Internat. J. Math. & Math. Sci.
Vol. 7 No. 3 (1984) 599-614

UNSTEADY FREE CONVECTION OF A STAGNATION POINT OF ATTACHMENT ON AN ISOTHERMAL SURFACE

D.B. INGHAM and J.H. MERKIN

Department of Applied Mathematical Studies
The University of Leeds
Leeds LS2 9JT, England

and

I. POP

Faculty of Mathematics
University of Cluj
Cluj, Romania

(Received July 21, 1982)

ABSTRACT: The time development free convection flow near a three-dimensional stagnation point of attachment on an isothermal surface is studied at large Grashof numbers. A small time solution and an accurate numerical method is described for determining the solution of the time-dependent boundary-layer equations. For a range of values of parameter c, which describes the local geometry, the development of the various physical properties of the flow are calculated and compared with their values at small and large values of time. In another range of values of c the numerical results suggest the development of a singularity in the boundary-layer equations at a finite value of time. An anlysis is presented which is consistent with the numerical results and confirms the presence of this singularity.

KEY WORDS AND PHRASES: free connection, boundary layer flows
1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. 76R10, 76N05.

1. INTRODUCTION

Recent studies on the unsteady free and mixed convection heat transfer over two-dimensional or axi-symmetric bodies with various initial and boundary conditions have been reported by a number of workers; Elliott (1), Gupta and Pop (2), Ingham (3), Kikkawa and Ohnishi (4), Katagiri and Pop (5, 6), Jain and Lohar (7), Arunachalam and Rajappa (8) and Ingham and Merkin (9).

The purpose of the present paper is to furnish some information on the time dependent free convection flow near a three-dimensional stagnation point of attachment on an isothermal surface. The boundary-layers are of both nodal and saddle-point type (for details see Banks (10, 11)). We assume that the surface is initially at the same temperature T_∞ as the surrounding fluid which is at rest.

Then, at time $t = 0$, the temperature of the body is suddenly changed to T_w, a constant, and then maintained at this value throughout. The governing equations for this situation comprise an eighth-order system of partial differential equations involving the Prandtl number Pr and, in addition, a parameter c which describes the local geometry of the surface. The equations are integrated numerically and the results of these integrations are presented for various positive values of c, which correspond to nodal points of attachment and for negative values of c, corresponding to saddle-points of attachment. In order to reduce the number of computations the Prandtl number has been taken to be unity throughout this paper.

Series valid for small times are also derived for the stream functions and the temperature. These are used to determine the skin friction and heat transfer parameters for small times, and these are compared with the numerical solution.

For values of c greater than some value, say c^*, (where $c^* = -0.1637$ for unit Prandtl number) the numerical calculations could be continued until the steady state has been reached. These solutions agree with the solutions obtained when solving the steady state equations. For c less than c^* it was found that a steady state solution did not appear to be approached. A theory is presented which is consistent with the numerical results and which leads to the presence of a singularity at a finite time.

2. EQUATIONS OF MOTION

Consider the unsteady three-dimensional free convection flow of an incompressible viscous fluid near a regular surface. Dissipation effects are assumed to be negligble and all physical properties of the fluid are taken as constant except the density variations in the buoyancy terms. Under these assumptions the boundary-layer equations in the vicinity of the stagnation point may be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 , \tag{2.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = g\beta(T-T_\infty)ax + \nu\frac{\partial^2 u}{\partial z^2} , \tag{2.2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = g\beta(T-T_\infty)by + \nu\frac{\partial^2 v}{\partial z^2} , \tag{2.3}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \kappa\frac{\partial^2 T}{\partial z^2} , \tag{2.4}$$

where (x,y,z) are Cartesian coordinates with origin O at the stagnation point, and such that z measures distance normal to the surface at O, (u,v,w) are the respective velocity components, t is the time, T is the temperature with T_∞ the temperature of the ambient fluid, g is the acceleration due to gravity, β is the coefficient of thermal expansion, κ is the thermal diffusivity and ν is the kinematic viscosity. The parameters a and b are the principal curvatures at O.

We look for a solution of equations (2.1)-(2.4) subject to the initial and boundary conditions that

$$\begin{aligned} &\text{for } t \leqslant 0, \quad u = v = w = 0, \quad T = T_\infty \quad \text{everywhere}, \\ &\text{for } t \geqslant 0, \quad u = v = w = 0, \quad T = T_w \quad \text{on} \quad z = 0, \\ &\qquad\qquad u \to 0, \ v \to 0, \ T \to T_\infty \quad \text{as} \quad z \to \infty, \end{aligned} \tag{2.5}$$

where T_w is the constant body temperature.

To find the solution of (2.1)-(2.5) it is convenient to introduce the stream functions f and g, where, for the present problem, in terms of the Grashof number $G_r = g\beta(T_w - T_\infty)/a^3\nu^2$ we write

$$\left.\begin{aligned} &u = \nu a^2 G_r^{1/2}\, x \frac{\partial \bar{f}}{\partial \bar{z}}, \quad v = \nu a^2 G_r^{1/2}\, y \frac{\partial \bar{g}}{\partial \bar{z}}, \quad w = -\nu a G_r^{1/4}(\bar{f} + \bar{g}) \\ &h = (T - T_\infty)/(T_w - T_\infty), \qquad \tau = \nu a^2 G_r^{1/2}\, t, \\ &\bar{z} = a\, G_r^{1/4}\, z. \end{aligned}\right\} \tag{2.6}$$

