FINITE PROPAGATION SPEED AND
KERNELS OF STRICTLY ELLIPTIC OPERATORS

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ABSTRACT. We establish estimates of the resolvent and other related kernels
and discuss L^p-theory for a class of strictly elliptic operators on $\mathbb{R}^n$.
The class of operators considered in the paper is of the form $A_0 + B$ with the
leading elliptic part A_0 and a "singular" perturbation B, whose coefficients
have L^p-type and are modeled after Schrödinger operators.

1. INTRODUCTION.

In this paper we shall study the resolvent kernel $R_\zeta(x,y) = (\zeta-A)^{-1}$
and other related "functions of A" for a class of strictly elliptic operators
$A_0 = \sum a_\alpha(x)D^\alpha$ on $\mathbb{R}^n$ and their perturbations $A_0 + B$.

We are particularly interested in L^p-theory of elliptic operators,
$1 \leqslant p \leqslant \infty$, and typical problems that arise here are spectral properties of A,
closedness, essential selfadjointness, accretivity, semigroup generation, etc.
Underlying all those is the question of existence and estimates of the resolvent
kernel R_ζ.

Typically the estimate has the form of a convolution-type bound

$$|R_\zeta(x,y)| \leqslant c(\zeta)H(|x-y|)$$

with an L^1-radial decreasing function H.

In the uniformly elliptic case kernels of the resolvent $R_\zeta = (\zeta-A)^{-1}$ and
semigroup $K_t = e^{-tA}$ were studied extensively ([1], [2], [3] et al.). In
particular, Eidelman [1] derived the following radial bound of K_t

$$|K_t(x,y)| \leqslant ct^{-n/m} \exp\left(- a\frac{|x-y|^{m'}}{m\sqrt{t}}\right) ;$$

where $m = $ order A, $m' = \frac{m}{m-1}$, and I. Gelfand and G. Shilov [2] applied this estimate to study generalized eigenfunction expansions of A.

In [4], [5] we obtained similar estimates in a different way starting with the resolvent kernel. The latter approach allows us to treat perturbations of A, and consequently operators with "nonregular" coefficients. We applied these results to L^p-spectral theory of elliptic operators.

In the present paper we extend the results of [4] to a class of strictly elliptic operators with possibly unbounded coefficients, obtained by linear deformations of uniformly elliptic symbols in the ξ-variable, $a = a(x,\xi) \to \tilde{a}(x,\delta_x(\xi))$. Here $x \to \delta_x$ means a matrix function $(\delta_{ij}(x))$ depending smoothly on $x \in \mathbf{R}^n$.

Natural examples of such deformations are second order elliptic operators $A = \sum a_{ij}(x)D_{ij}^2$. Indeed any quadratic form $a(x,\xi) = \sum a_{ij}(x)\xi_i\xi_j$ can be viewed as a deformation of the simplest one: $|\xi|^2 \to |\delta_x(\xi)|^2$ with $\delta_x = (a_{ij}(x))^{\frac{1}{2}}$. Other examples appear as right (left) invariant operators on nilpotent Lie groups (see [6]).

For the sake of presentation we shall restrict ourselves to the simplest case of deformations when $\delta_x = \delta(x)I$ is scalar ("conformal dilations"). Such transformations correspond to a multiplication of the leading symbol of A with a positive function $\delta(x)^m$, $m = $ order of A. The dilating factor $\delta(x)$ is subject to a certain constraint, called "finite propagation speed" condition. Its precise definition is given in § 2. This condition limits the rate of growth of δ at ∞, which can not exceed $O(|x|)$. In other words leading coefficients of A can not grow faster than $|x|^m$ for m-th order operators.

Let us observe that growth restrictions on the leading coefficients are well known in both ordinary (Sturm-Liouville) theory, $A = -a(x)\frac{d^2}{dx^2} + \ldots$ (see for instance [7], Ch. 9) and also for partial differential operators $A = \sum a_{ij}(x)D_{ij}^2 + \ldots$ ([8]). A sufficient condition for A to be well defined (essentially selfadjoint or Weyl's "limit circle case") is the divergence of the integral

$$\int^{+\infty} \frac{dx}{\sqrt{a(x)}} \quad \text{on} \quad \mathbf{R}; \quad \int^{+\infty} \frac{dr}{\sqrt{a(r)}}; \quad (a(r) = \sup_{|x| \leqslant r} \|a_{ij}(x)\|) \quad \text{on} \quad \mathbf{R}^n . \qquad (1.2)$$

The latter condition was called in [8] "finite propagation speed." Physically, $\|a(x)\|$ can be interpreted as a "local propagation speed," then integrals (1.2) measure the "amount of time required to get from a finite point to ∞ ".

It is not clear to what extent (1.2) is necessary to get a well defined operator A. Results of [9], [10], [11] indicate that it might be "too restrictive," yet in special cases ([12], [13]) it proved to be necessary in a certain sense.

The form of "finite propagation speed" used in our paper is close but somewhat stronger than (1.2). It is expressed in terms of derivatives of $\delta(x)$ rather than integrals. Also the way we apply it in our setting differs from the standard method of "hyperbolic wave propagators" (cf. [8], [14]).

Under assumptions of § 2 we show that m^{-th} order elliptic operators are well defined in different L^p-spaces $(1 < p \leqslant \infty)$ and their kernels (resolvent, semigroup, etc.) share many properties of "constant coefficient" and "uniformly elliptic" kernels (cf. [15], [16], [5], [4]). There is one notable exception however: the radial bound (1) central for the argument of [5], [4] is no longer valid. Instead we utilize another Fourier analytic tool – The <u>Hardy-Littlewood maximal function</u>:

$$f \rightarrow f^*(x) = \sup_{r>0} \frac{c_n}{r^n} \int_{|y| \leqslant r} |f(x+y)| dy \quad . \tag{1.3}$$

Maximal functions appear in this context when "deformations" of symbols (of resolvent and related operators) are translated into "deformations" of kernels. As a result one gets kernels of the type

$$K_\delta(x,y) = \delta^{-n}(x) K(\frac{x-y}{\delta(x)}) \; ;$$

where K is L^1-radial, $\delta(x) > 0$ – the dilating factor. If $\delta(x) \leqslant c$ is bounded (uniformly elliptic case) K_δ is estimated by the usual "L^1-dilation"

$$K_\delta(x,y) \leqslant \frac{1}{c^n} K(\frac{x-y}{c}) \quad . \tag{1.4}$$

In our case (unbounded $\delta(x)$) K_δ has no longer a single convolution type bound (1.4). But the maximal function (1.3) applies to give

$$(K_\delta f)(x) \leqslant \| K \|_1 f^*(x) \quad .$$

The maximal operator $f \rightarrow f^*$ is well known to be L^p $(1 < p < \infty)$ (see for inst. [17]), which yields a bound for the L^p-operator norm of K_δ. Such bounds are used throughout the paper as a substitute for the radial bound (1.1). Otherwise our argument is similar to [4].

