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MARKOVIAN EXTENSIONS AND REDUCTIONS OF A FAMILY OF σ-ALGEBRAS

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ABSTRACT. In the first part of this paper different properties of two special realizations of a stochastic dynamic system, as well as relationships between the realization problem and a problem derived from it are investigated. In the second part, solutions of the following two problems (that follow directly from the realization problem) are given: how to find the minimal (resp. maximal) markovian flow of information (understood as a family of σ-algebras) that contains (resp. is contained in) given output of a system, and is such that each of these two flows of information is equally predictable by the other one.

KEY WORDS AND PHRASES: *Markovian family of σ-algebras; equal predictability; stochastic dynamic system; extension and reduction.*
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1. INTRODUCTION.

Let (Ω,S,P) be an arbitrary probability space and let $H = (H_t)_{t\in R}$ be a family of σ-subalgebras of S; new families P_t, P_{t-}, F_t, F_{t+} are defined in the usual way: $P_t = \bigvee_{s\le t} H_s$, $P_{t-} = \bigvee_{s<t} H_s$, $F_t = \bigvee_{s\ge t} H_s$, $F_{t+} = \bigvee_{s>t} H_s$. One can think about P_{t-} and P_{t+} (resp. P_t and F_t) as about real past and real future (resp. past and future) at the moment t of a flow of information carried by H.

The fact that a real-valued random variable X with finite expectation, defined on (Ω,S,P), is measurable with respect to some σ-algebra K will be written as $X \in K$. If K_1, K_2 are arbitrary sub-σ-algebras of S, then the smallest σ-algebra with respect to which all conditional expectations $E(X|K_2)$, $X \in K_1$, are measurable, will be denoted as $E(K_1|K_2)$, [1].

The smallest σ-algebra that contains K_1 and K_2 will be denoted $K_1 \vee K_2$. If K_1 and K_2 are independent, we shall write $K_1 \underline{\vee} K_2$ instead of $K_1 \vee K_2$.

If for every event $A_2 \in K_2$ there exists an event $A_1 \in K_1$ such that $P(A_1 \Delta A_2) = 0$, then it will be said that $K_1 \supseteq K_2$ a.s. [P]. If $K_1 \supseteq K_2$ a.s. [P] and $K_2 \supseteq K_1$ a.s. [P], then it will be said that $K_1 = K_2$ a.s. [P] (compare with [2]). All relationships indicated among σ-algebras will hold only a.s. [P].

Family H will be called markovian if

$$E(F_t|P_t) = H_t, \ t \in R$$

It will be said that H is the minimal (resp. maximal) family having some certain property if any other family H* with the same property is such that $H_t^* \supseteq H_t$ (resp. $H_t^* \subseteq H_t$), $t \in R$.

Let A, A_1, A_2 be σ-subalgebras of S. A_1 and A_2 are said to be conditionally independent given A if

$$E(X_1X_2|A) = E(X_1|A)E(X_2|A) \text{ for all } X_1 \in A_1, \ X_2 \in A_2; \tag{1.1}$$

we write $A_1 \perp\!\!\!\perp A_2|A$ and say, also, that A is splitting for A_1 and A_2 (compare with [1] and [3]).

Let A and B be σ-subalgebras of S such that $A \subseteq B$. It is said that $A_{i,B}$ is the independent complement of A in B if $A_{i,B} \subseteq B$ and

(i) A and $A_{i,B}$ are independent σ-algebras;

(ii) $A \vee A_{i,B} = B$.

(see [2, Part II]). If $A = H_t$, we shall write $H_{t,i,B}$ instead of $(H_t)_{i,B}$.

If $A_{i,A_k \vee A}$ exists for k=1,2, then (1.1) is equivalent to the independence of $A_{i,A_1 \vee A}$ and $A_{i,A_2 \vee A}$.

We shall say that families H and H* are equally predictable by each other if

$$E(F_t^*|P_t) = E(F_t|P_t^*), \ t \in R. \tag{1.2}$$

If $H_t^* \subseteq H_t$, $t \in R$, equality (1.2) becomes

$$E(F_t^*|P_t) = E(F_t^*|P_t^*) = E(F_t|P_t^*), \ t \in R. \tag{1.3}$$

Following [4, p. 181], we might say that the first of these equalities means that H does not anticipate H*; in that case, the second equality can be interpreted to mean that H is not anticipated by H*.

