gamma^{\mu}_{k} (\phi), \tag{5.35}$$

for all $\phi \in B^{o}_{\mu,b}$.

<u>Proof</u>. Using (5.32), the proof follows trivially for members of $H^{(1)}_{\mu,b}$. Now use Lemma 5.9 to complete the proof.

<u>Proposition 5.12</u>. For $f \in B'_{\mu,b}$ satisfying

$$|< f,\phi >| \le A \sup_{o\le k\le 1} \gamma^{\mu}_{k} (\phi), \quad \forall \phi \in B^{o}_{\mu,b},$$

we have

$$< f,\phi > = -\int_0^b h(t) (t^{-1}D_t)^3(t^{-\mu-\frac{1}{2}} \phi(t))dt, \tag{5.36}$$

for each ϕ in $B^{(1)}_{\mu,b}$ and where h is a continuous function.

<u>Proof</u>. We know that the theorem is true for $r = 0$. For $\phi \in B^{(1)}_{\mu,b}$,

$$< f,\phi > = -(\mu+\tfrac{1}{2}) < x^{-\mu+\frac{1}{2}} [x^{\mu+\frac{1}{2}} f]^{(-1)}, \eta >, \eta \in H^{(1)}_{\mu,b} \text{ (equation (5.32))}$$

$$= -(\mu+\tfrac{1}{2}) \int_0^b h(t)(t^{-1}D_t)^2(t^{-\mu-\frac{1}{2}}\eta(t))dt,$$

by Theorem (5.7), since

$$| < x^{-\mu+\frac{1}{2}}[x^{\mu+\frac{1}{2}} f]^{(-1)}, \eta > | \le A\ \gamma_0^\mu(\eta), \text{ (see Proposition (5.11))}$$

and

$$\eta(x) = o(x^{\mu+\frac{1}{2}}), \quad \text{as} \quad x \to 0+.$$

Therefore,

$$< f,\phi > = -(\mu+\tfrac{1}{2}) \int_0^b h(t)(t^{-1}D_t)^2[t^{-\mu-\frac{1}{2}} \cdot \frac{t^{\mu+\frac{1}{2}}}{\mu+\frac{1}{2}} \frac{1}{t} \frac{d}{dt}(t^{-\mu-\frac{1}{2}}\phi(t))]\, dt$$

$$= -\int_0^b h(t)(t^{-1}D_t)^3(t^{-\mu-\frac{1}{2}}\phi(t))dt.$$

If we replace both $B^0_{\mu,b}$ and $B^{(1)}_{\mu,b}$ by $B^\infty_{\mu,b}$ and $H^{(1)}_{\mu,b}$ by $H^\infty_{\mu,b}$, Lemmas 5.9, 5.10 and Propositions 5.11, 5.12 are still true. Hence by induction on r, we get

<u>Theorem 5.13</u>. For $f \in B'_{\mu,b}$ satisfying

$$|< f,\phi >| \le A \sup_{0\le k\le r} \gamma_k^\mu(\phi), \quad (\forall\ \phi \in B^0_{\mu,b})$$

for some non-negative integer r, we have

$$< f,\phi > = (-1)^r \int_0^b h(t)(t^{-1}D_t)^{r+2}[t^{-\mu-\frac{1}{2}}\phi(t)]dt, \tag{5.37}$$

for each $\phi \in B^0_{\mu,b}$ where h is a continuous function.

The above structure theorem helps us to say more about the finite Hankel transform of elements in $B'_{\mu,b}$.

Let $\phi \in B^\infty_{\mu,b}$ and F,Φ be the finite Hankel transforms of $f \in B'_{\mu,b}$ and ϕ respectively. Then

$$< F,\Phi > = < f,\phi >$$

$$= (-1)^{r-2} \int_0^b h(x) \int_0^\infty (y)^{\frac{1}{2}}\ \phi(y)(\frac{1}{x}\frac{d}{dx})^r(x^{-\mu}J_\mu(xy))dydx \tag{5.38}$$

(using equations (2.4) and (5.37)).