First we construct the "free" resolvent $R^0 = (\zeta - A_0)^{-1}$, A_0 is the leading (homogeneous) part of A, via the "parametrix series"

$$R^0_\zeta = K_\zeta \sum_0^\infty L^k_\zeta = K_\zeta (I - L_\zeta)^{-1} \quad . \tag{1.5}$$

Here K is a pseudodifferential operator (ψDO) with symbol $\frac{1}{\zeta - a(x,\xi)}$, $a(x,\xi)$ being the leading symbol of A, and L a ψDO of order $\leqslant -1$, whose symbol is computed explicitly (§ 4).

The "perturbed" resolvent $R = (\zeta - A)^{-1}$ is constructed via the perturbation series

$$R_\zeta = \sum_0^\infty R^0_\zeta (B R^0_\zeta)^k \quad . \tag{1.6}$$

The coefficients of perturbation $B = \sum b_\alpha(x)D^\alpha$ are taken in L^p-classes. Such classes are well known in the theory of Schrödinger operators $-\Delta + V(x)$ (cf. [18]) and were studied for higher order operators in [15], [16], [5], [4]. In our case $b_\alpha(x)$ can be allowed certain growth at $\{\infty\}$ depending on $\delta(x)$. Such conditions are naturally formulated in terms of weighted L^p-spaces, L^p_w, with weights w, that are fractional powers of δ. Also the L^p-domain of A_0 and A can be characterized as a weighted L^p-Sobolev space (Theorem 2).

The central result of the paper, Lemma 1, analogous to Lemma 1 of [5], gives conditions (in terms of L^p-classes of coefficients) for relative boundedness of B with respect to A_0 and estimates the norm of BK_ζ. This result applies in § 4 to sum series(1.5)and(1.6),and to derive different corollaries. Among them we get

    a) L^p-closedness of $A = A_0 + B$ and bounds on its L^p-spectrum;

    b) a priori estimates and essential selfadjointness of A in L^2;

    c) resolvent summability, i.e., convergence $\zeta R_\zeta f(x) \to f(x)$ in L^p and a.e.

    d) existence of a strongly continuous holomorphic semigroup $\{e^{-tA}\}_{\mathrm{Re}\ t > 0}$.

Let us remark that due to the maximal function techniques adopted in the paper the L^1-space is excluded from consideration. The L^1-theory of integral kernels $K_\delta(x,y) = \delta(x)^{-n}K(\delta(x)^{-1}(x-y))$ with $K \in L^1$ poses an interesting harmonic analysis question. At the end of § 4 we give two examples which have bearing on L^1-theory.

In conclusion let us mention that the basic L^2-theory of elliptic operators ([19], [20], [3]) was extended later in a fairly general setting of pseudodifferential calculus ([21], [9], [22], [6]). But much less is known about L^p, $p \neq 2$. Two recent works that discuss specifically L^p-theory are those by Nagel and Stein [6] and Beals [23]. Both methods allow to treat a variety of "nonelliptic problems" (subelliptic Laplacians, degenerate elliptic operators, etc.), but remain entirely within the pseudodifferential framework in the sense that symbols (coefficients) are assumed sufficiently smooth.

Another trend ([15], [16], [5]) was to explore what "amount of nonregularity" of coefficients yields "well defined" L^p-operators. The present paper continues the latter trend, but our approach seems to be extendable to various "nonelliptic" situations as well. This extension will be discussed elsewhere.

2. THE CLASS OF OPERATORS.

Throughout this paper we shall consider operators of the form $A = A_0 + B$ with the leading elliptic part $A_0 = \sum\limits_{|\alpha|=m} a_\alpha(x)D^\alpha$ and a "singular" perturbation $B = \sum\limits_{|\alpha|\leqslant m} b_\alpha(x)D^\alpha$.

The following assumptions will be made on A_0 and B.

I. The leading part A_0 is assumed to be homogeneous of degree m and have a real symbol $a(x,\xi)$ obtained by the "deformation" of a uniformly elliptic symbol $\tilde{a}(x,\xi)$

$$a(x,\xi) = \tilde{a}(x,\delta(x)\xi) = \delta^m(x)\tilde{a}(x,\xi) .$$

Uniform ellipticity as usual means that

$$C_1 |\xi|^m \leq \tilde{a}(x,\xi) \leq C_2 |\xi|^m, \quad \text{all} \quad \xi .$$

The dilating factor $\delta(x)$ and the coefficients of $\tilde{a}(x,\xi) = \sum \tilde{a}_\alpha(x)\xi^\alpha$ are subject to the following constraints

$$|\delta^{(\beta)}(x)| \leq \text{const } \delta^{1-|\beta|}; \quad |\beta| \leq m \tag{2.1}$$

$$|\tilde{a}_\alpha^{(\beta)}(x)| \leq \text{const } \delta^{-|\beta|}; \quad |\beta| \leq m \tag{2.2}$$

Here and elsewhere $u^{(\beta)}$ denote a partial derivative $D^\beta u$. Condition (2.1) is close but somewhat stronger than the "finite propagation speed" condition [8]. The latter adjusted for $m^{-\text{th}}$ order operators of the above type requires

$$\int^{+\infty} \frac{dr}{\delta(r)} = +\infty, \quad \delta(r) = \sup_{|x| \leq r} \delta(x) \tag{2.3}$$

whereas (2.1) implies

$$|\frac{\delta'}{\delta}| \leq \frac{C}{\delta} \Rightarrow \int^{+\infty} \frac{dr}{\delta(r)} \geq \ln\delta|^{+\infty} = +\infty .$$

Both conditions do not allow $\delta(x)$ to grow faster than $0(|x|)$ as $x \to \infty$.