If H*, $H_t^* \subseteq H_t$, $t \in R$, is a markovian family, then (1.3) becomes

$$E(F_t^*|P_t) = H_t^* = E(F_t|P_t^*), \ t \in R. \tag{1.4}$$

It is easy to see that in this case (1.4) is equivalent to

$$F_t^* \perp\!\!\!\perp P_t|H_t^*, \ F_t \perp\!\!\!\perp P_t^*|H_t^*, \ t \in R, \tag{1.5}$$

which proves the following result.

LEMMA 1. Let H* be a markovian family such that $H_t^* \subseteq H_t$, $t \in R$. Then H and H* are equally predictable by each other if and only if H_t^* is splitting for F_t^* and P_t as well as for F_t and P_t^* for every t. □

If $H_t^* \supseteq H_t$, $t \in R$, then (1.2) becomes

$$E(F^*_t|P_t) = E(F_t|P_t) = E(F_t|P_t^*), \ t \in R;$$

thus, H* is not anticipated by H and does not anticipate it. The following simple result will later be needed.

LEMMA 2. Let H* be a markovian family such that $H_t^* \supseteq H_t$, $t \in R$. If H* does not anticipate H, then $H^* \supseteq E(F_t|P_t)$, $t \in R$.

PROOF. Markovian property of H* and $F_t \subseteq F_t^*$ imply

$$H_t^* = E(F_t^*|P_t^*) \supseteq E(F_t|P_t^*) = E(F_t|P_t). \quad \square$$

In this paper we shall be concerned with the following two problems. The first one is establishing a relationship between the equal predictability of two families by each other and realizations of a stochastic dynamic system (which is, as we shall see, closely related to the problem of determining a σ-algebras splitting for two given σ-algebras) whose output is represented by one of these two families. The second one consists of finding, when H is given, the maximal (resp. minimal) markovian family H*, such that H and H* are equally predictable by each other and $H_t^* \subseteq H_t$ (resp. $H_t^* \supseteq H_t$), $t \in R$; those two families will be called, respectively, maximal markovian reduction and minimal markovian extension of family H.

2. PREDICTABILITY AND STOCHASTIC DYNAMIC SYSTEM.

A family $H = (H_t)_{t\in R}$ of σ-algebras from S can be considered as a flow of information representing output of a stochastic dynamic system. More precisely, a stochastic dynamic system consists of two families, H and H*, which satisfy condition

$$P_{t-} \vee P_t^* \perp\!\!\!\perp F_{t+} \vee F_t^* | H_t^*, \quad t \in R; \tag{2.1}$$

here, H represents outputs and H* states of that system. For a given family of outputs H, any family H* satisfying (2.1), is called a realization of the system with those outputs. In practice, two additional families $H^1 = (H_t^1)_{t\in R}$ and $H^2 = (H_t^2)_{t\in R}$, $H_t^1 \subseteq H_t^2$, $t \in R$, are given and it is required that a realization H* is to be such that

$$H_t^1 \subseteq H_t^* \subseteq H_t^2, \quad t \in R. \tag{2.2}$$

(for detailed definitions of a stochastic dynamic system and the realization problem see e.g. [1]).

If H* is a realization of a stochastic dynamic system, that is if (2.1) holds, then the following relations are true:

$$P_{t-} \perp\!\!\!\perp F_t^* | H_t^*, \quad P_t^* \perp\!\!\!\perp F_{t+} | H_t^*, \quad P_t^* \perp\!\!\!\perp F_t^* | H_t^*; \tag{2.3}$$

the third of these relations simply means that family H* is markovian. It is clear that, generally, (2.1) and (2.3) are not equivalent.

In the previous section, in connection with equal predictability of H and H* by each other, we were, when H was given, considering cases $H_t^* \subseteq H_t$ and $H_t^* \supseteq H_t$, $t \in R$. Corresponding cases concerning a realization H* of a stochastic dynamic system with outputs H are connected with families H^1 and H^2 from (2.2) in the following ways:

$$H_t^1 = \{\phi,\Omega\}, \quad H_t^2 = H_t, \quad t \in R; \tag{2.4}$$

$$H_t^1 = H_t, \quad H_t^2 = S, \quad t \in R. \tag{2.5}$$

Obviously, (2.4) means $H_t^* \subseteq H_t$ and (2.5) means $H_t^* \supseteq H_t$, $t \in R$.