Equations (2.1)-(2.4) then become

$$\frac{\partial^2 \bar{f}}{\partial \bar{z}\partial \tau} + \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^2 - (\bar{f} + \bar{g})\frac{\partial^2 \bar{f}}{\partial \bar{z}^2} = h + \frac{\partial^3 \bar{f}}{\partial \bar{z}^3}, \tag{2.7}$$

$$\frac{\partial^2 \bar{g}}{\partial \bar{z}\partial \tau} + \left(\frac{\partial \bar{g}}{\partial \bar{z}}\right)^2 - (\bar{f} + \bar{g})\frac{\partial^2 \bar{f}}{\partial \bar{z}^2} = c\,h + \frac{\partial^3 \bar{g}}{\partial \bar{z}^3}, \tag{2.8}$$

$$\frac{\partial h}{\partial \tau} - (\bar{f} + \bar{g})\frac{\partial h}{\partial \bar{z}} = \frac{1}{Pr}\frac{\partial^2 h}{\partial \bar{z}^2}. \tag{2.9}$$

where $c = b/a$ is a parameter describing the local geometry of the body. When $c = 1$, clearly $f \equiv g$ and we recover the solution for the stagnation point flow on a body of revolution. If $c = b = o$, then $g \equiv 0$ and the problem reduces to the two dimensional stagnation point flow.

Equations (2.7)-(2.9) are not in a form appropriate for solving for small τ and, to do this, we must first make the further substitution

$$\bar{f} = 2\tau^{3/2} f(\eta,\tau) , \quad \bar{g} = 2\tau^{3/2} g(\eta,\tau) , \quad \eta = \bar{z}/2\tau^{1/2} . \tag{2.10}$$

Equations (2.7)-(2.9) then become

$$\frac{\partial^3 f}{\partial\eta^3} + 2\eta \frac{\partial^2 f}{\partial\eta^2} - 4 \frac{\partial f}{\partial\eta} + 4h = 4\tau \frac{\partial^2 f}{\partial\eta\partial\tau} + 4\tau^2 \left[\left(\frac{\partial f}{\partial\eta}\right)^2 - (f+g) \frac{\partial^2 f}{\partial\eta^2} \right] , \tag{2.11}$$

$$\frac{\partial^3 g}{\partial\eta^3} + 2\eta \frac{\partial^2 g}{\partial\eta^2} - 4 \frac{\partial g}{\partial\eta} + 4ch = 4\tau \frac{\partial^2 g}{\partial\eta\partial\tau} + 4\tau^2 \left[\left(\frac{\partial g}{\partial\eta}\right)^2 - (f+g) \frac{\partial^2 g}{\partial\eta^2} \right] \tag{2.12}$$

$$\frac{1}{Pr} \frac{\partial^2 h}{\partial\eta^2} + 2\eta \frac{\partial h}{\partial\eta} = 4\tau \frac{\partial h}{\partial\tau} - 4\tau^2 (f+g) \frac{\partial h}{\partial\eta} . \tag{2.13}$$

The initial and boundary conditions corresponding to (2.5) are

$$\left.\begin{aligned} &\tau < 0 , \quad f = g = h = 0 \quad \text{everywhere.} \\ &\tau \geq 0 . \quad f = \frac{\partial f}{\partial\eta} = g = \frac{\partial g}{\partial\eta} = 0 , \quad h = 1 \quad \text{on} \quad \eta = 0 , \\ &\qquad \frac{\partial f}{\partial\eta} \to 0 , \quad \frac{\partial g}{\partial\eta} \to 0 \quad \text{as} \quad \eta \to \infty . \end{aligned}\right\} \tag{2.14}$$

3. SMALL TIME SOLUTION

We can obtain a solution to equations (2.11)-(2.13) valid for small τ by expanding $f(\eta,\tau)$, $g(\eta,\tau)$ and $h(\eta,\tau)$ in powers of τ which have the forms

$$\left.\begin{aligned} f(\eta,\tau) &= f_0(\eta) + \tau^2 f_1(\eta) + \dots \\ g(\eta,\tau) &= g_0(\eta) + \tau^2 g_1(\eta) + \dots \\ h(\eta,\tau) &= h_0(\eta) + \tau^2 h_1(\eta) + \dots \end{aligned}\right\} \tag{3.1}$$

Upon substituting (3.1) into equations (2.11)-(2.13) we find that

$$\left.\begin{aligned} &\frac{1}{Pr} h_0'' + 2\eta h_0' = 0 , \\ &h_0(0) = 1 , \quad h_0(\infty) = 0 . \end{aligned}\right\} \tag{3.2}$$

$$\left.\begin{aligned} &f_0''' + 2\eta f_0'' - 4 f_0' + 4 h_0 = 0 , \\ &f_0(0) = f_0'(0) = 0 , \quad f_0'(\infty) = 0 . \end{aligned}\right\} \tag{3.3}$$

$$\left.\begin{aligned} &g_0''' + 2\eta\, g_0'' - 4g_0' + 4\,c\,h_0 = 0\,, \\ &g_0(0) = g_0'(0) = 0\,,\quad g_0'(\infty) = 0\,. \end{aligned}\right\} \quad (3.4)$$

$$\left.\begin{aligned} &\frac{1}{Pr}\, h_1'' + 2\eta\, h_1' - 8h_1 = -4(1+c)\, f_0 h_0'\,. \\ &h_1(0) = h_1(\infty) = 0\,. \end{aligned}\right\} \quad (3.5)$$

$$\left.\begin{aligned} &f_1''' + 2\eta\, f_1'' - 12\, f_1' + 4h_1 = 4\, f_0'^2 - 4(1+c)\, f_0 f_0''\,, \\ &f_1(0) = f_1'(0) = f_1'(\infty) = 0\,. \end{aligned}\right\} \quad (3.6)$$

$$\left.\begin{aligned} &g_1''' + 2\eta\, g_1'' - 12\, g_1' + 4ch_1 = 4\, c^2 f_0'^2 - 4c(1+c)\, f_0 f_0'' \\ &g_1(0) = g_1'(0) = 0\,,\quad g_1'(\infty) = 0\,. \end{aligned}\right\} \quad (3.7)$$