II. Perturbation $B = \sum\limits_{|\alpha| \leq m} b_\alpha(x)D^\alpha$ has coefficients with local L^p-type singularities, $b_\alpha \in L^{p_\alpha}_{\text{loc}}$. For each term $b_\alpha(x)D^\alpha$ of B we introduce its "fractional order"

$$d_\alpha = \frac{n}{p_\alpha} + |\alpha| \quad \text{and require} \quad d_\alpha \leq m . \tag{2.4}$$

The latter is needed in order to have B bounded relative to A_0.

We also need some control of b_α at $\{\infty\}$, depending on δ. A convenient way to express it is in terms of weighted L^p-spaces, $L^p_w = \{f: \int |fw|^p dx < \infty\}$. Precisely, we take $b_\alpha \in L^{p_\alpha}_w$ with the weight

$$w = w_\alpha = \delta^{-d_\alpha} . \tag{2.5}$$

Let us notice that the above class of operators is closed under the adjunction $A \to A^*$, provided the coefficients of B are sufficiently smooth, $(b_\alpha \in \mathcal{L}^p_{|\alpha|,w}$ - the L^p-weighted Sobolev space of order $|\alpha|)$. Indeed the adjoint of $a_\alpha(x)D^\alpha$ is

$$D^\alpha(a_\alpha \cdot) = \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} a_\alpha^{(\alpha-\beta)} D^\beta .$$

Remembering that $a_\alpha = \delta^m \tilde{a}_\alpha$ with $\tilde{a}_\alpha$ and δ subject to (2.1) - (2.2), we estimate the β-order coefficient of the adjoint

$$a_\alpha^{(\alpha-\beta)} = \sum_{\gamma'+\gamma=\alpha-\beta} (\delta^m)^{(\gamma)} (\tilde{a}_\alpha)^{(\gamma')}$$

as

$$|a_\alpha^{(\alpha-\beta)}| \leq \delta^{m-|\gamma|}\delta^{-|\gamma'|} = \delta^{m-|\alpha-\beta|} = \delta^{|\beta|} \ , \quad \text{q.e.d.}$$

So the lower order terms of A^* coming from the top order coefficients:
$b_\alpha = a_\alpha^{(\alpha-\beta)}$, are $L_w^\infty(w = \delta^{-|\beta|})$, i.e. belong to the above class of
perturbations. Similar argument applies to derivatives of the coefficients b_α of
B . Namely, interpolating in 3 parameters: p (for L^p-class of b_α) , Sobolev
(fractional) order s and the exponent t of the weight $w = \delta^t$, one can show
that

$$b \in L_{s;\delta^t}^p \Rightarrow b^{(\gamma)} \in L_{s-|\gamma|; \ \delta^{t-|\gamma|}}^q \ , \quad \text{where} \ \frac{n}{q} = \frac{n}{p} + |\gamma| \ .$$

This implies in particular that the class of perturbations $A_0 + B$ includes
all symmetric operators with the leading part A_0 . The latter is important both
for the argument below and in various applications.

3. THE MAIN LEMMA.

In this section we shall prove the main Lemma, which gives condition for
relative boundedness of the operator $B = b(x)D^\alpha$ with respect to A_0 of § 1, and
provides an estimate of the operator norm of the product $B \times$ "parametrix of $(\zeta-A_0)^{-1}$"
This result will enable us to prove the convergence of series (1.5) and (1.6)
and to derive all consequences in § 4 .

Let $A_0 = a(x,D)$ be an elliptic operator of § 2 with symbol
$a(x,\xi) = \tilde{a}(x,\delta(x)\xi)$. Define a ψDO $K = K_\zeta(x,z), z = x - y$, with symbol
$\frac{1}{\zeta-a(x,\xi)}$, a parametrix of $(\zeta-A_0)$,

$$K(x,z) = (2\pi)^{-n/2} \int \frac{e^{i\xi \cdot z}}{\zeta-a(x,\xi)} \, d\xi \tag{3.1}$$

LEMMA 1. Let $a(x,\xi) = \tilde{a}(x,\delta(x)\xi)$ and $b(x) \in L_w^{p_\alpha}$ satisfy assumptions
(2.1) - (2.5) of § 2, then for all $\zeta \in$ range $a(x,\xi) = \mathbf{R}_+$, the operator BK_ζ
is L^p-bounded, $1 < p \leq p_\alpha$, and its norm is estimated as follows

$$\| BK_\zeta \| \leq \frac{C}{|\sin \theta/2|^{n+1}} r^{-1+d/m}; \ \zeta = re^{i\theta}, d = d_\alpha = \frac{n}{p_\alpha} + |\alpha| \leq m \tag{3.2}$$

where C depends on p ; the $L_w^{p_\alpha}$-norm of $b(x)$ and the leading symbol
$a(x,\xi)$.

PROOF. The operator BK consists of a ψDO $D^\alpha K$ followed by a
multiplication with $b(x)$. We start with $D^\alpha K$. By the usual product formula of
ψDO's its symbol is equal

$$\sigma(x,\xi) = \sum \binom{\alpha}{\beta}\xi^{\alpha-\beta}\partial_x^\beta(\frac{1}{\zeta-a}) \ . \tag{3.3}$$

We apply the "iterated chain rule" to partial derivatives of $(\frac{1}{\zeta-a})$

$$\partial_x^\beta(\frac{1}{\zeta-a}) = \sum_{\beta^1 \ldots \beta^k} C_{\beta^1 \ldots \beta^k} (\zeta-a)^{-k-1} \prod_1^k (\partial_x^{\beta^j} a) \ , \tag{3.4}$$

summation taken over all partitions $\beta = \sum_1^k \beta^j$ into the sum of multi-indices $\{\beta^j\}$ with certain combinatorial coefficients $C_{\beta^1 \ldots \beta^k}$. As a result σ is represented as

$$\sum \phi_i(x)\sigma_i(x,\xi) \tag{3.5}$$

where

$$\phi_i(x) = \prod_1^k a_{\alpha^j}^{(\beta^j)} \quad - \text{ product of } k \text{ partial derivatives of the coefficients of } a(x,\xi) \tag{3.6}$$

and

$$\sigma_i(x,\xi) = \frac{\xi^\nu}{(\zeta-a)^{k+1}}, \quad \text{where the multiindex } \nu = \sum_1^k \alpha^j + (\alpha-\beta) .$$
$$|\nu| = km + |\alpha-\beta|$$