Since

$$(-1)^r(\frac{1}{x}\frac{d}{dx})^r(x^{-\mu}J_\mu(xy)) = y^r x^{-\mu-r}J_{\mu+r}(xy), \tag{5.39}$$

then

$$< F,\Phi > = \int_0^b h(x) \int_0^\infty (y)^{\frac{1}{2}}\phi(y)y^r x^{-\mu-r}J_{\mu+r}(xy)dydx$$

$$= \int_0^\infty y^r\ \Phi(y) \int_0^b \frac{h(x)}{\mu+\frac{1}{2}+r} \sqrt{xy}\ {}_x J_{\mu+r}(xy)dxdy. \tag{5.40}$$

Now let $g_r(x) = \dfrac{h(x)}{x^{\mu+\frac{1}{2}+r}}$.

Since

$$\frac{\sqrt{xy}\ J_{\mu+r}(xy)}{x^{\mu+\frac{1}{2}+r}} \sim \frac{(xy)^{\mu+r+\frac{1}{2}}}{x^{\mu+\frac{1}{2}+r}} \quad \text{as } x \to 0^+,$$

so that

$$G_r^{\mu+r}(y) = {}_0\int^b g_r(x)\ (xy)^{\frac{1}{2}} J_{\mu+r}(xy)dx$$

is well defined.

Thus (5.40) becomes

$$< F,\Phi > = {}_0\int^\infty y^r \Phi(y)\ G_r^{\mu+r}(y)dy$$

$$= < y^r\ G_r^{\mu+r}(y),\ \Phi(y) >$$

Consequently,

$$F = y^r\ G_r^{\mu+r}(y) \quad \text{(in the functional sense)}$$

is of slow growth since $|G_r^{\mu+r}(y)| < \infty$. We list the above as

Theorem 5.14. For any $F \in Y'_{\mu,b}$, there exists a continuous function $G(y)$ $(= y^r G_r^{\mu+r}(y))$ of slow growth such that $F|_{Y^\infty_{\mu,b}}$ is equivalent to $G(y)$, i.e.,

$$< F,\Phi > = < G(y),\Phi(y) >,$$

for $\Phi \in Y^\infty_{\mu,b}$, where $Y^\infty_{\mu,b} = h_\mu\ [B^\infty_{\mu,b}]$.

Furthermore, $G(y)$ may be extended to an analytic function.

6. FURTHER PROPERTIES OF THE FINITE HANKEL TRANSFORM

Notation: Let us write $\hat{f} = f|_{B^\infty_{\mu,b}}$, for any $f \in B'_{\mu,b}$, and likewise $\hat{F} = F|_{Y^\infty_{\mu,b}}$ for any $F \in Y'_{\mu,b}$. Also write $\hat{\phi}$ to denote the members of $B^\infty_{\mu,b}$ and $\hat{\Phi}$ for the members of $Y^\infty_{\mu,b}$.

For any $\hat{f}$, we have for some integer $r \geq 2$,

$$< \hat{f},\hat{\phi} > = (-1)^r\ {}_0\int^b h(x)\left(\frac{1}{x}\frac{d}{dx}\right)^r (x^{-\mu-\frac{1}{2}}\ \phi(x))dx,$$

where $h(x)$ is a continuous function.

Definition 6.1. We define a new space $H^\infty_\mu(I)$ by

$$H^\infty_\mu(I) = \{\phi \in H_\mu : \phi^{(k)}(x) = o(x^{\mu+3/2}) \text{ as } x \to 0^+, \text{ for } k = 0,1,2,\ldots\}, \tag{6.1}$$

where $I = (0,\infty)$.