(Here dashes denote differentiation with respect to η). The linear differential equations (3.2)-(3.7) are solved successively. However, the solutions of the higher order equations rapidly become complicated and much laborious work is needed to obtain solutions in an analytic form and so we give here only the first two terms in each expansion and limit attention to the case of $Pr = 1$. We find that

$$h_0 = \operatorname{erfc} \eta = \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta^2}\, d\eta\,. \quad (3.8)$$

$$f_0 = g_0 c^{-1} = -\frac{2}{3} \operatorname{erfc} \eta + \frac{1}{3\sqrt{\pi}} (2\,\eta^2 - 1)\, e^{-\eta^2} + \frac{1}{3\sqrt{\pi}}\,, \quad (3.9)$$

$$\begin{aligned} h_1 = (1+c)\,\{&-(\tfrac{1}{4} + \eta^2 + \tfrac{1}{3}\eta^4)\operatorname{erfc}^2\eta + \frac{1}{\sqrt{\pi}}(\tfrac{11}{6}\,\eta + \eta^3)e^{-\eta^2}\operatorname{erfc}\eta \\ &- \frac{2}{3\sqrt{\pi}}(1+\eta^2)\, e^{-2\eta^2} - \frac{4}{15\pi}\, e^{-\eta^2} + (\tfrac{1}{4} + \tfrac{14}{15\pi})\,[(1+4\eta^2 \\ &+ \tfrac{4}{3}\eta^4)\operatorname{erfc}\eta - \frac{2}{3\sqrt{\pi}}(5\eta + 2\eta^3)\, e^{-\eta^2}]\}\,, \end{aligned} \quad (3.10)$$

$$f_1 = -[\frac{1}{2} + \frac{74}{45\pi} + (\frac{1}{4} + \frac{44}{45\pi})c][(1 + 6\eta^2 + 4\eta^3) + \frac{8}{15}\eta^6)\,\mathrm{erfc}\,\eta$$

$$- \frac{8}{15\sqrt{\pi}}(\eta^5 + 7\eta^3 + \frac{33}{4}\eta)\,e^{-\eta^2}] + [\frac{1}{4} + (2 + \frac{1}{2}c)\eta^2$$

$$+ \frac{1}{3}(5 + 2c)\eta^4 + \frac{2}{9}(2-c)\eta^6]\,\mathrm{erfc}^2\eta$$

$$+ \{-\frac{1}{9\sqrt{\pi}}[\frac{3}{2}(13+c)\eta + (23+11c)\eta^3 + 4(2-c)\eta^5]\,e^{-\eta^2}$$

$$+ (\frac{1}{4} + \frac{14}{15\pi}(1+c)(1 + 4\eta^2 + \frac{4}{3}\eta^4) - \frac{8}{15\sqrt{\pi}}(1+c)\eta\}\,\mathrm{erfc}\,\eta$$

$$+ \frac{1}{9\pi}[2(2-c) + (8+5c)\eta^2 + 2(2-c)\eta^4]\,e^{-2\eta^2}$$

$$- \frac{2}{3\sqrt{\pi}}(\frac{1}{4} + \frac{14}{15\pi})(1+c)(5\eta + 2\eta^3)\,e^{-\eta^2} + \frac{4}{15\pi}(1+c)\,e^{-\eta^2}\,, \qquad (3.11)$$

$$g_1' = c\{-[\frac{1}{4} + \frac{44}{45\pi} + (\frac{1}{2} + \frac{74}{45\pi})c][(1 + 6\eta^4 + \frac{8}{15}\eta^6)\,\mathrm{erfc}\,\eta$$

$$- \frac{8}{15\sqrt{\pi}}(\eta^5 + 7\eta^3 + \frac{33}{4}\eta)\,e^{-\eta^2}] + [\frac{1}{4}c + (2c + \frac{1}{2})\eta^2$$

$$+ \frac{1}{3}(2+5c)\eta^4 + \frac{2}{9}(2c-1)\eta^6]\,\mathrm{erfc}^2\eta$$

$$+ (-\frac{1}{9\sqrt{\pi}}[\frac{3}{2}(1+13c)\eta + (23c+11)\eta^3 + 4(2c-1)\eta^5]\,e^{-\eta^2}$$

$$+ (1+c)(\frac{1}{4} + \frac{14}{15\pi})(1 + 4\eta^2 + \frac{4}{3}\eta^4) - \frac{8}{15\sqrt{\pi}}(1+c)\eta)\,\mathrm{erfc}\,\eta$$

$$+ \frac{1}{9\pi}[2(2c-1) + (5+8c)\eta^2 + 2(2c-1)\eta^4]\,e^{-2\eta^2}$$

$$- \frac{2}{3\sqrt{\pi}}(1+c)(\frac{1}{4} + \frac{14}{15\pi})(5\eta+2\eta^3)\,e^{-\eta^2} + \frac{4}{15\pi}(1+c)\,e^{-\eta^2}\}\,, \qquad (3.12)$$