In order to treat a possible L^p-type singularity of the coefficient $b(x)$ we shall use two interpolation inequalities for convolution and multiplication.

$$\|K*f\|_{p'} \leq \|K\|_q \|f\|_p ; \quad \frac{1}{p'} = \frac{1}{p} - (1-\frac{1}{q}) \quad \text{(Young)}$$

$$\|bf\|_{p'} \leq \|b\|_r \|f\|_p ; \quad \frac{1}{p'} = \frac{1}{p} + \frac{1}{r} \quad \text{(Hölder)} \tag{3.7}$$

including a stronger version of Young's inequality the Hardy-Littlewood-Sobolev inequality, where $K \in L^q$-weak (see for instance [17], Ch. 3). The relation between p, p' and q is the same but p = 1 is excluded in the latter case.

Rephrasing (3.7) in the reciprocal scale $\frac{1}{p} \in [0,1]$, one can say that a convolution with $K \in L^q$ (or L^q-weak) shifts the L^p-class to the left by the amount $(1 - \frac{1}{q})$, while a multiplication with $b \in L^r$ shifts it to the right by $\frac{1}{r}$. The product of two operations $f \to bK*f$ is bounded if the L^p-classes of b and K are "Young-dual": $\frac{1}{r} = 1 - \frac{1}{q}$.

This relation explains the limitation on the scale of L^p-spaces in Lemma 1, $p \leq p_\alpha$ (we do not want a "convolution-type" term K to "push" the L^p-class of f out of the scale [0,1]). It also explains the definition of the "fractional order" (2.3). Indeed, a multiplication with $b \in L^p$ is "equivalent" by (3.7) to a convolution with the "fractional Laplacian" $\Lambda^s = (-\Delta)^{s/2}$; $s = \frac{n}{p}$).

Now we return to the operator $b(x)D^\alpha K$ and want to make the above heuristic argument accurate. It is convenient to multiply and divide BK by the fractional Laplacian Λ^s, $s = \frac{n}{p_\alpha}$ (s = 0, of $p_\alpha = \infty$). Then we get from (3.5)

$$BK\Lambda^s\Lambda^{-s} = \sum (b\phi_i)M_i\Lambda^{-s}$$

a combination of multiplication operators $(b\phi_i)$, ψDO's M_i with symbols

$$\sigma_i = \frac{\xi^\nu |\xi|^s}{(\zeta-a)^{k+1}} \tag{3.8}$$

and a fractional Laplacian Λ^{-s}. The order of σ_i is $|\nu| + s - m(k+1)$, which

by (3.6) is equal to

$$\ell = |\alpha| - |\beta| + \frac{n}{p_\alpha} - m = d_\alpha - m - |\beta| \leqslant 0 . \qquad (3.9)$$

We "pull out" the absolute value of complex $\zeta = re^{i\theta}$ in the denominator of
(3.8), and by the homogenuity of a write σ_i in terms of a "uniformly elliptic"
symbol $\tilde{\sigma}_i = \dfrac{\xi^\nu |\xi|^\zeta}{e^{i\theta} \tilde{a}(x,\xi)}$,

$$\sigma_i(x,\xi) = \frac{r^\mu}{\delta^t} \tilde{\sigma}_i(x; \ r^{-1/m}\delta(x)\xi) \qquad (3.10)$$

where

$$\mu = -1 + d/m - |\beta|/m; \qquad d = d_\alpha = \frac{n}{p_\alpha} + |\alpha|$$

$$t = |\nu| + s = mk + d - |\beta| .$$

The weight $\delta^{-t}(x)$ will joint the left (multiplication) factor of
$(b\phi_i)M_i\Lambda^{-s}$. Remembering that ϕ_i is a product of derivatives
$\prod_{\alpha^j} a^{(\beta^j)}, \ \sum_1^k \beta^j = \beta$ we get

$$\frac{b \cdot \prod_1^k a_{\alpha^j}^{(\beta^j)}}{\delta^{d+km-|\beta|}} = (b/\delta^d)\prod(a_{\alpha^j}^{(\beta^j)}/\delta^{m-|\beta^j|}) .$$

By hypothese (2.1), (2.2), (2.5) of § 2 the first factor b/δ^d is L^{p_α},
whereas all terms of the product   are L^∞. Thus the "multiplication: (left)
factor $\psi_i = \dfrac{(b\phi_i)}{\delta^t} \in L^{p_\alpha}$ and the "convolution" (right) factor Λ^{-s} of the
product $\psi_i M_i \Lambda^{-s}$ "complete" each other in the sense of interpolation formulas
(3.7). It remains to study the middle term. We shall show that a ψDO M_i is
L^p-bounded $(1 < p < \infty)$ and estimate its norm.
    We first observe that the dilation of symbols $\tilde{\sigma}_i \rightarrow \sigma_i = \tilde{\sigma}(x, \rho\xi)$ with
$\rho = r^{-1/m}\delta$, results in the dual L^1-dilation of kernels, i.e.

$$M_i(x,z) = \rho^{-n}\tilde{M}_i(x; \ \rho^{-1}z) \qquad (3.11)$$

which represents M_i in terms of the "uniformly elliptic" kernel $\tilde{M}_i$.
    The order ℓ of a ψDO σ_i is nonpositive by (3.9). So two cases are
possible.
    1°. Strictly negative order, $\ell < 0$.
    A negative order ψDO $\tilde{\sigma}$ $S^\ell_{1,0}$ is well known to have an L^1-radial
convolution-type bound

$$|\tilde{M}_i(x,z)| \leqslant cH(|z|) \qquad (3.12)$$

where c depends on a finite number $(N = n + 1)$ symbol class seminorms of $\tilde{\sigma}$
(see for inst. [5]).

A straightforward evaluation of seminorms yields

$$\sup_{x,\xi} |\xi|^N |(\partial_\xi^\beta \tilde{\sigma}_i)(x,\xi)| \le \frac{C}{|\sin \theta/2|} N, \quad N = |\beta| ,$$

whence the constant in estimate (3.2) of Lemma 1.