Now define $h_b(x)$ to be the periodic extension of period b of $h(x)$ on $(0,b]$. Then for any $x \in R^+$, $h_b(x) = h(x-nb)$ for some positive integer n such that $0 < x-nb \leq b$. Associated with $h_b(x)$ is the regular distribution in $H^{\infty\prime}_\mu(I)$ having the value

$$< h_b\ (x),\ \phi(x) > = \sum_{n=0}^{\infty}\ {}_0\int^b h(x)\ \phi(x+nb)dx, \tag{6.2}$$

for any $\phi \in H^\infty_\mu(I)$.

The right-hand side of (6.2) converges, since ϕ is of rapid descent as $x \to \infty$. Now define a functional f_b on $H^\infty_\mu(I)$ by

$$< f_b,\phi > = (-1)^r \int_0^\infty h_b(x)(\tfrac{1}{x} D_x)^r (x^{-\mu-\frac{1}{2}}\phi(x))dx$$
$$= \sum_{n=0}^{\infty} (-1)^r [\int_0^\infty h(x-nb)(\tfrac{1}{x} D_x)^r \cdot (x^{-\mu-\frac{1}{2}}\phi(x))dx]$$
$$= \sum_{n=0}^{\infty} (-1)^r \int_{nb}^{(n+1)b} h(x-nb)(\tfrac{1}{x} D_x)^r (x^{-\mu-\frac{1}{2}}\phi(x))dx. \qquad (6.3)$$

This defines a linear continuous functional on $H_\mu^\infty(I)$. Also for $\hat{\phi}$ in $B_{\mu,b}^\infty$, we have

$$< f_b,\hat{\phi} > = (-1)^r \int_0^b h(x)(\tfrac{1}{x} D_x)^r (x^{-\mu-\frac{1}{2}}\phi(x))dx, \quad (\text{since } \hat{\phi} = 0 \text{ for } x > b)$$
$$= < \hat{f},\hat{\phi} >.$$

So we see that f_b is a periodic extension of f.

Theorem 6.2. Every $\hat{f}$ in $B'_{\mu,b}$ may be extended to a periodic linear functional, with period b, on $H_\mu^\infty(I)$ which is continuous in the topology of H_μ.

Theorem 6.3. For every ε in $(0,b/4)$, and each f in $B'_{\mu,b}$, the function

$$F_\varepsilon(y) = < f(x), \lambda_\varepsilon(x)\ (xy)^{\frac{1}{2}}\ J_\mu(xy) >, \qquad (6.4)$$

where $\lambda_\varepsilon(x)$ is defined by (2.7), is a smooth function of slow growth and defines a regular generalized function in $Y'_{\mu,b}$.

Proof. The proof of the above theorem is similar to the proof of [1, Lemma 12] given by Zemanian.

Theorem 6.4. The finite Hankel transform, $h_\mu f$, of a generalized function f in $B'_{\mu,b}$ is the limit, as $\varepsilon \to 0$, of the family $F_\varepsilon(z)$ of regular generalized functions defined in Theorem 6.3.

Proof. Since $F_\varepsilon(z)$ is a regular functional in $Y'_{\mu,b}$, it is sufficient to show that

$$< h_\mu f,\Phi > = < F_\varepsilon,\Phi >$$

for each Φ in $Y_{\mu,b}$, as $\varepsilon \to 0$. For each Φ in $Y_{\mu,b}$ there exists a unique ϕ in $B_{\mu,b}$ given by Equation (2.4). As $\varepsilon \to 0+$, $\lambda_\varepsilon(x) = 1$ on $(0,b)$.

Now we have

$$< F_\varepsilon,\Phi > = << f(x), \lambda_\varepsilon(x)(xy)^{\frac{1}{2}}J_\mu(xy) >, \Phi(y) >$$
$$= \int_0^\infty < f(x), \lambda_\varepsilon(x)(xy)^{\frac{1}{2}}J_\mu(xy) > \Phi(y)dy$$
$$= < f(x), \lambda_\varepsilon(x) \int_0^\infty \Phi(y)(xy)^{\frac{1}{2}}J_\mu(xy)dy > \quad (\text{by [8, Corollary 5.3.2b, p. 121]})$$
$$= < f(x), \lambda_\varepsilon(x)\ \phi(x) >$$
$$\to <f(x),\phi(x) >, \quad \text{as } \varepsilon \to 0+.$$

Consequently, $< F_\varepsilon, \Phi > \to < f(x), \phi(x) > = < h_\mu f, \Phi >$, as $\varepsilon \to 0+$, for each $\Phi \in Y_{\mu,b}$.

