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ON THE SPECTRUM OF WEAKLY ALMOST PERIODIC SOLUTIONS OF CERTAIN ABSTRACT DIFFERENTIAL EQUATIONS

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ABSTRACT. In a sequentially weakly complete Banach space, if the dual operator of a linear operator A satisfies certain conditions, then the spectrum of any weakly almost periodic solution of the differential equation $u' = Au + f$ is identical with the spectrum of f except at the origin, where f is a weakly almost periodic function.

KEY WORDS AND PHRASES. *Strongly (weakly) almost periodic function, sequentially weakly complete Banach space, densely defined linear operator, dual operator, Hilbert space, nonnegative self-adjoint operator.*
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1. INTRODUCTION.

Suppose X is a Banach space and X^* is the dual space of X. Let J be the interval $-\infty < t < \infty$. A continuous function $f : J \to X$ is said to be strongly almost periodic if, given $\varepsilon > 0$, there is a positive real number $\ell = \ell(\varepsilon)$ such that any interval of the real line of length ℓ contains at least one point τ for which

$$\sup_{t\in J} ||f(t+\tau)-f(t)|| \leq \varepsilon. \tag{1.1}$$

We say that a function $f : J \to X$ is weakly almost periodic if the scalar-valued function $\langle x^*, f(t)\rangle = x^*f(t)$ is almost periodic for each $x^* \in X^*$.

It is known that, if X is sequentially weakly complete, $f : J \to X$ is weakly almost periodic, and λ is a real number, then the weak limit

$$m(e^{-i\lambda t}f(t)) = \underset{T\to\infty}{w\text{-}\lim} \frac{1}{T}\int_0^T e^{-i\lambda t}f(t)dt \tag{1.2}$$

exists in X and is different from the null element Θ of X for at most a countable set $\{\lambda_n\}_{n=1}^{\infty}$, called the spectrum of $f(t)$ (see Theorem 6, p. 43, Amerio-Prouse [1]). We denote by $\sigma(f(t))$ the spectrum of $f(t)$.

2. RESULTS

Our first result is as follows (see Theorem 9, p. 79, Amerio-Prouse [1] for the spectrum of an S^1-almost periodic function).

THEOREM 1. Suppose X is a sequentially weakly complete Banach space, A is a densely defined linear operator with domain $D(A)$ and range $R(A)$ in X, and the dual operator A^* is densely defined in X^*, with $R(i\lambda - A^*)$ being dense in X^* for all real $\lambda \neq 0$. Further, suppose $f : J \to X$ is a weakly almost periodic (or an S^1-almost periodic continuous) function. If a differentiable function $u: J \to D(A)$ is a weakly almost periodic solution of the differential equation

$$u'(t) = Au(t) + f(t) \tag{1.3}$$

on J, with u' being weakly continuous on J, then $\sigma(u(t)) \setminus \{0\} = \sigma(f(t)) \setminus \{0\}$.

PROOF OF THEOREM 1. First we note that u is bounded on J, since u is weakly almost periodic. Hence, for $x^* \in X^*$, we have

$$\frac{1}{T}\int_0^T e^{-i\lambda t}x^*u'(t)dt = x^*\frac{1}{T}\left\{ [e^{-i\lambda t}u(t)]_0^T + \frac{i\lambda}{T}\int_0^T e^{-i\lambda t}u(t)dt\right\} \tag{2.1}$$

$$\to i\lambda x^* m(e^{-i\lambda t}u(t)) \quad \text{as} \quad T \to \infty .$$

So, for $x^* \in D(A^*)$, it follows from (1.3) that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T e^{-\lambda t}x^*Au(t)dt = \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{-i\lambda t}(A^*x^*)u(t)dt$$

$$= \lim_{T\to\infty} (A^*x^*)\left[\frac{1}{T}\int_0^T e^{-i\lambda t}u(t)dt\right]$$

$$= (A^*x^*)m(e^{-i\lambda t}u(t))$$

$$= i\lambda x^*m(e^{-i\lambda t}u(t)) - x^*m(e^{-i\lambda t}f(t)). \tag{2.2}$$

Consequently, we have

$$x^*m(e^{-i\lambda t}f(t)) = (i\lambda x^* - A^*x^*)m(e^{-i\lambda t}u(t)). \tag{2.3}$$

Now suppose that $\lambda \in \sigma(f(t)) \setminus \{0\}$. Then, since $D(A^*)$ is dense in X^*, there exists $x_1^* \in D(A^*)$ such that

$$0 \neq x_1^*m(e^{-\lambda t}f(t)) = (-\lambda x_1^* - A^*x_1^*)m(e^{-\lambda t}u(t)). \tag{2.4}$$

Therefore $m(e^{-i\lambda t}u(t)) \neq 0$ and so $\lambda \in \sigma(u(t)) \setminus \{0\}$.