From (3.11), (3.12) it follows that the kernel $M_i(x,z)$ is bounded by the
so-called ρ-dilation of the radial L^1-function H with $\rho = \rho(x) = r^{-1/m}\delta(x)$, i.e.

$$|M_i(x,z)| \le c\rho^{-n}(x)H(|\rho^{-1}(x)z|) = H_\rho; \quad z = x - y .$$

If ρ were bounded $\rho(x) \le \varepsilon$ (uniformly elliptic case), then $H_\rho \le H_\varepsilon$ and
the usual L^1-convolution would immediately imply the result: L^p-boundedness of
M_i for all $1 \le p \le \infty$ (cf. [4], [5]). In case of unbounded $\delta(x)$ the maximal
function estimate applies,

$$(H_\rho f)(x) \le \|H\|_1 f^*(x) .$$

This yields L^p-boundedness of M_i in all spaces $1 < p < \infty$ and proves
Lemma 1 in the first case.

2°. Order $\ell = 0$. We split $\tilde{\sigma}_i \in S^0_{1,0}$ into the sum of 3 terms:
$\tilde{\sigma}_i = c(x)I + \sigma_0 + \sigma_-$, a multiple of the "identity" (in the ξ-variable) function;
a homogeneous of order zero (in ξ) symbol $\sigma_0(x,\xi)$ with the zero mean-value
over the unit sphere $|\xi| = 1$; and a negative order symbol $\sigma_- \in S^{-\ell}_{1,0}$.

Let us illustrate this splitting for $k = 0$, i.e.
$\tilde{\sigma} = \frac{\xi^\nu}{\tilde{\zeta - a}} \ (|\nu| = m)$. Then

$$\tilde{\sigma} = \frac{\xi^\nu}{\tilde{a}} - \frac{\zeta\xi^\nu}{(\zeta-\tilde{a})\tilde{a}} ,$$

the first term splits into the sum of $c(x) = \int_{|\xi|=1} \xi^\nu/\tilde{a}(x,\xi)d\xi$ and
$\sigma_0 = \xi^\nu/\tilde{a} - c(x)$, while the second is obviously in $S^{-m}_{1,0}$. Notice that $\tilde{\sigma}_-$ is
sufficiently regular at $\{0\}$, which is important for the existence of "global"
radial bound H of the kernel $\tilde{M}_- = \tilde{\sigma}_-(x,D)$ (cf. [5]).

Correspondingly kernel $\tilde{M}$ splits into the sum of three kernels:
$c(x)I + \tilde{M}_0 + \tilde{M}_-$, a multiplication with "nice" (bounded) function $c(x)$; a
Calderon-Zygmund kernel $\tilde{M}_0(x;z)$ (homogeneous of degree $-n$ in z) and an
L^1-radially bounded kernel $\tilde{M}_-$.

The first two of them are invariant under the ρ-dilation

$$M(z) \rightarrow \rho^{-n}M(\rho^{-1}z) ,$$

while the third is dominated by the maximal function as in case 1° . This
completes the proof.

4. RESOLVENT KERNELS OF A_0 AND A AND THEIR APPLICATIONS.

     After Lemma 1 we can study the convergence of series (1.5) and (1.6), that give resolvent kernels $R^0 = (\zeta - A_0)^{-1}$ and $R = (\zeta - A)^{-1}$. We start with R^0 .

     THEOREM 1. Series (1.5) converges absolutely in the complement of a parabolic region Ω about positive real axis in $\mathbb{C}$, in all L^p -spaces $1 < p < \infty$, and defines the resolvent R^0_ζ of A.

     PROOF. We recall that the operator $L = (\zeta - A_0)K - I$. By the product formula of ψDO's its symbol

$$\sigma_L(x,\xi) = \sum_{1 \leqslant |\beta| \leqslant m} \frac{1}{\beta!} D^\beta_\xi(a) \partial^\beta_x (\frac{1}{\zeta-a}) ,$$

expands into the sum $\sum \phi_i \sigma_i$ of the type that appear in Lemma 1. Namely,

$$\phi_i(x) = a_\alpha \prod_1^k a^{(\beta^j)}_{\alpha^j}; \quad \sum_1^k \beta^j = \beta; \quad (a_\alpha \in L^\infty_{\delta -m} \text{ plays the role of } b(x))$$

and

$$\sigma_i(x,\xi) = \frac{\xi^\nu}{(\zeta-a)^{k+1}}; \quad \nu = \alpha - \beta + \sum_1^k \alpha^j; \quad |\nu| = (k+1)m - |\beta| .$$

     So the whole argument of Lemma 1 can be repeated for σ_L . Notice that all ψDO's $\sigma_i(x,D)$ have strictly negative order, $\leqslant -1$.

     Hence by Lemma 1 each term ϕM of L and consequently the whole operator L is estimated as

$$\|L\| \leqslant \sum \|\phi_i M_i\| \leqslant \frac{C}{|\sin \theta/2|^{n+1}} r^{-\lambda}; \quad \lambda = 1 - |\beta|/m \geqslant \frac{1}{m} , \quad (4.1)$$

with C depending on L^∞ -norms of ϕ_i and p $(1 < p < \infty)$. Returning to series expansion (1.5), we see that $\sum_0^\infty L^k$ converges absolutely if the right hand side of (4.1)

$$\frac{C}{|\sin \theta/2|^{n+1}} r^{-\lambda} < 1 . \quad (4.2)$$

This condition gives a parabolic region Ω about positive real axis in $\mathbb{C}$, in whose complement we have a well defined bounded operator $R^0_\zeta = K_\zeta(I-L_\zeta)^{-1}$.

     We want to show that R^0_ζ is the resolvent of A_0 , that is

(I) $(\zeta - A_0)R^0_\zeta f = f$; $f \in L^p$

(II) $R^0(\zeta - A_0)f = f$;   for all f in the L^p -domain of A_0 .

     The first formula immediately follows from the relation $(\zeta - A_0)K = I - L$. It shows that R^0 is the right resolvent of A_0 . To show the existence of the left resolvent (II) we shall use duality between L^p -spaces. Namely: the adjoint of the right resolvent of A_0 in L^p becomes the left resolvent of the adjoint operator A^*_0 in $L^{p'}$ $(\frac{1}{p'} + \frac{1}{p} = 1)$, and vice versa.

