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TWO LARGE SUBSETS OF A FUNCTIONAL SPACE

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ABSTRACT. Let P_1 denote the Banach space composed of all bounded derivatives f of everywhere differentiable functions on $[0,1]$ such that the set of points where f vanishes is dense in $[0,1]$. Let D_0 consist of those functions in P_1 that are unsigned on every interval, and let D_1 consist of those functions in P_1 that vanish on dense subsets of measure zero. Then D_0 and D_1 are dense G_δ-subsets of P_1 with void interior. Neither D_0 nor D_1 is a subset of the other.

KEY WORDS AND PHRASES. Banach space of functions, derivative dense G_δ*-subset.*
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1. INTRODUCTION.

The real vector space D of all bounded derivatives of everywhere differentiable functions on $[0,1]$ is a Banach space [1] under the norm

$$\| f \| = \sup_{0 \le x \le 1} |f(x)|.$$

(At the endpoints 0 and 1 we require that the one sided derivatives exist.) Tibor Salat essentially proved [1] that the set

$$D_0 = \{f \in D: \ f \text{ is unsigned on any interval}\}$$

is a nowhere dense subset of D. To do this, he observed that

$$P_1 = \{f \in D: f = 0 \text{ on a dense subset of } [0,1]\}$$

is a nowhere dense Banach subspace of D and $D_0 \subset P_1$. Since P_1 is a Banach space in its own right, it is natural to study D_0 as a subset of P_1. Put

$$D_1 = \{f \in P_1: f \neq 0 \text{ almost everywhere on } [0,1]\}.$$

In this note, we prove that D_0 and D_1 are "large" subsets of the Banach space P_1.

THEOREM 1. D_0 is a dense G_δ-set in P_1 with void interior.

THEOREM 2. D_1 is a dense G_δ-set in P_1 with void interior.

Now put $E = \{f \in P_1 : f$ is not almost everywhere discontinous$\}$.

If $f \in P_1$, then f must vanish at every point where f is continuous, and $E \subset P_1 \setminus D_1$. We obtain from Theorem 2, a result of Clifford Weil [2].

COROLLARY 1 (C. Weil). E is a first category subset of P_1.

Clifford Weil [3] proved most of Theorem 1. Finally, we show that neither of the sets D_0 or D_1 is a subset of the other, and we prove an analogue of Theorem 2 for spaces of nonnegative derivatives. (I take this opportunity to thank the referee for simplifying many of my arguments.)

Proof of Theorem 1. The proof that D_0 is a dense G_δ-set in P_1 is essentially given in [3], so we leave it.

It remains to prove that D_0 has void interior. Let $f \in P_1$ and $\varepsilon > 0$. It is easy to find an interval $[a,b]$ for which $f(a) = f(b) = 0$ and $|f(x)| < \varepsilon$ for $x \in [a,b]$. Now let $g(x) = f(x)$ if $x \notin [a,b]$, and $g(x) = 0$ if $x \in [a,b]$. Clearly $g \in P_1$ but $g \notin (D_0 \cup D_1)$. Finally, $\|f-g\| \le \varepsilon$. Thus every open set in P_1 contains functions $\notin D_0 \cup D_1$, so $D_0 \cup D_1$ has void interior in P_1. □

Proof of Theorem 2. For each positive integer n, define

$$E_n = \{f \in P_1 : m\{x : f(x) = 0\} \ge n^{-1}\}.$$

Then $D_1 = P_1 \setminus \bigcup_n E_n$. We claim that each E_n is closed in P_1. Let $f_k \in E_n$ and $f \in P_1$ and $\lim_{k\to\infty} \|f_k - f\| = 0$. Say

$$A_k = \{x : f_k(x) = 0\}$$

and $mA_k \ge n^{-1}$. Then at each $x \in A = \bigcap_j \bigcup_{k\ge j} A_k$, $f(x) = 0$. But $mA \ge n^{-1}$ so $f \in E_n$. Thus E_n is closed in P_1.