Thus we have

$$\sigma(f(t)) \setminus \{0\} \subseteq \sigma(u(t)) \setminus \{0\}. \tag{2.5}$$

Now assume that $\lambda \in \sigma(u(t) \setminus \{0\}$. Then, since $R(i\lambda - A^*)$ is dense in X^*, there exists $x_2^* \in D(A^*)$ such that

$$0 \neq (-i\lambda x_2^* - A^* x_2^*)m(e^{-i\lambda t}u(t)) = x_2^* m(e^{-i\lambda t}f(t)). \tag{2.6}$$

Therefore $m(e^{-i\lambda t}f(t)) \neq 0$ and so $\lambda \in \sigma(f(t)) \setminus \{0\}$.

Consequently, we have

$$\sigma(u(t)) \setminus \{0\} \subseteq \sigma(f(t)) \setminus \{0\}. \tag{2.7}$$

It follows from (2.5) and (2.7) that $\sigma(u(t)) \setminus \{0\} = \sigma(f(t)) \setminus \{0\}$, which completes the proof of the theorem.

REMARK 1. The conclusion of Theorem 1 remains valid if $D(A^*)$ is total and $R(i\lambda - A^*)$ is total for all real $\lambda \neq 0$, instead of dense in X^*.

We indicate the proof of the following result.

THEOREM 2. In a sequentially weakly complete Banach space X, suppose A is a densely defined linear operator, the dual operator A^* is densely defined in X^*, with $R(\lambda^2 + A^*)$ being dense in X^* for all real $\lambda \neq 0$, and $f : J \to X$ is a weakly almost periodic (or an S^1-almost periodic continuous) function. If a twice differentiable function $u : J \to D(A)$ is a weakly almost periodic solution of the differential equation

$$u''(t) = Au(t) + f(t) \tag{3.1}$$

on J, with u'' being weakly continuous and u' bounded on J, then

$$\sigma(u(t)) \setminus \{0\} = \sigma(f(t)) \setminus \{0\}.$$

PROOF. For $x^* \in D(A^*)$, we have

$$\frac{1}{T}\int_0^T e^{-i\lambda t}x^*u''(t)dt = x^* \left\{\frac{1}{T}[e^{-i\lambda t}u'(t)]_0^T + \frac{i\lambda}{T}\int_0^T e^{-i\lambda t}u'(t)dt\right\}$$

$$= x^* \left\{\frac{1}{T}[e^{-i\lambda t}u'(t)]_0^T + \frac{i\lambda}{T}[e^{-i\lambda t}u(t)]_0^T - \frac{\lambda^2}{T}\int_0^T e^{-i\lambda t}u(t)dt\right\}$$

$$\to -\lambda^2 x^*m(e^{-i\lambda t}u(t)) \text{ as } T \to \infty. \tag{3.2}$$

Hence it follows from (3.1) that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T e^{-i\lambda t}x^*Au(t)dt = (A^*x^*)m(e^{-i\lambda t}u(t))$$

$$= -\lambda^2 x^*m(e^{-i\lambda t}u(t)) - x^*m(e^{-i\lambda t}f(t)). \tag{3.3}$$

Thus we have

$$- x^*m(e^{-i\lambda t}f(t)) = (\lambda^2 x^* + A^*x^*)m(e^{-i\lambda t}u(t)). \tag{3.4}$$

Now the rest of the proof parallels that of Theorem 1.

REMARK 2. The conclusion of Theorem 2 also remains valid if $D(A^*)$ is total and $R(\lambda^2 + A^*)$ is total for all real $\lambda \neq 0$, instead of dense in X^*.

REMARK 3. If X is a Hilbert space and A is a nonnegative self-adjoint operator, then the hypotheses on A in Theorem 2 are verified (see Corollary 2, p. 208, Yosida [2]) and so Theorem 2 is a generalization of a result of Zaidman [3].

NOTE. As a consequence of our Theorem 1, we have the following result:

THEOREM 3. In a Hilbert space H, suppose A is a self-adjoint operator and $f : J \to H$ is a weakly almost periodic (or an S^1-almost periodic continuous) function. If a differentiable function $u : J \to D(A)$ is a weakly almost periodic solution of the differential equation

$$u'(t) = Au(t) + f(t)$$

on J, with u' being weakly continuous on J, then

$$\sigma(u(t)) \setminus \{0\} = \sigma(f(t)) \setminus \{0\}.$$

PROOF. By Example 4, p. 210, Yosida [2], $R(i\lambda - A) = H$ for all real $\lambda \neq 0$.

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