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# ON A CLASS OF FOURTH ORDER LINEAR RECURRENCE EQUATIONS

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ABSTRACT. This paper is concerned with sequences that satisfy a class of fourth order linear recurrence equations. Basic properties of such sequences are derived. In addition, we discuss the oscillatory and nonoscillatory behavior of such sequences.

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## 1. INTRODUCTION.

This paper is concerned with sequences that satisfy the recurrence equation

$$y(k+4)-4y(k+3)+(6+p(k+2))y(k+2)-4y(k+1)+y(k) = 0 \tag{1.1}$$

for $k\geq 0$, where the coefficient function p is real, positive and defined on the set of consecutive integers $\{2,3,4,\dots\}$. Using the difference operator in Finite Calculus, we may write (1.1) in the form

$$\Delta^4 y(k)+p(k+2)y(k+2) = 0, \quad k = 0,1,2,\dots \tag{1.2}$$

A sequence or (discrete) function which satisfies (1.1) for all nonnegative integers is said to be a solution of (1.1). We shall discuss in this paper various properties of the solutions of (1.1) over the set of positive integers $I^+$. In particular, we shall discuss the oscillatory properties of the solutions over $I^+$. In this paper, the oscillatory behavior of a solution is described by means of the distribution of its nodes. The concept of a node is defined as follows. Let f be a real valued function defined on a set $\{a, a+1, \dots, b\}$ of consecutive integers. If the points $(k, f(k))$, $a \leq k \leq b$, are joined by straight line segments to form a broken line, then this broken line gives a representation of a continuous function, henceforth denoted by $f^\circ(t)$, such that $f^\circ(k) = f(k)$ for $k = a,\ldots,b$. The zeros of $f^\circ(t)$ are called the nodes of $f(k)$.

Studies concerning our equation (1.1) do not seem to appear anywhere in literature. Fortunately, the properties of the continuous analogue of equation (1.2),

$$y^{(4)}(t)+p(t)y(t) = 0, \tag{1.3}$$

have been explored to some extent in a number of studies. We shall thus model part of our investigations after some of these studies. Needless to say, techniques different from those employed to deal with (1.3) have to be developed in order to study (1.1). For related studies which provide background material and motivation to write this paper, we refer the readers to the works listed in the references [1-6].

In the sequel, $I$ denotes the set of nonnegative integers and $I^+$ the set of positive integers. If $r \in [k,k+1)$ where $k$ is a nonnegative integer, we define $r^+$ to be $k+1$. If $s \in (k,k+1]$ where $k$ is a nonnegative integer, we define $s^-$ to be $k$. A node $t$ of $f(k)$ is said to be simple if $f(t^-) \neq 0$ and $f(t^+) \neq 0$.

The following result is elementary but fundamental throughout our subsequent development.

LEMMA 1.1. Two vectors $(x_1,y_1)$, $(x_2,y_2)$ and the origin are collinear if and only if $x_1y_2-x_2y_1 = 0$. Furthermore, $x_1y_2-x_2y_1$ is positive if and only if the vectors $(0,0,1)$, $(x_1,y_1,0)$ and $(x_2,y_2,0)$ form a right handed triad in $R^3$.

LEMMA 1.2. Let $f$ and $g$ be real functions defined on a set of consecutive integers. If $f(c+1)$ is positive, $f(c)$ is nonpositive and $g(c+1)$ is positive for some integer $c$, and if $g(k)f(k+1)-g(k+1)f(k)$ is nonpositive at $k = c$, then $g(k)$ has a simple node $t$ in $[r, c+1)$, where $r$ is the (simple) node of $f(k)$ in $[c,c+1)$.

PROOF. The vector $(f(c+1),g(c+1))$ is in the interior of the first quadrant of the plane. By Lemma 1.1, the vector $(f(c),g(c))$ must lie in the set $\{(x,y)|0 \geq x > y\}$. Consequently, $g^\circ(t)$ must have a unique zero in $[r,c+1)$. Q.E.D.

We shall assume the reader is familiar with the notions used in the Finite Calculus and the elementary theory of linear recurrence equations (see Fort [2]). In particular, we note the existence and uniqueness theorem hold for our equation (1.1) and that the solutions of (1.1) are continuously dependent on their initial values. We shall use $[f(k)]^{(m)}$ to denote the generalized factorial function, i.e.

$$[f(k)]^{(m)} = f(k)f(k-1)f(k-2)\ldots f(k-m+1).$$

We shall also need the following discrete analogue of Rolle's Theorem, the proof of which is elementary.

LEMMA 1.3. Suppose the function $f(k)$ has two nodes $s$ and $t$ where $t > N > s$ for some integer $N$, then the function $\Delta f(k)$ has at least one node in $[s^+-1, t^+-1]$.