where

$$f_0''(0) = -h_0'(0) = \frac{2}{\sqrt{\pi}}\,, \quad g_0''(0) = \frac{2}{\sqrt{\pi}}c\,,$$

$$h_1'(0) = -0.04766(1+c),\ f''(0) = -0.03824 - 0.01461c\,,$$

$$g_1''(0) = -0.01461c - 0.03824c^2\,.$$

The skin friction and heat transfer parameters C_x, C_y and C_h, defined by

$$C_x = \frac{\mu}{\rho\nu^2 a^3 x}\left(\frac{\partial u}{\partial z}\right)_{z=0}, \quad C_y = \frac{\mu}{\rho\nu^2 a^3 y}\left(\frac{\partial v}{\partial z}\right)_{z=0} \text{ and } C_h = -\frac{1}{a(T_w - T_\infty)}\left(\frac{\partial T}{\partial z}\right)_{z=0}$$

are then, for small τ,

$$C_x = \tau^{1/2} G_r^{3/4} (0.56419 - (0.01912 + 0.00741\, c)\, \tau^2 + \ldots) \qquad (3.13)$$

$$C_y = \tau^{1/2} G_r^{3/4} (0.56419 - (0.00741c + 0.01912c^2)\, \tau^2 + \ldots) \qquad (3.14)$$

$$C_h = \tau^{-1/2} G_r^{1/4} (0.56419 + 0.02383(1+c)\, \tau^2 + \ldots)\,. \qquad (3.15)$$

4. NUMERICAL SOLUTION

Equations (2.11), (2.12) and (2.13) were solved numerically using essentially the same method as that described in (9). The method involves first replacing the τ-derivatives by forward differences and averaging all the other terms over the step length from τ to $\tau + \Delta\tau$. This leads to three non-linear ordinary differential equations which are then differenced using central differences and the resulting non-linear algebraic equations solved iteratively by the Newton-Raphson process. Equations (2.11), (2.12) and (2.13) are in a form suitable only for solving for small values of τ and consequently it is more appropriate for large τ to solve equations (2.7), (2.8) and (2.9). The change over from one set of equations to the other was done at $\tau = 1$. To keep a check on the errors introduced by differencing in the τ-direction, the step from τ to $\tau + \Delta\tau$ was covered in one and then two steps and then insisting that the difference between the two solutions thus obtained was less than some small pre-assigned quantity (5.10^{-5} in the present context). Also this check could be used for increasing the step length as required. All the integrations were carried out with a transverse step length of $\Delta\eta = 0.025$ (for $\tau < 1$) and $\Delta\bar{z} = 0.05$ (for $\tau > 1$) with the outer boundary condition applied at $\eta = 5$ and $\bar{z} = 10$ (except for the case of $c = -0.5$ which is described below). Initially $\Delta\tau = 0.1$ but this could be increased considerably as the steady state was approached.

Values of C_x, C_y and C_h obtained from the numerical integration are given in tables 1, 2 and 3 respectively, (note that for $c = 0$, $C_y \equiv 0$, and for $c = 1$, $C_x = C_y$) for various values of τ and c. From these we can see that the steady state is rapidly approached as $\tau \to \infty$ and that the rate at which this steady state is approached does not appear to depend significantly on the value of c, with the approach being slightly slower for $c < 0$. Also, for the case $c = 0.5$ values of C_x, C_y and C_h as calculated from the series (3.13), (3.14) and (3.15) and from the numerical integrations are shown in figure 1. From this we can see that the series for small τ give a good approximation even up to values of $\tau \sim 2.0$.

In considering the steady state problem, Banks (11) showed that for each c in the range $c \geqslant c^*$ (where $c^* = -0.1637$ for Pr = 1) there were two solutions, and that there were no solutions for $c < c^*$. The results presented here (for all $c > c^*$) all tend to that steady state which has the greater heat transfer.

A numerical integration of the equations was also performed for $c = -0.5$ as representative of the cases where no steady state solution exists. Here it was found that the boundary layer thickness increased rapidly as the integration proceeded and in order to satisfy the outer bounder condition with sufficient accuracy this was put at $\bar{z} = 40$ now using a step length $\Delta\bar{z} = 0.1$. The numerical results indicate strongly the existence of a singularity in the solution at a finite time τ_s (say). The nature of this singularity is similar to those found by Banks and Zaturska (12), Simpson and Stewartson (13,14) and Brown and Simpson (15) in related problems. The solution near the singularity has the characteristsic three region strucure, with inner and outer viscous regions and a thick inviscid middle region. This is shown clearly in figure 2 where graphs of $\partial\bar{f}/\partial\bar{z}$, $\partial\bar{g}/\partial\bar{z}$ and h are plotted against $\bar{z}$ for $\tau = 6.3$. From this we can infer that $\partial\bar{f}/\partial\bar{z}$ and h both remain finite at $\tau \to \tau_s$, but that $\bar{g}$ becomes infinite.