Since $\lambda_\varepsilon(x)(xy)^{\frac{1}{2}}J_\mu(xy) \to \chi_{(0,b)}(x)(xy)^{\frac{1}{2}}J_\mu(xz)$, as $\varepsilon \to 0+$, (where $\chi_{(0,b)}$ is the characteristic function of the interval $(0,b)$) and the latter is not a testing function in $B_{\mu,b}$, it is not true, in general, that the limit of $F_\varepsilon(z)$, as $\varepsilon \to 0$, exists as an ordinary function. Where the limit does exist as an ordinary function, it will be denoted by $F_0(z)$.

Corollary 6.5. If f is a regular generalized function in $B'_{\mu,b}$, then the limiting function $F_0(z)$ exists and is equivalent to the finite Hankel transform of f.

Proof. If f is regular, then for each ε in $(0, b/4)$,

$$F_\varepsilon(z) = {}_0\int^b f(x)\lambda_\varepsilon(x)(xy)^{\frac{1}{2}} J_\mu(xz)dx.$$

As $\varepsilon \to 0^+$, we obtain

$$F_0(z) = {}_0\int^b f(x)(xy)^{\frac{1}{2}}J_\mu(xz)dx, \tag{6.5}$$

and from Example 4, we see that $F_0(z)$ is equivalent to $h_\mu f$.

Using the function $\chi_{(0,b)}(x)$, (6.5) can be written as

$$F_0(z) = < \chi_{(0,b)}(x)f(x), \lambda(x) (xz)^{\frac{1}{2}}J_\mu(xz) >,$$

where $\lambda(x)$ is a testing function in $D(R)$ such that $\lambda(x) = 1$ on $(0,b]$. In this case, to calculate the finite Hankel transform we merely truncate the regular distribution in $B'_{\mu,b}$ at $x = b$, as would be expected. In a similar way one might interpret the limit as $\varepsilon \to 0+$ of the family of the functions $F_\varepsilon(z)$ as a process of truncation for distributions in general, for one is replacing $f(x)$ by the distributional limit $\lambda_\varepsilon(x) f(x)$ as $\varepsilon \to 0+$.

Corollary 6.6. If the generalized function $f \in B'_{\mu,b}$ has support in $(0,b]$, then the function $F_0(z)$ exists and is equivalent to the finite Hankel transform of f.

Proof. Let $\lambda(x)$ be a testing function in $D(I)$ such that $\lambda(x) = 1$ on a neighborhood of the support of f. Then

$$F_\varepsilon(z) = < f(x), \lambda_\varepsilon(x) \lambda(x)(xz)^{\frac{1}{2}}J_\mu(xz) >$$

such that,

$$\lim_{\varepsilon\to 0+} F_\varepsilon(z) = F_0(z) = < f(x), \lambda(x)(xz)^{\frac{1}{2}}J_\mu(xz) >,$$

since $\lambda_\varepsilon(x) = 1$ on the support of f as $\varepsilon \to 0+$. But Zemanian [1, Theorem 2] has proved that

$$h_\mu f = < f(x), \lambda(x)(xz)^{\frac{1}{2}}J_\mu(xz) >,$$

for every functional in H'_μ having compact support.