## 2. PRELIMINARY CONSIDERATIONS.

It is helpful to view equation (1.1) as a pair of second order difference equations

$$\Delta^2 x(k) = -p(k+1)y(k+1)$$
$$\Delta^2 y(k) = x(k+1), \qquad k = 1,2,... \tag{2.1}$$

A solution of the system (2.1) is a vector valued function $Z(k) = \{x(k),y(k)\}$ defined on $I^+$ and satisfies (2.1) there. Clearly, (2.1) is equivalent to (1.1) in the sense that y(k) satisfies (1.1) on $I^+$ if and only if $\{\Delta^2 y(k-1),y(k)\} \equiv \{x(k),y(k)\}$ satisfies (2.1) on $I^+$. The system (2.1) is introduced because of its evident geometrical significance. A solution Z(k) of the system (2.1) is a sequence of points in the x,y-plane whose behavior is evidently dependent upon the initial values as well as the function p(k). To facilitate discussion, we shall join the sequence of points Z(k), k = 1,2,..., whenever possible, by straight line segments. The resulting continuous polygonal curve shall be termed the continuous motion of the solution Z(k). Since the parametric representation of the motion is $x = x^\circ(t)$, $y = y^\circ(t)$, we shall therefore use $Z^\circ(t)$ to denote it.

Suppose $Z(k) = \{x(k),y(k)\}$ is a nontrivial solution of (2.1). For each $k \geq 1$, denote

$$x(k)y(k+1)-y(k)x(k+1) = x(k)\Delta y(k)-y(k)\Delta x(k)$$

by W(k) or W(k,Z). The geometrical interpretation of W(k) is clear. It is the signed area of the triangle with vertices (0,0), (x(k),y(k)) and (x(k+1),y(k+1)) multiplied by two.

LEMMA 2.1. W(k) = 0 at k = N if and only if the points (0,0), (x(N),y(N)) and (x(N+1),y(N+1)) are collinear. Furthermore, W(k) > 0 at k = N if and only if the vectors (0,0,1), (x(N),y(N),0) and (x(N+1),y(N+1),0) form a right-handed triad.

The above Lemma follows directly from Lemma 1.1. Note that

$$\Delta W(k) = x(k+1)\Delta^2 y(k)-y(k+1)\Delta^2 x(k) = x^2(k+1)+p(k+1)y^2(k+1), \qquad k = 1,....$$

Since p(k) is positive for k = 2,3,..., W(k) is monotone increasing for k = 1,2,... and $\Delta W(k) = 0$ if and only if $Z(k+1) = 0$. Consequently, $\Delta W(k)$ cannot vanish at two consecutive integers. It follows that

THEOREM 2.2. W(k) cannot vanish at three consecutive integers, furthermore, it is a strictly increasing function for $k \geq 1$ with the possible exception of two consecutive integers N and N+1 at which W(N) = W(N+1).

It is possible that W(k) = W(k+1) = 0 at k = N (so that Z(N+1) = 0), however,

THEOREM 2.3. If W(k) = W(k+1) = 0 at k = N, then $Z^\circ(t) \neq 0$ for any $t \in [N,N+1)$.

For otherwise Z(N) must vanish which contradicts our assumption that Z(k) is nontrivial. Since the vectors (x(k),y(k)), (x(k),x(k+1)) and (y(k),y(k+1)) cannot be zero whenever $W(k,Z) \neq 0$, as a corollary of Theorem 2.2, we have

THEOREM 2.4. Let $Z(k) = \{x(k),y(k)\}$ be a nontrivial solution of (2.1). Then the vectors (x(k),y(k)), (x(k),x(k+1)) and (y(k),y(k+1)) cannot vanish at two nonconsecutive integers.

In particular, if one of these vectors vanishes at k = N, then the nodes of x(k) and y(k) in [1,N-1) and $(N+2,\infty)$ must be simple.

If the continuous motion of a nontrivial solution $Z(k)$ is free of (vector) zeros over a subinterval $J$ of $[1,\infty)$, we may introduce the usual polar coordinate $(R(t),\theta(t))$ of any point on the curve over this subinterval. Since $Z^\circ(t)$ is continuous over $J$, $\theta(t)$ is also continuous there. Furthermore, if $W(k,Z)$ is positive at $k = N$, then $\theta(t)$, in view of Lemma 2.1, is continuous and strictly increasing on $[N, N+1]$. By finite induction, we easily conclude that if $W(k,Z)$ is positive ($W(k,Z)$ is negative) for $k = M,\ldots,N$ where $M$, $N$ belongs to $I$, then $\theta(t)$ is continuous and strictly increasing (resp. strictly decreasing) on $[M,N+1]$. In view of Theorem 2.2, $W(k,Z)$ is negative for $k = 1,\ldots$ or else $W(k,Z)$ has at least one node in $[1,\infty)$. In the first case, $Z^\circ(t) \neq 0$ for any $t$ in $[1,\infty)$ so that $\theta(t)$ is well defined, continuous and strictly decreasing on $[1,\infty)$. In the latter case, we infer from Theorem 2.2 and Theorem 2.3 that $Z^\circ(t)$ may only vanish at some point $T$ but $Z^\circ(t) = 0$ for $t \geq T^+$ and $1 \leq t \leq T^+-1$. Suppose $\theta(T^+-1)$ belongs to $[M\pi,(M+2)\pi)$ for some integer $M$, and $\theta_0 = \arctan(y(T^+)/x(T^+))$, we shall define $\theta(t)$ for $t \in [T^+-1,T)$ to be $\theta(T^+-1)$ and $\theta(t)$ for $t \in [T,T^+]$ to be $\theta_0+(M-2)\pi$. The function $\theta(t)$ so defined on $[1,\infty)$ shall be called the phase function of $Z(k)$. We shall also call $R(t)$ the norm function of $Z(k)$. Note that $R(t)$ and $\theta(t)$ are piecewise differentiable functions on $[1,\infty)$.