     We can not apply the above result directly to A^*_0 to show the existence of the right resolvent, since A^*_0 is not of homogeneous order. But $A^*_0 = A_0 + B$, whose lower order terms $\{b_\alpha D^\alpha\}_{|\alpha| \leqslant m}$ have coefficients bounded "relative to A_0 ".

Therefore the "perturbation" Theorem (Theorem 4 below) applies to prove the existence of the right resolvent for A_0^* outside of another parabolic region Ω'. Once both resolvents, right and left, are shown to exist they must be equal. This completes the proof of Theorem 1.

Next we apply Lemma 1 to characterize the L^p-domain of A_0, $\mathcal{D}_p(A_0)$, $1 < p < \infty$. For constant coefficient elliptic operators $\mathcal{D}_p(A_0)$ is known to be the m-th Sobolev space $L^{p,m} = (1-\Delta)^{m/2} L^p$ and the same is true for uniformly elliptic operators (see [4]). For strictly elliptic operators of the above type we have

THEOREM 2. The L^p-domain of A_0 is equal to the m-th weighted Sobolev space with the weight $w = \delta^m$

$$L_w^{p,m} = \{f: \sum_{|\alpha| \leq m} \int |w f^{(\alpha)}|^p \, dx < \infty\},$$

and the norms $\|(\zeta - A_0)f\|$ in $\mathcal{D}_p(A_0)$ and $\sum_{|\alpha| \leq m} \|w D^\alpha f\|_p$ in $L_w^{p,m}$ are equivalent.

PROOF. We have to show that the operator $w(1-\Delta)^{m/2} R_\zeta^0$ is bounded and invertible in L^p. But $R_\zeta^0 = K(I-L)^{-1}$. It suffices to show boundedness and invertibility of the operator $T = w(1-\Delta)^{m/2} K$. Boundedness follows directly from Lemma 1. Indeed

$$w(1 - \Delta)^{m/2} K = \sum_{|\alpha| \leq m} c_\alpha w D^\alpha K.$$

is the sum of terms that appear in Lemma 1.

To prove invertibility it is convenient to use $(\lambda - \Delta)^{m/2}$ for sufficiently large $\lambda > 0$ instead of $(1-\Delta)^{m/2}$ in the definition of $L_w^{p,m}$ - norm and also to reverse two terms in the product $w(\lambda - \Delta)^{m/2}$.

We observe that

$$w(\lambda - \Delta)^{m/2} = [(\lambda - \Delta)^{m/2} + B]w, \qquad (4.3)$$

where order $B \leq m - 1$, and the coefficients $b_\alpha = \sum_{|\nu| \leq m} c_\nu w^{(\nu)}/w$ are all bounded. Then for sufficiently large λ the operator norm

$$\| B(\lambda - \Delta)^{-m/2} \| < 1/2 .$$

Hence the inverse $[(\lambda - \Delta)^{m/2} + B]^{-1}$ exists and is bounded.

Now we can invert $T = w(\lambda - \Delta)^{m/2} R^0$ using (4.3),

$$T^{-1} = (\zeta - A_0)w^{-1} [(\lambda - \Delta)^{m/2} + B]^{-1} . \qquad (4.4)$$

But $A_0 w^{-1} = \tilde{A}_0 + \tilde{B}$ with uniformly elliptic $\tilde{A}_0 = w^{-1} A_0$ and $\tilde{B}$ of the same type as B.

Finally "pulling out" $(\lambda - \Delta)^{m/2}$ from the right factor in (4.4),
$(\lambda - \Delta)^{-m/2} [I + (\lambda - \Delta)^{-m/2} B]^{-1}$ the problem reduces to L^p-boundedness of the

operator $S = (\tilde{A}_0 + \tilde{B})(\lambda - \Delta)^{-m/2}$ for uniformly elliptic $\tilde{A}_0$ and "relatively small" $\tilde{B}$, which is well known (cf. [4]). Indeed, the operator S is a combination of zero- and negative order ψDO's $\{\tilde{a}_\alpha(x)D^\alpha(\lambda - \Delta)^{m/2}: |\alpha| \leq m\}$. The former are given by Calderon-Zygmund kernels, hence L^p for all $1 < p < \infty$ (see [17], ch. 2). The latter have L^1-radially bounded kernels, hence L^p for all $1 \leq p \leq \infty$.

In many applications, including the perturbation series (1.6), one needs to estimate the operator norm of BR^0_ζ. By Theorem 1 $R^0_\zeta = K_\zeta(I - L_\zeta)^{-1}$ and $(I-L_\zeta)^{-1}$ is easily shown to converge to I as $\zeta \to \infty$ uniformly in any sector $\{|\arg \zeta| \geq \theta > 0\}$. The problem is thus reduced to estimating $\|BK_\zeta\|$. The latter is provided by Lemma 1.

COROLLARY 1. If A_0 and B satisfy assumptions of § 2, then

$$\|BR^0_\zeta\| \leq (C \sum \|b_\alpha\|) \frac{r^{-1+d/m}}{|\sin \theta/2|^{k+1}}; \tag{4.5}$$

for all L^p, $1 < p \leq \min \{p_\alpha\}$ and $\zeta \in \mathbb{C}\backslash\Omega$ (where R^0_ζ exists). Here $d = \max \{d_\alpha\}$ is the "fractional order of B, constant C depends on p and the symbol $a(x,\xi)$.

From (4.5) immediately follows

COROLLARY 2 (cf. [5], [4], [16]). The operator B is A_0-bounded and the L^p-domains of A and A_0 are equal

$$\mathcal{D}_p(A) = \mathcal{D}_p(A_0) = L_w^{p,m}; \; w = \delta^{-m}, \; 1 < p \leq \min \{p_\alpha\}.$$

Let us notice that the relative bound in the right hand side of (4.5) can be made as small as one likes, taking sufficiently large r, provided the "fractional" order $d < m$. In this case the perturbation series (1.6) converges absolutely in the complement of a parabolic region

$$\Omega' = \{\zeta: \text{relative bond} \leq 1\}.$$

The limit case, $d = m$, is important, as it gives the "optimal amount" of L^p-type singularity of lower order coefficients and also allows "top order" perturbations. In this case one can claim somewhat less.