It remains to prove that E_n is nowhere dense. Let $f \in E_n$ and $\varepsilon > 0$. Use [4] to get a function $g \in D_1$ such that $0 \le g \le 1$. But there is a number $c > 0$ such that $c < \varepsilon$ and $m\{x : 0 < |f(x)| \le c\} < n^{-1}$. It follows that

$$m\{x : f(x) = -cg(x)\} < n^{-1}.$$

Finally,

$$m\{x : f(x) + cg(x) = 0\} < n^{-1}$$

and $f + cg \notin E_n$. Moreover, $\|(f + cg) - f\| = \|cg\| \le c < \varepsilon$. Thus E_n is nowhere dense.

In the proof of Theorem 1 we saw that $D_0 \cup D_1$, and hence D_1, has void interior. □

It follows from Theorems 1 and 2 that $D_0 \cap D_1$, the set of all functions in P_1 that vanish on dense sets of measure zero and are unsigned in any interval, is a dense G_δ-subset of P_1. Next we show that the sets D_0 and D_1 are quite different. Neither is a subset of the other.

THEOREM 3. The Sets $D_0 \setminus D_1$ and $D_1 \setminus D_0$ are nonvoid.

PROOF. Let h be a function in D_0. We construct a sequence of intervals (a_n, b_n), with mutually disjoint closures, such that $b_n - a_n < 2^{-n}$ and $h(a_n) = h(b_n) = 0$ for each n, and $\bigcup_n (a_n, b_n)$ is dense in $[0,1]$. Let $h_n = h\psi_{(a_n,b_n)}$ where ψ means characteristic function: for $0 \le x \le 1$, $h_n(x) = h(x)\,\psi_{(a_n,b_n)}(x)$. It follows that $h_n \in P_1$, and the sequence $(\|h_n\|)$ is bounded. Put

$$f = \Sigma_n 2^{-n} h_n \in P_1.$$

Now f cannot be signed on any subinterval of an (a_n,b_n), so $f \in D_0$. Clearly for $x \in [0,1]\setminus\cup_n(a_n,b_n)$, $h_n(x) = 0$ for all n and $f(x) = 0$. But

$$m\{[0,1]\setminus\cup_n(a_n,b_n)\} = 1 - \Sigma_n(b_n-a_n) > 0.$$

Thus $f \in D_0\setminus D_1$.

We use [4] to obtain a function in $D_1\setminus D_0$. □

Now put

$$P_2 = \{f \in P_1\text{: } f \text{ is nonnegative}\},$$

$$D_2 = \{f \in P_2\text{: } f > 0 \text{ almost everywhere on } [0,1]\}.$$

Then P_2 is a complete metric space in its own right. We conclude by showing that D_2 is a "large" subset of P_2.

THEOREM 4. D_2 is a dense G_δ-subset of P_2 with void interior.

PROOF. Define E_n as in the proof of Theorem 2. Then $E_n \cap P_2$ is a closed subset of P_2. It remains to prove that $D_2 = P_2\setminus\cup_n(E_n \cap P_2)$ is dense in P_2. We use [4] to construct $g \in D_2$ such that $0 \le g \le 1$. For any $f \in P_2$ and any $c > 0$, we have $f + cg \in D_2$ and $\|(f + cg) - f\| = \|cg\| \le c$. So D_2 is a dense G_δ-subset of P_2.

That D_2 has void interior follows from the same proof (for Theorem 1) that D_0 has void interior, so we leave this point. □

Compare Theorem 4 to the work in [5]. There it is shown that the singular functions form a dense G_δ-subset of the complete metric space of continuous nondecreasing functions on [0,1] under the sup metric. When the primitives of the functions in P_2 are taken, [5] suggests that D_2 is a "small" subset of P_2. Of course the metric used in [5] was different from the one used here.

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