Example 8. For the distribution $\delta(x-k)$, from (6.4) we obtain

$$\begin{aligned} F_\varepsilon(z) &= < \delta(x-k), \lambda_\varepsilon(x)(xz)^{\frac{1}{2}}J_\mu(xz) > \\ &= \lambda_\varepsilon(k)(kz)^{\frac{1}{2}}J_\mu(kz). \end{aligned}$$

But for $0 < k < b$, $\lambda_\varepsilon(k) = 1$ as $\varepsilon \to 0+$. And for $k \geq b$, $\lambda_\varepsilon(k) = 0$. Therefore,

$$F_0(z) = \begin{cases} \sqrt{kz}\, J_\mu(kz), & \text{for } 0 < k < b \\ 0, & \text{for } k \geq b, \end{cases}$$

$$= h_\mu\,[\delta(x-k)].$$

7. THE FOURIER-BESSEL SERIES

Classically the inverse finite Hankel transform is considered to be the Fourier-Bessel series

$$\frac{2}{b^2} \sum_{n=1}^{\infty} (x/\lambda_n)^{\frac{1}{2}} \frac{J_\mu(x\lambda_n)}{J^2_{\mu+1}(b\lambda_n)} \cdot \Phi(\lambda_n), \tag{7.1}$$

where

$$\Phi(\lambda_n) = {}_0\!\int^b \phi(x)(x\lambda_n)^{\frac{1}{2}} J_\mu(x\lambda_n)\, dx, \tag{7.2}$$

is the finite Hankel transform [7] of some function ϕ.

In section 2, we showed that for any $\phi \in B_{\mu,b}$ and $\Phi(\lambda_n)$ given by (7.2), we have

$$\phi(x) = \frac{2}{b^2} \sum_{n=1}^{\infty} \lambda_\varepsilon(x)\left(\frac{x}{\lambda_n}\right)^{\frac{1}{2}} \cdot \frac{J_\mu(x\lambda_n)}{J^2_{\mu+1}(b\lambda_n)}\, \Phi(\lambda_n), \quad \text{as } \varepsilon \to 0. \tag{7.3}$$

We obtain an inversion theorem similar to (7.1) for generalized finite Hankel transforms of members in $B'_{\mu,b}$.

Theorem 7.1 (inversion). Let f be an arbitrary generalized function in $B'_{\mu,b}$, where $\mu \geq -\frac{1}{2}$. Let $F = h_\mu(f)$, be the finite Hankel transform. Then in the sense of convergence in $B'_{\mu,b}$, we have

$$f(x) = \lim_{N\to\infty} \frac{2}{b^2} \sum_{n=1}^{\infty} \left(\frac{x}{\lambda_n}\right)^{\frac{1}{2}} [J_\mu(x\lambda_n)/J^2_{\mu+1}\,(b\lambda_n)]F(\lambda_n). \tag{7.4}$$

Proof. Let $\lambda(x)$ be an arbitrary testing function in $B_{\mu,b}$. We wish to prove that

$$< \frac{2}{b^2} \sum_{n=1}^{N} \left(\frac{x}{\lambda_n}\right)^{\frac{1}{2}} \frac{J_\mu(x\lambda_n)}{J^2_{\mu+1}(b\lambda_n)} F(\lambda_n), \phi(x) > \to < f(x), \phi(x) >, \text{ as } N \to \infty.$$

Since $(x)^{\frac{1}{2}} J_\mu(x\lambda_n)$ is locally integrable over $(0,b)$,

$$< \frac{2}{b^2} \sum_{n=1}^{N} \left(\frac{x}{\lambda_n}\right)^{\frac{1}{2}} \frac{J_\mu(x\lambda_n)}{J^2_{\mu+1}(b\lambda_n)} F(\lambda_n), \phi(x) >$$

$$= {}_0\!\int^b \frac{2}{b^2} \sum_{n=1}^{N} \frac{F(\lambda_n)}{J^2_{\mu+1}(b\lambda_n)} \left(\frac{x}{\lambda_n}\right)^{\frac{1}{2}} \cdot J_\mu(x\lambda_n)\, \phi(x)dx$$

$$= \sum_{n=1}^{N} \frac{2}{b^2} \frac{F(\lambda_n)}{\lambda_n J^2_{\mu+1}\,(b\lambda_n)} \Phi(\lambda_n), \quad \text{(from (2.2))}$$

$$= \sum_{n=1}^{N} \frac{2}{b^2} \frac{\Phi(\lambda_n)}{\lambda_n J^2_{\mu+1}(b\lambda_n)} \lim_{\varepsilon \to 0+} < f(x), \lambda_\varepsilon(x)(x\lambda_s)^{\frac{1}{2}} J_\mu(x\lambda_n) >$$