If family H is given, then family H* that satisfies (2.3) is a markovian family which is not anticipated by the real past of H and does not anticipate the real future of H. It is clear that, if H and H* are equally predictable by each other, $H_t^* \subseteq H_t$, $t \in R$, and H* is markovian, then (2.3) holds, although the converse is not always true.

In this section we shall investigate relationships between (1.2), (2.1) and (2.3), when families H^1 and H^2 from (2.2) are given by (2.4) and (2.5). Some of the results obtained will lead to the main results, given in the next section.

Family H*, defined by $H_t^* = P_t$, $t \in R$, satisfies (1.2), (2.1), (2.3) and (2.5). However, family H*, defined by $H_t^* = \{\phi,\Omega\}$, satisfies (1.2), (2.3) and (2.4), but not

(2.1). It is clear that, just as it is of interest to look for the minimal realization of a dynamic system, in the case of a family H* satisfying (1.2) (or (2.3)), it is of interest to find maximal among such families in case (2.4), and minimal among them in case (2.5).

LEMMA 3. Let H* be a family satisfying (2.1). If $H^*_t \subseteq H_t$ and $H_t \perp\!\!\!\perp F_{t+}|P_{t-} \vee H^*_t$, $t \in R$, then family H itself is markovian.

PROOF. From (2.1) it follows $P_{t-} \perp\!\!\!\perp F_{t+}|H^*_t$, which is equivalent to $E(F_{t+}|P_{t-} \vee H^*_t) \subseteq H^*_t$, so that, because $H^*_t \subseteq P_{t-} \vee H^*_t$,

$$E(F_{t+}|P_{t-} \vee H^*_t) = E(F_{t+}|H^*_t),\ t \in R. \tag{2.6}$$

The assumption of Lemma 3 is equivalent to $E(F_{t+}|P_t) \subseteq P_{t-} \vee H^*_t$, which, because of $P_{t-} \vee H^*_t \subseteq P_t$ and (2.6), gives

$$E(F_{t+}|P_t) = E(F_{t+}|P_{t-} \vee H^*_t) = E(F_{t+}|H^*_t),\ t \in R$$

that is, equivalently,

$$F_{t+} \perp\!\!\!\perp P_t|E(F_{t+}|H^*_t),\ t \in R. \tag{2.7}$$

Since $E(F_{t+}|H^*_t) \subseteq H_t$, (2.7) implies $F_{t+} \perp\!\!\!\perp P_t|H_t$, that is

$$F_t \perp\!\!\!\perp P_t|H_t,\ t \in R,$$

which is, because $H_t \subseteq F_t$, $H_t \subseteq P_t$, equivalent to the fact that H is a markovian family.□

LEMMA 4. Let family H be an output of a stochastic dynamic system. The minimal realization H^m of that system, satisfying condition

$$H_t \subseteq H^m_t \subseteq P_t,\ t \in R, \tag{2.8}$$

is defined by

$$H^m_t = E(F_t|P_t),\ t \in R.$$

PROOF. It can be shown that H^*_t is splitting for P_{t-} and F_{t+} if and only if it is splitting for $P_{t-} \vee \bar{H}_t$ and $F_{t+} \vee \hat{H}$, where $\bar{H}_t \subseteq H^*_t$ and $\hat{H}_t \subseteq H^*_t$ (see [1]). Thus, because of (2.8), H^*_t must be splitting for P_t and F_t. But, the minimal σ-algebra from P_t that splits P_t and F_t is defined by $E(F_t|P_t)$, $t \in R$ (see [5]). Put $H^m_t = E(F_t|P_t)$, $t \in R$. From $H_s \subseteq H^*_s \subseteq P_s \subseteq P_t$ for $s \leq t$, it follows $P_t = P^m_t$, which gives

$$P_{t-} \vee P^m_t \perp\!\!\!\perp F_t|H^m_t,\ t \in R.$$

It is easy to check that H^m is a markovian family, which, because $F_t \subseteq F^m_t$, implies

$$P_{t-} \vee P^m_t \perp\!\!\!\perp F_{t+} \vee F^m_t|H^m_t,\ t \in R. \qquad \square$$

LEMMA 5. Let $A = (A_t)_{t \in R}$ be a family of σ-algebras such that $A_t \subseteq P_t$ and A_t is independent of H_t for every $t \in R$. If H* is a markovian family of σ-algebras such that $H^*_t \subseteq H_t$, $t \in R$ and that (1.5) is satisfied, then H^*_t is independent of $E(F_u|A_u)$ for all $t,u \in R$.