5. ANALYSIS OF SINGULARITY

The nature of the solution near the singularity is dictated by the inviscid middle region, and to obtain a solution in this region we first put

$$\bar{f} = \xi^{-1/2} F(\zeta,\xi) , \quad \bar{g} = \xi^{-3/2} G(\zeta,\xi) , \quad h = h(\zeta,\xi) ,$$
$$\xi = \tau_s - \tau \quad \text{and} \quad \zeta = (\bar{z} - \beta(\xi))\, \xi^{1/2} . \tag{5.1}$$

The choice of ζ in (5.1) reflects the shift of origin of this middle region with $\beta(\xi)$ a function to be determined. Equations (2.7)-(2.9) become, for Pr = 1,

$$\begin{aligned}(G + \tfrac{\zeta}{2}) \frac{\partial^2 G}{\partial \zeta^2} - \frac{\partial G}{\partial \zeta}\left(1 + \frac{\partial G}{\partial \zeta}\right) &= \xi^{3/2} \frac{d\beta}{d\xi} \frac{\partial^2 G}{\partial \zeta^2} \\ \xi^2 \left(\frac{\partial^3 G}{\partial \zeta^3} + c\, h\right) &- \xi F \frac{\partial G}{\partial \zeta} - \xi \frac{\partial^2 G}{\partial \xi \partial \zeta} ,\end{aligned} \tag{5.2}$$

$$\begin{aligned}(G + \tfrac{\zeta}{2}) \frac{\partial^2 F}{\partial \zeta^2} &= \xi^{3/2} \frac{d\beta}{d\xi} \frac{\partial^2 F}{\partial \zeta^2} - \xi^2 \frac{\partial^3 F}{\partial \zeta^3} - \xi\, h \\ &- \xi \left(F \frac{\partial^2 F}{\partial \zeta^2} - \left(\frac{\partial F}{\partial \zeta}\right)^2\right) - \xi \frac{\partial^2 F}{\partial \xi \partial \zeta} ,\end{aligned} \tag{5.3}$$

$$(G + \tfrac{\zeta}{h}) \frac{\partial h}{\partial \zeta} = \xi^{3/2} \frac{d\beta}{d\xi} \frac{\partial h}{\partial \zeta} - \xi^2 \frac{\partial^2 h}{\partial \zeta^2} - \xi F \frac{\partial h}{\partial \zeta} - \xi \frac{\partial h}{\partial \xi} . \tag{5.4}$$

We can see from equations (5.2)-(5.4) that F and h play only a passive role in this solution; the main problem is in determining G which we now expand as

$$G(\zeta, \xi) = G_0(\zeta) + \xi^{1/2} \log \xi \, G_1(\zeta) + \xi^{1/2} G_2(\zeta) + \ldots \tag{5.5}$$

There are corresponding expansions for F and h. The reason for taking this expansion for G is discussed below. The equation for G_0 is

$$(G_0 + \tfrac{\zeta}{2}) G_0'' - G_0' (G_0' + 1) = 0 , \tag{5.6}$$

(dashes now denote differentiation with respect to ζ). The general solution of equation (5.6) is $G_0 = - \zeta/2 - 1/2\alpha \sin(\alpha\zeta+\gamma)$ for constants α and γ. Now, in the viscous inner region (next to the body surface) the governing equations are equations (2.7)-(2.9) (with τ replaced by ξ). Consequently, the solution for G for small ζ must not generate any inverse powers of ξ in $\bar{g}$. This requires G_0 to be $O(\zeta^3)$ for small ζ and so $\gamma = \pi$ and

$$G_0 = - \frac{\zeta}{2} + \frac{1}{2\alpha} \sin(\alpha\zeta) . \tag{5.7}$$

The solutions for F_0 and h_0 are then, from equations (5.3) and (5.4),

$$F_0 = A_0 \zeta , \qquad h_0 = B_0 . \tag{5.8}$$

(for some constants A_0 and B_0). Again the solution for F_0 must be such that F should not contribute any inverse powers of ξ to $\bar{f}$.

There is a problem in determining the next term in the expansion for G.

This is generated by the term involving $\xi^{3/2}\, d\beta/d\xi$ in equation (5.2). If this term is $O(\xi^{m})$, say, then β will be $O(\xi^{m-1/2})$ if $m \neq 1/2$ and $O(\log \xi)$ if $m = 1/2$. The numerical solution indicates that the thickness of the inner region is growing as the singularity is approached (i.e. as $\xi \to 0$) which requires $m \leq 1/2$, but the solution of the equation thus generated gives rise to fractional powers of $\bar{z}$ in the inner region unless $m = 1/2$. Consequently we take $m = 1/2$ and $\beta = \beta_0 \log \xi$ (for some constant β_0). However, we find it is not sufficient just to include a term of $O(\xi^{1/2})$ in (5.5), but we also need a term of $O(\xi^{1/2} \log \xi)$ as will become apparent below.

Substituting (5.5) into equation (5.2) and equating like powers gives rise to the equations for G_1 and G_2 as

$$(G_0 + \tfrac{\zeta}{2})\, G_1'' - (\tfrac{1}{2} + 2\, G_0')\, G_1' + G_0''\, G_1 = 0 \, , \tag{5.9}$$

$$(G_0 + \tfrac{\zeta}{2})\, G_2'' - (\tfrac{1}{2} + 2\, G_0')\, G_2' + G_0''\, G_2 = \beta_0\, G_0'' - G_1' \, . \tag{5.10}$$

The general solution of equation (5.9) is

$$G_1 = C_1(1 - \cos(\alpha\zeta)) + D_1\left[\cos^2(\tfrac{\alpha\zeta}{2}) + 2\, \sin^2(\tfrac{\alpha\zeta}{2})\, \log(\sin\alpha\zeta)\right] \tag{5.11}$$

where C_1 and D_1 are constants. To avoid the logarithmic singularity in G_1 at $\zeta = \pi/\alpha$ we must have $D_1 = 0$ and so

$$G_1 = C_1(1 - \cos(\alpha\zeta)) \, . \tag{5.12}$$

The complementary function for equation (5.10) is the same as (5.11) and so avoiding the logarithmic singularity, we find that

$$G_2 = (\beta_0 + 2C_1) + C_2(1 - \cos(\alpha\zeta)) \, , \tag{5.13}$$

for some constant C_2. Now (5.13) generates a term of $O(\xi^{-1})$ in $\bar{g}$ unless $C_1 = -\beta_0/2$. Then $G_2 = C_2(1-\cos(\alpha\zeta))$. We can seen that without the term of $O(\xi^{1/2}\log \xi)$ in (5.5) (i.e. $C_1 = 0$) then equation (5.13) would give rise to a term of $O(\xi^{-1})$ in $\bar{g}$ which could then not be matched onto the inner solution.