PROPOSITION 1. Operator $\|BR^0_\zeta\| < 1$ in any region

$$\Omega_{R,\theta} = \{\zeta: |\zeta| \geq R; |\arg \zeta| > \theta\} \tag{4.6}$$

with arbitrarily small θ and large R, depending on θ.

To prove Proposition 1 we cut each coefficient $b_\alpha \in L^{p_\alpha}$ into two parts: $b'_\alpha \in L^\infty$ and "very small" $b''_\alpha \in L^{p_\alpha}$, $\|b''_\alpha\| < \epsilon$. Consequently, the whole perturbation B splits into the sum $B' + B''$. The first term has order $d \leq m - 1$, hence $\|B'R^0_\zeta\|$ becomes small outside of a parabolic region Ω', while the second $\|B''R^0_\zeta\| < 1$ in some region $\Omega_{R,\theta}$. Here θ depends on $\sum \|b''_\alpha\|$ and can be made as small as one likes. This proves the Proposition.

From Proposition 1 easily follows

COROLLARY 3 (a priory estimate). For any $\epsilon > 0$ there exists
$C = C_\epsilon > 0$ s.t.

$$\| Bf \|_p \leq \epsilon \| A_0 f \|_p + C \| f \|_p ; \quad f \in L_w^{p,m} \tag{4.7}$$

Along with Kato-Rellich Theorem a priory estimate (4.7) implies essential
selfadjointness in L^2 of a formally symmetric operator $A_0 + B$. Proposition 1
also yields a large class ("optimal" in the sense of L^p-type singularity of
coefficients) of "well defined" elliptic operators in L^p-spaces. Namely,

THEOREM 3. 1) If an operator $A = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ of § 1 has leading

coefficients s.t. $\left\{ \dfrac{a_\alpha(x)}{\delta^m(x)} \right\}_{|\alpha|=m}$ - uniformly continuous, and the lower order

coefficients $a_\alpha \in L_w^{p_\alpha}$, satisfy (2.3) - (2.4), then A is "well defined" in all
L^p-spaces $1 < p \leq \min p_\alpha$ in the sense that

$$\mathcal{D}_p(A) = L_w^{p,m} \quad \text{and} \quad A | L_w^{p,m} = \text{closure} \{ A_{\min} = A | C_0^\infty \} .$$

2) If A is formally symmetric and $\min \{p_\alpha\} \geq 2$ then A is essentially
selfadjoint on C_0^∞ .

Theorem 3 extends many of earlier known results [8], [9], [24], [4], [11],
[15], [16] on essential selfadjointness and L^p-theory.

Now we turn to the resolvent R_ζ of A .

THEOREM 4. (cf. [5], [4]) Let $A = A_0 + B$ be an operator of Theorem 3.
Then

(I) for all $1 < p \leq \min \{p_\alpha\}$ the resolvent $R_\zeta = (\zeta - A)^{-1}$ exists in $\Omega_{R,\theta}$
and is given by an absolutely convergent series (1.6).

(II) (resolvent summability) for all $f \in L^p$ $(1 < p \leq \min \{p_\alpha\})$ the family

$$(\zeta R_\zeta f)(x) \to f(x) \quad \text{as} \quad \zeta \to \infty \quad \text{uniformly in} \quad \Omega'$$

in L^p-norm.

The first statement is already proved, the second follows from the estimates
of Lemma 1 (cf. [4], [5]).

After Theorem 2 we can obtain a variety of other "analytic multipliers" $\phi(A)$
and "summation families" $\{\phi_\epsilon(A)\}_\epsilon$ by Cauchy integration of R_ζ . Indeed, the
formula

$$\phi(A) = \frac{1}{2\pi i} \int_\Gamma \phi(\zeta) (\zeta - A)^{-1} d\zeta$$

defines a nice (bounded) operator $\phi(A)$, provided ϕ is integrable over the
contour Γ with respect to the measure $d\mu = \dfrac{Cr^{-1+d/m}}{|\sin \theta/2|^{n+1}} d\zeta$. One such example
is the family of functions $\{\phi_t(\zeta) = e^{-t\zeta}\}_t$. Here contour Γ consists of two rays
$\{re^{\pm i\psi}; r > r_0\}$, $0 < \psi \leq \dfrac{\pi}{2}$ and a circle $\{r_0 e^{i\theta}: |\theta| \geq \psi\}$. Angle ψ can be
chosen arbitrarily small due to the shape of $\Omega_{R,\theta}$.

Thus we get

THEOREM 5. An operator A of Theorem 3 generates a holomorphic semigroup e^{-tA} in the right half plane Re t > 0 in all L^p, $1 < p \leq \min \{p_\alpha\}$.

Moreover, the family of functions $e^{-tA}f(x) \to f(x)$ as $t \to 0$ in L^p-norm.

REMARK 1. As in [4] (§ 5), all the above results extend to strictly elliptic operators on certain Riemannian manifolds diffeomorphic to $\mathbb{R}^n$ and strictly elliptic systems.

REMARK 2. More general classes of coefficients $\{b_\alpha\}$ of the perturbation B can be allowed in § 3. One such class consists of $b_\alpha(x)$ defined on quotient spaces $E = \mathbb{R}^n/E$ (E is a subspace of $\mathbb{R}^n$), which are L^p on E, like "N-body" potentials: $V = \sum\limits_{i<j} V(x^i - x^j)$ on $\mathbb{R}^{3N}$ ($x^i \in \mathbb{R}^3$). In this case condition (2.4) of § 2 should read: $\dfrac{\dim E}{p_\alpha} + |\alpha| \leq m$, (see [5]). Another extension of L^p classes is discussed in [15], [16]. It consists of all functions $b_\alpha(x)$, whose convolution with $H_\alpha(z) = |z|^{-s}$ ($|z| \leq 1$, and $s = n - m + \alpha$) (a "local singularity" of $D^\alpha(\zeta - A_0)^{-1}$) is bounded. This condition is close to so called "Rollnik condition" in the theory of Schrödinger operators (see [25]).

We shall conclude this section with two examples, which have bearing on L^1-theory of the above class of operators. This theory poses the following interesting harmonic analysis problem: given an L^1 function K(x) and a dilating factor $\delta(x) > 0$, when the kernel $K_\delta = \dfrac{1}{\delta^n(x)} K(\dfrac{x-y}{\delta(x)})$ is L^1 bounded.