PROOF. The second equality in (1.5) and $A_u \subseteq P_u$ imply $E(E(F_u|A_u)|P^*_u) \subseteq E(F_u|P^*_u) = H^*_u$, but because $H^*_u \subseteq H_u$, this gives $E(E(F_u|A_u)|P^*_u) = \{\phi,\Omega\}$. Thus, H^*_t is independent of $E(F_u|A_u)$ for all $t \leq u$.

By using the first equality in (1.5) one gets $E(F^*_u|E(F_u|A_u)) \subseteq E(F^*_u|P_u) = H^*_u$, which, because H^*_u and A_u are independent, means that $E(F^*_u|E(F_u|A_u)) = \{\phi,\Omega\}$. Thus, H^* is independent of $E(F_u|A_u)$ for all $t \geq u$. □

COROLLARY 1. Let H be a family of σ-algebras such that H_{t,i,P_t} exists for every $t \in R$. If H^* is a markovian family of σ-algebras such that $H^*_t \subseteq H_t$, $t \in R$, and (1.5) is satisfied, then H^*_t is independent of $E(F_u|H_{u,i,P_u})$ for all $t,u \in R$.

3. MAIN RESULTS.

Let H be a given family of σ-algebras. The solution of the problem of finding its minimal markovian extension, is given by the following result.

THEOREM 1. If a family $H^m = (H^m_t)_{t\in R}$ is such that $H^m_t \supseteq H_t$, $t \in R$, then H^m is the minimal markovian extension of H if and only if H^m is the minimal realization (satisfying (2.2), where H^1 and H^2 are as in (2.5)) of a stochastic dynamic system whose output is H.

PROOF. Let H^m be the minimal realization (satisfying (2.2) and (2.5)) of a stochastic dynamic system whose output is H. According to Lemma 4, H^m is defined by $H^m_t = E(F_t|P_t)$, $t \in R$. From obvious equality $P^m_t = P_t$, it follows $E(F_t|P_t) = E(F_t|P^m_t)$, and, because H^m is markovian, $E(F^m_t|P_t) = (F^m_t|P^m_t) = H^m_t = E(F_t|P_t)$, which proves that H and H^m are equally predictable by each other.

If H^* is some other markovian extension of H (such that H^* and H are equally predictable by each other), then, according to Lemma 2, it follows that $H^*_t \supseteq E(F_t|P_t) = H^m_t$, which proves the minimality of H^m.

The other half of the proof is obvious. □

THEOREM 2. Let H be a family of σ-algebras such that H_{t,i,P_t} exists for every $t \in R$, and let A be a σ-algebra from P that is independent of $E(F_t|H_{t,i,P_t})$ for each $t \in R$. Then family $H^* = (H^*_t)_{t\in R}$, defined by

$$H^*_t = E(A|H_t),\ t \in R, \tag{3.1}$$

is a markovian reduction of H. If independent complement $\sum_{t,i,P}$ of $\sum_t = E(F_t|H_{t,i,P_t})$ exists for every $t \in R$, and if A from (3.1) is replaced by

$$A^M = \bigcap_{t\in R} \sum_{t,i,P}, \tag{3.2}$$

then family H^* is the maximal markovian reduction of H.