The solutions for F_1, h_1, F_2 and h_2, again chosen so that F and h can be matched onto the inner solution, are

$$F_1 = A_1 \, , \quad F_2 = A_2 - 2(A_1 + \beta_0\, A_0)\, \log\, (\sin \tfrac{\alpha\zeta}{2}) \, , \tag{5.14}$$

$$h_1 = 0 \, , \quad h_2 = 0 \, , \tag{5.15}$$

for some constants A_1 and A_2.

As noted above, the inner region is governed by equations (2.7)-(2.9) (with τ replaced by ξ). The leading order terms in an expansion of the solution in powers of ξ cannot be determined from the equations. We know only that they must satisfy certain compatability conditions on $\bar{z} = 0$ and match with the solution in the middle region, which, from (5.7) and (5.8) gives $\bar{g} \sim -\alpha^2 z^3/12$, $\bar{f} \sim A_0 z$ and $h \to B_0$ as $\bar{z} \to \infty$.

The solution in the middle region holds only upto $\zeta = 2\pi/\alpha$; beyond this point the diffusion terms will again be important and a further outer (viscous) region is needed in order that the outer boundary conditions be satisfied smoothly. The equations for this outer region are again (2.7)-(2.9), now centred on $\zeta = 2\pi/\alpha$ or $\bar{z} = (2\pi/\alpha)\xi^{-1/2} - \beta_0 \log \xi$. As in the inner region, the leading terms cannot be determined from the equations.

The theory outlined above shows that, from (5.7), $\bar{g}(\infty,\tau) \sim -(\tau_s-\tau)^{-3/2}\pi/\alpha$ as $\tau \to \tau_s$. This is confirmed by the numerical results shown in figure 3 where $[-\bar{g}(\infty,\tau)]^{-2/3}$ is plotted against τ for $c = -0.5$, resulting in the required straight line. Also from this graph we can get an estimate for τ_s as $\tau_s = 6.355$ and, from the slope, estimate the value of α as $\alpha \doteq 0.950$. Numerical calculations have been performed for several other values of $c < c^*$ and all show similar characteristics to those presented for $c = -0.5$, with the time at which the singularity appears decreasing as $|c|$ increases.

Thus it is concluded that if $c > c^*$ the steady state boundary-layer solution can be approached from an unsteady situation whereas if $c < c^*$ the solution of the unsteady boundary-layer equations develops a singularity at a finite value of time.

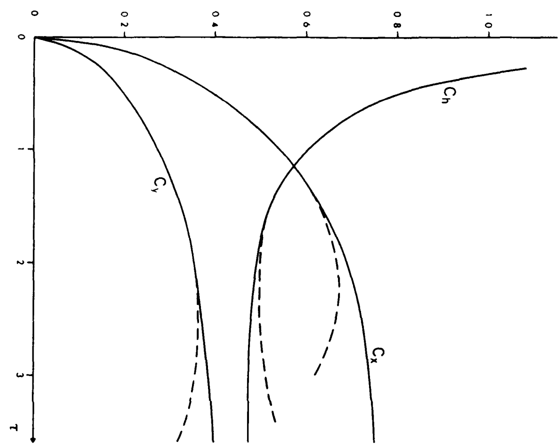

Figure 1: Graphs of C_x, C_y and C_h for c = 0.5
[full line is numerical solution, broken line series solutions (3.13), (3.14) and (3.15), respectively.]

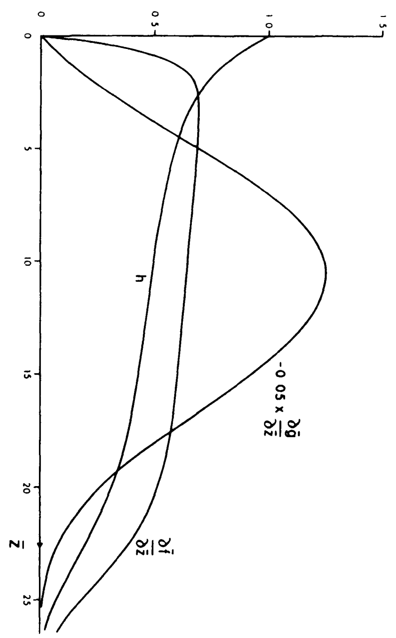

Figure 2: Graphs of h, $\partial \bar{f}/\partial \bar{z}$ and -0.05 $\partial \bar{g}/\partial \bar{z}$ against x for c = -0.5 at I = 6.3