Two examples below indicate that the situation in L^1-case becomes more subtle compared to L^p-theory (p > 1). Namely, an L^1-radially bounded K and a "finitely propagating" δ are no longer sufficient for L^1: the growth of δ must be "slower" than $O(|x|)$, and some additional relation between K and δ appears.

EXAMPLE 1. Take $\delta(x) = |x|$ on $\mathbb{R}$. Then $K(x,y) = \dfrac{1}{x} K(1 - \dfrac{y}{x})$ is homogeneous of degree -1. It is well known (see, for instance, [17], Appendix) that K is L^p if and only if $\int y^{-p} K(1-y)\, dy < \infty$. The latter is obviously true for p > 1, but fails for p = 1.

EXAMPLE 2. Now we take $\delta(x) = 2\sqrt{x}$ on $\mathbb{R}$ and write the operator $K_\delta f$ as

$$(K_\delta f)(x) = \int K(y) f(x - \delta(x)y)\, dy .$$

Then

$$\| Kf \|_1 = \int K(y) \int f(x - \delta(x)y)\, dx\, dy .$$

We introduce a new variable $u = x - \delta(x)y = \phi_y(x)$. The range of u is the whole real line $\mathbb{R}$, but the function $u = \phi_y(x)$ has a critical point $x_0 = y^2$, and on the interval $-y^2 \leq u \leq 0$ u is "three-fold" (see figure). We divide the line $\mathbb{R}_x$ into three intervals $I_1 = (-\infty; 0]$; $I_2 = (0; 2y^2]$; $I_3 = (2y^2; +\infty)$. On I_1 and I_3 the Jacobian $(\dfrac{du}{dx})$ is bounded from both sides,

$$0 < c_1 \leq (\dfrac{du}{dx}) \leq c_2 < \infty .$$

Hence

$$\int_{I_1} + \int_{I_3} f(x - \delta(x)y) \, dx \leqslant \frac{1}{c_1} \| f \|_1 \, .$$

It is only a neighborhood of the critical point $x_0 = y^2$ that causes $(\frac{du}{dx})$ to "blow up". The Jacobian $(\frac{du}{dx})$ is computed explicitely.

$$(\frac{du}{dx}) = 1 \pm \frac{y}{\sqrt{y^2+u}} \quad \text{("-" on } [0,y^2]$$
$$\text{"+" on } [y^2,2y^2]) \, .$$

Hence

$$\int_{I_2} f(x - \delta(x)y) \, dx = 2 \int_{-y^2}^{0} f(u) \, \frac{y}{\sqrt{y^2+u}} \, du + 2 \int_{-y^2}^{0} f(u) \, du \, .$$

Finally, the change of variable $y \to \sqrt{y}$, and the order of integration yields

$$\int_{0}^{\infty} K(y) \int_{-y^2}^{0} f(u) \, \frac{y}{\sqrt{y^2+u}} \, du = \int_{0}^{\infty} f(u) \int_{-u}^{\infty} \frac{K(\sqrt{y})}{\sqrt{y+u}} \, dy \, du \, .$$

This shows that K is L^1-bounded iff the convolution $K(\sqrt{y}) * \frac{1}{\sqrt{y}}$ is bounded. Then

$$\| K_\delta(f) \|_1 \leqslant (\| K \|_1 + \| K(\sqrt{y}) * \frac{1}{\sqrt{y}} \|_\infty) \, \| f \|_1 \, .$$

A similar condition on K can be derived for other polynomial functions $\delta(x) = |x|^s$ ($s < 1$). Namely, the convolution of $y^{1/2-s} K(y^{1-s})$ and $\frac{1}{\sqrt{y}}$ must be L^∞.

These results can be used to study L^1 properties of the resolvent kernel R_ζ for operators $A = \delta(x)^m \frac{d^m}{dx^m} + \ldots$ on $\mathbf{R}$. Indeed, the kernel

$$K(z) = \int_{-\infty}^{\infty} \frac{e^{iz \cdot \xi}}{\zeta - \xi^m} \, d\xi \quad \text{is bounded near } \{0\} \text{ and decays exponentially at}$$

$\{\infty\}^{-\infty}$, $K(z) \sim e^{-\gamma |z|}$. If $\delta(x) \sim |x|^s$ with $s < m$, the condition $y^{-1/2-s} K(y^{1-s}) * \frac{1}{\sqrt{y}} \, L^\infty$ is obviously satisfied. Hence K_δ and consequently R_ζ are L^1. On the other hand if $\delta(x) \sim |x|$ has the maximal allowed growth both kernels fail to be L^1.
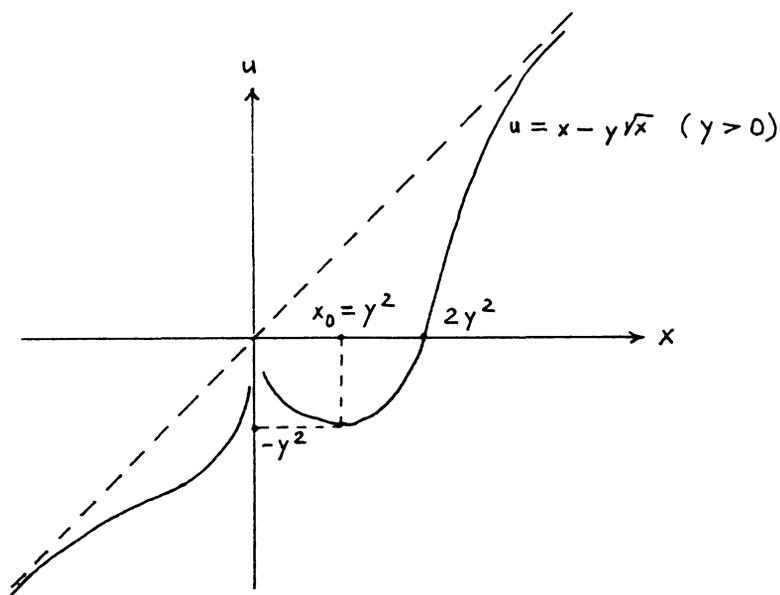
$$u = x - y\sqrt{x} \quad (y > 0)$$

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