PROOF. Equation (3.1) implies $H^*_t \subseteq H_t$, $t \in R$, so that H^* is a reduction. If we define A_t by $A_t = E(A|P_t)$, then we get $H^*_t \subseteq A_t$, which gives $E(A|H_t) \subseteq E(H_t|A_t)$. But, on the other side, if an event $A \in H_t$ is independent of H^*_t, then it is independent of A, so that $E(H_t|A_t) \subseteq H^*_t$, which, together with the previous inclusion, gives

$$H^*_t = E(H_t|A_t),\ t \in R. \tag{3.3}$$

Let s,t be such that $s \geq t$. The markovian property of H^* is implied by (3.3) and

$$E(H^*_s|P^*_t) \subseteq E(H^*_s|A_t) = E(E(H_s|A_s)|A_t) = E(H_s|A_t) = E(E(H_s|P_t)|A_t) \subseteq E(H_t|A_t) = H^*_t.$$

For $s \geq t$, we get $E(H_s|P^*_t) \subseteq H^*_t$, which implies

$$E(F_t|P^*_t) = H^*_t\ . \tag{3.4}$$

Also, we have $E(H^*_s|P_t) = E(H^*_s|H_t) \subseteq E(A|H_t) = H^*_t$, which, together with (3.4), prove that H and H^* are equally predictable by each other. Thus, family H^* is a markovian reduction of H.

Let us prove that, if $H^M_t = E(A^M|H_t)$, where A^M is defined by (3.2), then H^M is the maximal such family. Let H^1 be any markovian reduction of H such that H^1 and H are equally predictable by each other. From Corollary 1 it follows $H^1_t \subseteq \Sigma_{u,i,P}$ for all t,u, that is $H^1_t \subseteq A^M$, which implies $H^1_t \subseteq E(A^M|H_t) = H^M_t$, so that maximality of H^M is proved. □

4. <u>EXAMPLES AND COMMENTS</u>.

Let $H = (H_t)_{t=1,2,3}$ be a family of three arbitrary σ-algebras, and let families $H^* = (H^*_t)_{t=1,2,3}$ and $H^{**} = (H^{**}_t)_{t=1,2,3}$ be defined by

$$H^*_1 = H_1,\ H^*_2 = H_1 \vee H_2,\ H^*_3 = H_3$$

$$H^{**}_1 = H_1,\ H^{**}_2 = H_2 \vee H_3,\ H^{**}_3 = H_3$$

It is easy to see that both families H^* and H^{**} are markovian and contain information carried by H at any given t=1,2,3. However, there is no way in which H^* and H^{**} could be compared and, also, neither one of them and H are equally predictable by each other. Both of these families are realizations of a stochastic dynamic system with output H, but they are not minimal. According to Theorem 1, family H^m, defined by $H^m_1 = H_1$, $H^m_2 = E(H_2 \vee H_3|H_1 \vee H_2)$, $H^m_3 = H_3$ is the minimal realization of that system and the minimal markovian extension of H. □

Let output $H = (H_t)_{t=1,2,3}$ of a stochastic dynamic system be defined by $H_1 = \mathcal{B}$, $H_2 = C$, $H_3 = \mathcal{B}$, where $C \subseteq \mathcal{B}$. Realization of this system that satisfies (2.4) does not exist. But the maximal markovian reduction H^M of H is given by $H^M_1 = H^M_2 = H^M_3 = \mathcal{A}$. Note that families H^* and H^{**}, defined by $H^*_1 = H^*_2 = \mathcal{A}$, $H^*_3 = \mathcal{B}$, and $H^{**}_1 = \mathcal{B}$, $H^{**}_2 = H^{**}_3 = \mathcal{A}$ are also markovian reductions but neither one of them and H are equally predictable by each other. □

Let us now suppose that $H = (H_t)_{t=1,2,3,4}$ is a family of σ-algebras which are all mutually independent except H_1 and H_3. If $\Sigma = E(H_3|H_1)$ and if $\Sigma_{i,P}$ exists, then family H^*, defined by

$$H^*_1 = \{\phi,\Omega\},\ H^*_2 = H_2,\ H^*_3 = E(H_3|\Sigma_{i,P}),\ H^*_4 = H_4,$$

is a markovian reduction of H, but it is not the maximal one. By using Theorem 2, we can show that the maximal reduction is defined by

$$H^M_1 = E(H_3|H_1)_{i,H_1},\ H^M_2 = H_2,\ H^M_3 = E(H_3|E(H_3|H_1)_{i,H_1 \vee H_3}),\ H^M_4 = H_4.$$

Note that dynamic system with output H does not have realization satisfying (2.4).

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