(from Theorem 6.4)

$$\to < f(x), \phi(x) >, \quad \text{as} \quad N \to \infty.$$

We verify this inversion Theorem by means of a numerical example.

Example 9. For $0 < k < b$, $\delta(x-k) \varepsilon E'(I) \subset H'_\mu \subset B'_{\mu,b}$. The finite Hankel transform of $\delta(k-x)$ is,

$$h_\mu \delta(x-k) = (kz)^{\frac{1}{2}} J_\mu(kz) = F(z).$$

Hence,

$$F(\lambda_n) = (k\lambda_n)^{\frac{1}{2}} J_\mu(k\lambda_n), \quad n = 1,2,3,\ldots.$$

Now

$$< \frac{2}{b^2} \sum_{n=1}^{N} (\frac{x}{\lambda_n})^{\frac{1}{2}} \frac{J_\mu(x\lambda_n)}{J^2_{\mu+1}(b\lambda_n)} (k\lambda_n)^{\frac{1}{2}} J_\mu(k\lambda_n), \phi(x) >$$

$$= \frac{2}{b^2} \sum_{n=1}^{N} \frac{J_\mu(k\lambda_n)}{(\lambda_n)^{\frac{1}{2}} J^2_{\mu+1}(b\lambda_n)} (k)^{\frac{1}{2}} \int_0^b (x\lambda_n)^{\frac{1}{2}} J_\mu(x\lambda_n) \phi(x) dx$$

$$= \frac{2}{b^2} \sum_{n=1}^{N} \frac{J_\mu(k\lambda_n)}{(\lambda_n)^{\frac{1}{2}} J^2_{\mu+1}(b\lambda_n)} (k)^{\frac{1}{2}} \Phi(\lambda_n)$$

$$= \phi(k)$$

$$= < \delta(x-k), \phi(x) >.$$

This also yields

$$\delta(x-k) = \lim_{N \to \infty} \frac{2}{b^2} \sum_{n=1}^{N} \sqrt{kx} \frac{J_\mu(k\lambda_n) J_\mu(x\lambda_n)}{J^2_{\mu+1}(b\lambda_n)}$$

in the sense of convergence in $B'_{\mu,b}$.

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REFERENCES

1. Zemanian, A. H.. A distributional Hankel transform, Siam J. App. Math., vol. 14, No. 3, 1966, pp. 561-576.
2. Zemanian, A. H.. The Hankel transformation of certain distributions of rapid growth, Siam J. Appl. Math., vol. 14, No. 4, 1966.
3. Schwartz, L.. Theorie des distributions, Herman, Paris, 1978.
4. Zemanian, A. H.. Generalized integral transformations, Interscience Publishers, New York, 1968.
5. Zemanian, A. H.. Distribution theory and transform analysis, McGraw-Hill, New York, 1965.
6. Sneddon, I. N.. The use of integral transforms, McGraw-Hill Book Company, 1972.
7. Watson, G. N.. Theory of Bessel functions, 2nd ed., Cambridge University Press, 1958.

8. Schwartz, L.. Mathematics for the physical sciences, Addison Wesley Publishing Company.

9. Carmichael, R. D.. Analytic representation of the distributional finite Fourier transform, Siam J. Math Anal. 5(5), 1974, 737-761.

10. Warmbrod, G. K.. The distributional finite Fourier transform, Siam J. Appl. Math, 17(5), 1969, 930-956.