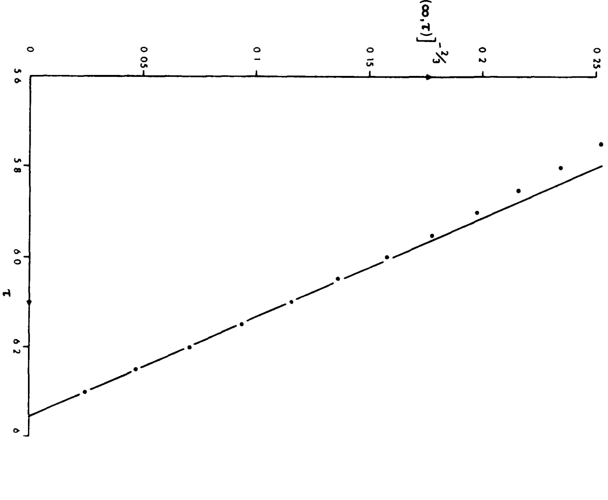

Figure 3: Graph of $[-\bar{g}(\infty, I)]^{-2/3}$ against I for c = 0.5

TABLE 1

C_x for various c

c / τ	-0.15	-0.1	0	0.25	0.5	0.75	1.0	2.0
0.1	0.1783	0.1783	0.1783	0.1783	0.1783	0.1783	0.1783	0.1782
0.2	0.2519	0.2518	0.2520	0.2519	0.2519	0.2519	0.2518	0.2517
0 3	0.3081	0.3081	0.3081	0.3080	0.3079	0.3078	0.3077	0.3073
0.4	0.3549	0.3550	0.3549	0.3547	0.3545	0.3543	0.3542	0.3537
0 5	0.3957	0.3957	0.3956	0.3953	0.3950	0.3946	0.3943	0.3930
0 6	0.4320	0.4320	0.4318	0.4313	0.4308	0.4303	0.4298	0.4277
0.7	0.4647	0.4647	0.4644	0.4637	0.4629	0.4622	0.4615	0.4586
0.8	0.4946	0.4945	0.4941	0.4930	0.4920	0.4910	0.4900	0.4860
0.9	0.5219	0.5217	0.5212	0.5198	0.5185	0.5172	0.5158	0.5106
1.0	0.5470	0.5468	0.5461	0.5444	0.5426	0.5409	0.5392	0.5325
1.1	0.5702	0.5699	0.5690	0.5668	0.5647	0.5625	0.5604	0.5519
1.3	0.6114	0.6109	0.6096	0.6064	0.6033	0.6001	0.5970	0.5848
1.4	0.6468	0.6460	0.6443	0.6398	0.6354	0.6311	0.6268	0.6105
1.9	0.7032	0.7018	0.6988	0.6913	0.6840	0.6770	0.6701	0.6449
2.3	0.7444	0.7422	0.7377	0.7267	0.7019	0.6934	0.6852	0.6558
2.7	0.7742	0.7711	0.7650	0.7403	0.7278	0.7164	0.7057	0.6690
3.1	0.7955	0.7915	0.7837	0.7654	0.7436	0.7298	0.7170	0.6750
3.5	0.8107	0.8058	0.7963	0.7746	0.7528	0.7369	0.7226	0.6774
4.3	0.8292	0.8225	0.8099	0.7829	0.7602	0.7418	0.7259	0.6781
5.1	0.8387	0.8303	0.8152	0.7847	0.7612	0.7420	0.7253	0.6777
6.7	0.8468	0.8340	0.8175	0.7840	0.7560	0.7408	0.7246	0.6775
8.3	0.8504	0.8363	0.8174	0.7833	0.7592	0.7403	0.7244	0.6774
9.9	0.8525	0.8380	0.8172	0.7830	0.7591	0.7402	0.7244	0.6774
∞	0.8589	0.8392	0.8170	0.7828	0.7591	0.7402	0.7244	0.6774

TABLE 2

C_y for various c

τ \ c	−0.15	−0.1	0.25	0.5	0.75	2.0
0.1	−0.0268	−0.0178	0.0446	0.0892	0.1338	0.3565
0.2	−0.0378	−0.0252	0.0630	0.1260	0.1889	0.5029
0.3	−0.0463	−0.0309	0.0771	0.1541	0.2310	0.6135
0.4	−0.0534	−0.0356	0.0889	0.1776	0.2660	0.7045
0.5	−0.0597	−0.0398	0.0992	0.1980	0.2964	0.7821
0.6	−0.0654	−0.0436	0.1084	0.2162	0.3230	0.8495
0.7	−0.0705	−0.0470	0.1168	0.2326	0.3475	0.9086
0.8	−0.0753	−0.0502	0.1244	0.2476	0.3694	0.0606
0.9	−0.0798	−0.0531	0.1315	0.2613	0.3894	1.0064
1.0	−0.0839	−0.0559	0.1381	0.2739	0.4076	1.0467
1.1	−0.0879	−0.0585	0.1442	0.2855	0.4242	1.0819
1.3	−0.0952	−0.0633	0.1552	0.3062	0.4534	1.1397
1.5	−0.1018	−0.0676	0.1648	0.3238	0.4770	1.1832
1.9	−0.1133	−0.0751	0.1804	0.3514	0.5143	1.2381
2.3	−0.1231	−0.0814	0.1921	0.3620	0.5277	1.2541
2.7	−0.1314	−0.0865	0.2008	0.3780	0.5468	1.2719
3.1	−0.1384	−0.0908	0.2067	0.3885	0.5583	1.2787
3.5	−0.1444	−0.0944	0.2111	0.3951	0.5648	1.2803
4.3	−0.1539	−0.0998	0.2158	0.4013	0.5697	1.2793
5.1	−0.1608	−0.1034	0.2177	0.4029	0.5703	1.2776
6.7	−0.1705	−0.1075	0.2185	0.4030	0.5696	1.2773
8.3	−0.1761	−0.1098	0.2186	0.4027	0.5693	1.2770
9.9	−0.1801	−0.1111	0.2186	0.4026	0.5692	1.2770
∞	−0.1928	−0.1129	0.2186	0.4026	0.5692	1.2770

