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THE COMPACTUM OF A SEMI-SIMPLE COMMUTATIVE BANACH ALGEBRA

ABDULIAH H. AL-MOAJIL

GULF ORGANIZATION FOR INDUSTRIAL CONSULTING
P. O. Box 5114
Doha - Qatar.

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ABSTRACT. Let A be a commutative semi-simple Banach algebra such that the set consisting of finite sums of elements from minimal left ideals coincides with that of finite sums of elements from minimal right ideals. Let S(A) (the socle of A) denote this set. Let C(A) denote the set of elements x in A such that the map $a \to xax$ is compact. It is shown that C(A) is the norm closure of S(A).

KEY WORDS AND PHRASES. *Commutative Banach algebra, semi-simple, socle, compactum, spectrum, carrier space, idempotent.*
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1. INTRODUCTION

Let A be a Banach algebra. For $x \varepsilon A$ let T_x denote the operator defined by $T_x(a)=xax$. The compactum of A is defined to be the set $\{x \varepsilon A : T_x$ is a compact operator on A$\}$. A Banach algebra A in which A=C(A) is called a compact Banach algebra. Compact Banach algebras were first introducted by J. C. Alexander in [1] . The author, in [3], investigated the properties of the compactum in Banach Algebras. It was shown in [3] that if A is semi-simple and S(A) denotes the socle of A, then C(A) is non-zero if and only if S(A) is non-zero, and in this case, $S(A) \subset C(A)$. Moreover, C(A) is a closed set, therefore it contains the closure of S(A). A problem of interest is to determine sufficient conditions on A which imply that C(A) coincides with $\overline{S(A)}$. In [2] , the author proved that for a primitive B* algebra A, we have $C(A) = \overline{S(A)}$.

The purpose of this note is to prove that for a semi-simple commutative Banach algebra A, we have $C(A) = \overline{S(A)}$.

2. MAIN RESULT

To prove our theorem we use a result from [3] which states that if $x \varepsilon C(A)$ then the spectrum of $x(\sigma(x))$ is at most countable and o is its only possible accumulation point. Our terminology and notation is consistent with that of [4] , and our algebras are over the field of complex numbers.

THEOREM: Let A be a semi-simple commutative Banach algebra. If C(A) exists, then $C(A)=\overline{S(A)}$.

PROOF: We need to show that $C(A) \subset \overline{S(A)}$, as the other inclusion was already proven in [3] .

Let Φ denote the space of multiplicative linear functionals in A, i.e. Φ is the carrier space of A.

Let $x \varepsilon C(A)$. We have, from general theory of commutative Banach algebras, $\sigma(x)=\{\hat{x}(\phi):\phi\varepsilon\Phi\}\cup\{o\}$ where $\hat{x}$ is the continuous function on Φ defined by $\hat{x}(\phi)=\phi(x)$.

We claim that if ϕ is not an isolated point of Φ, then $\hat{x}(\phi)=o$.

This is true because if $\hat{x}(\phi)\neq o$ and $\{\phi_n\} \subset \Phi$ with $\lim_{n\to\infty} \phi_n = \phi$, then by the continuity of $\hat{x}$ we get $\hat{x}(\phi)= \lim_{n\to\infty} \hat{x}(\phi_n)$. But $\hat{x}(\phi_n)$ and $\hat{x}(\phi)$ belong to $\sigma(x)$. Therefore, $\hat{x}(\phi)$ is a non-zero accumulation point of $\sigma(x)$ which is impossible since $x\varepsilon C(A)$.

Now since $\sigma(x)$ is countable, let $\{\phi_n\}$ be a sequance in Φ such that $\sigma(x)=\{\hat{x}(\phi_n)$: $n=1, 2, ..\}$ $\{o\}$, where $\hat{x}(\phi_n)\neq o$ for all n. Note that each ϕ_n is an isolated point of Φ.

Now, by Silov's idempotent theorem [4] , for each n = 1,2.. there exists an idempotent $e_n \varepsilon A$ such that $\hat{e}_n(\phi)_n = 1$ and $\hat{e}_n(\phi) = o$ if $\phi \neq \phi_n$. It is evident that e_n is a minimal idempotent for each n.

For each m, let $x_n = \sum_{i=1}^{n} \hat{x}(\phi_i)\, e_i$. Then by the minimality of e_n, we have $x_n\, \varepsilon S(A)$.

Now, if $\{\phi_n\}$ is a finite set, then $\hat{x}=\hat{x}_n$ for some n and therefore $\hat{x}=\hat{x}_n \varepsilon S(A)$. Otherwise, by the compactness of $\sigma(x)$ and the fact that o is the only possible accumulation point of $\sigma(x)$, we have $\lim_n \hat{x}(\phi_n)=o$.

Now, if $\phi\neq\phi_n$ for any n, then $\hat{x}(\phi)=o$ and $e_n(\phi) = o$ for all n, thus $\hat{x}_n(\phi) = o$ for all n. Moreover, $\hat{x}_n(\phi_m)=\hat{x}(\phi_m)$ if $n \geq m$ and o if $n < m$. Therefore, $\| \hat{x}_n - \hat{x} \| = \sup_{\phi\varepsilon\Phi} |\hat{x}_n(\phi)-\hat{x}(\phi)| = \sup_m | \hat{x}_n(\phi_m)-\hat{x}(\phi_m) |$, and we get $\lim_n \| \hat{x}_n - \hat{x} \| = \lim_n \sup_{n<m} | \hat{x}(\phi_m) | = 0$.

Therefore, $\lim_n \hat{x}_n = \hat{x}$ and since the representation of A as an algebra of continuous functions on Φ is a homeomorphism, we get $\lim_n x_n = x$.

Therefore, $x\varepsilon\overline{S(A)}$.

REFERENCES

1. Alexander, J.C. Compact Banach Algebras. Proc. London Math. Soc. (3) 18, 1968, 1-18
2. Al-Moajil, A.H. The Compactum of a Primitive B* Algebra. Math. Japonica 26 (4) (1981), 385-387.
3. Al-Moajil, A.H. The Compactum and Finite Dimensionality in Banach Algebras. Internat. J. Math. & Math. Sci., 5, (2), (1982), 275-280.
4. Rickart C.E., General Theory of Banach Algebras. Von Nostrand, Princeton, N.J., MR 22 No. 5903 (1960).