TABLE 3

C_h for various c

c / τ	-0.15	-0.1	0	0.25	0.5	0.75	1.0	2.0
0.1	1.7856	1.7850	1.7851	1.7852	1.7854	1.7856	1.7858	1.7871
0.2	1.2634	1.2636	1.2638	1.2644	1.2649	1.2654	1.2659	1.2685
0.3	1.0339	1.0337	1.0341	1.0350	1.0360	1.0370	1.0379	1.0422
0 4	0.8976	0.8976	0.8982	0.8997	0.9011	0.9026	0.9041	0.0103
0.5	0.8054	0.8055	0.8063	0.8084	0.8105	0.8126	0.8147	0.8231
0.6	0.7380	0.7383	0.7394	0.7422	0.7449	0.7476	0.7503	0.7612
0.7	0.6863	0.6868	0.6882	0.6916	0.6951	0.6985	0.7019	0.7154
0.8	0.6453	0.6459	0.6476	0.6518	0.6560	0.6602	0.6643	0.6805
0.9	0.6119	0.6127	0.6147	0.6197	0.6247	0.6296	0.6345	0.6535
1.0	0.5841	0.5852	0.5875	0.5934	0.5992	0.6049	0.6105	0.6323
1.1	0.5608	0.5620	0.5647	0.5714	0.5781	0.5846	0.5910	0.5923
1.3	0.5237	0.5253	0.5288	0.5373	0.5457	0.5539	0.5620	0.5781
1.5	0.4959	0.4980	0.5022	0.5127	0.5229	0.5328	0.5424	0.5698
1.9	0.4577	0.4608	0.4667	0.4813	0.4951	0.5082	0.5207	0.5654
2.3	0.4341	0.4381	0.4459	0.4643	0.4824	0.5016	0.5154	0.5635
2.7	0.4190	0.4240	0.4335	0.4556	0.4761	0.4949	0.5107	0.5635
3.1	0.4091	0.4151	0.4263	0.4517	0.4741	0.4933	0.5104	0.5657
3.5	0.4026	0.4095	0.4223	0.4504	0.4750	0.4939	0.5118	0.5677
4.3	0.3952	0.4038	0.4193	0.4512	0.4760	0.4969	0.5151	0.5704
5.1	0.3916	0.4017	0.4191	0.4531	0.4784	0.4993	0.5178	0.5716
6.7	0.3882	0.4006	0.4202	0.4554	0.4810	0.5011	0.5186	0.5718
8.3	0.3862	0.4001	0.4210	0.4561	0.4812	0.5014	0.5187	0.5719
9.9	0.3848	0.3998	0.4212	0.4563	0.4813	0.5015	0.5188	0.5719
∞	0.3798	0.3992	0.4214	0.4564	0.4813	0.5015	0.5188	0.5179

REFERENCES

1. ELLIOTT, L. Free convection on a two-dimensional or axisymmetric body, Quart. J. Mech. Appl. Math 23 (1970) 153-162.
2. GUPTA, A.S. and POP, I. Effects of curvature on unsteady free convection past a circular cylinder, Phys. Fluids 20 (1977) 162-163.
3. INGHAM, D.B. Free-convection boundary layer on an isothermal horizontal cylinder, Z.A.M.P. (1978) 871-883.
4. KIKKAWA, S. and OHNISHI, T. Unsteady combined forced and natural convective heat transfer from horizontal circular and elliptic cylinders, Sci. and Eng. Dev. of Doshisha Univ. 19 (1978) 122-137.
5. KATAGIRI, M. and POP, I. Transient free convection from an isothermal horizontal circular cylinder, Warme-und Staffiit, 12 (1979) 73-81.
6. KATAGIRI, M. AND POP, I. Unsteady combined convection from an isothermal circular cylinder, Z.A.M.M. 59 (1979) 51-60.
7. JAIN, P.C. and LOHAR, B.L. Unsteady mixed convection heat transfer from a horizontal circular cylinder, Trans A.S.M.E. J. Heat Transfer 101 (1979) 126-131.
8. ARUNACHALAM, M. and RAJAPPA, M.R. Transient free convectin flow around two-dimensional or axisymmetric bodies, Acta Mechanica 35 (1980) 231-243.
9. INGHAM, D.B. and MERKIN, J.H. Unsteady mixed convection from an isothermal circular cylinder, ACTA MECHANICA 38 (1981) 55-69.
10. BANKS, W.H.H. Three-dimensional free convection flow near a two-dimensional isothermal surface, J. Eng. Math. 6 (1972) 109-115.
11. BANKS, W.H.H. Laminar free convection flow at a stagnation point of attachment on an isothermal surface, J. Eng. Math. 6 (1972) 109-115.
12. BANKS, W.H.H. and Zaturska, M.B. The collision of unsteady laminar boundary layers, J. Eng. Math. 13 (1979) 193-212.
13. SIMPSON, C.J. and STEWARTSON, K. A singularity in an unsteady convection boundary layer, to be published in Quart. J. Mech. Appl. Math.
14. STEWARTSON, K. and SIMPSON, C.J. On a singularity initiating a boundary-layer collision, Quart. J. Mech. Appl. Math. 35 (1982) 1-16.
15. BROWN, S.N. and SIMPSON, C.J. Collision phenomena in free convective flow over a sphere, to be published in J. Fluid Mech.