Suppose $Z(k) = \{x(k),y(k)\}$ and $F(k) = \{u(k),v(k)\}$ are two solutions of (2.1). Consider the linear combination

$$H(k) = \alpha Z(k)+\beta F(k), \qquad \alpha,\beta \in R.$$

Straightforward calculations will yield the following two equalities:

$$\begin{aligned} &\Delta\{x(k)\Delta v(k)+y(k)\Delta u(k)-u(k)\Delta y(k)-v(k)\Delta x(k)\} \\ &= \Delta\{x(k)v(k+1)+y(k)u(k+1)-u(k)y(k+1)-v(k)x(k+1)\} \qquad (2.2) \\ &= 0 \end{aligned}$$

and

$$W(k,H) = \alpha^2W(k,Z)+\beta^2W(k,F)+\alpha\beta\{x(k)v(k+1)+u(k)y(k+1)-y(k)u(k+1)-v(k)x(k+1)\}. \quad (2.3)$$

THEOREM 2.5. Let $Z(k) = \{x(k),y(k)\}$ and $F(k) = \{u(k),v(k)\}$ be nontrivial solutions of (2.1). If there exists a number $\mu$ in $[1,\infty)$ such that the points $(0,0)$, $Z^\circ(\mu)$ and $F^\circ(\mu)$ are collinear, then there exists a nontrivial pair $\{\alpha,\beta\}$ of real numbers such that $H(k) = \alpha Z(k)+\beta F(k)$ is a solution of (2.1) satisfying $H^\circ(\mu) = 0$. If in addition $Z^\circ(\mu)$ is not zero, $F^\circ(\mu)$ is not zero and the corresponding phase functions $\theta(t)$ and $\sigma(t)$ of $Z(k)$ and $F(k)$, respectively, satisfy $\theta(\mu)-\sigma(\mu) = n\pi$ for some integer $n$, then $(-1)^n\alpha\beta < 0$.

PROOF. If $Z^\circ(\mu) = 0$ or $F^\circ(\mu) = 0$, we may choose $\alpha = \beta = 1$. Otherwise, the algebraic system

$$\begin{aligned} \alpha x^\circ(\mu)+\beta u^\circ(\mu) &= 0 \\ \alpha y^\circ(\mu)+\beta v^\circ(\mu) &= 0 \end{aligned}$$

has a nontrivial solution $\{\alpha,\beta\}$. Clearly, $H(k) = \alpha Z(k)+\beta F(k)$ is a solution of (2.1) with $Z^\circ(\mu) = 0$. Note that $\alpha\beta < 0$ if and only if $x^\circ(\mu)u^\circ(\mu) > 0$ or $y^\circ(\mu)v^\circ(\mu) > 0$. Consequently, if $\theta(\mu)-\sigma(\mu) = n\pi$, then $(-1)^n\alpha\beta < 0$.

If the behavior of a solution of (2.1) is known for $k \geq n$, then it is sometimes possible to deduce its behavior for $k = 1,\ldots, n-1$ by the following easily verified

THEOREM 2.6. Let $\{x(k),y(k)\}$ be a solution of (2.1) and let m, N be positive integers. Let $k = N+m-j$ and let $X(j) = x(N+m-j)$ and $Y(j) = y(N+m-j)$, then $\{X(j),Y(j)\}$ satisfies

$$\Delta^2X(j) = -p(N+m-j-1)Y(j+1)$$
$$\Delta^2Y(j) = X(j+1)$$

for $j = 1,2,\ldots,N+m-1$.

We close this section by the following remark. For any nontrivial solution $Z(k) = \{x(k),y(k)\}$ of (2.1), since the function $W(k,Z)$ can be written as a determinant,

$$W(k,Z) = \begin{vmatrix} x(k) & y(k) \\ x(k+1) & y(k+1) \end{vmatrix},$$

its value is thus invariant under simple coordinate rotations of $Z(k)$ in the x,y-plane

$$\tilde{x}(k) = x(k)\cos\theta+y(k)\sin\theta$$
$$\tilde{y}(k) = -x(k)\sin\theta+y(k)\cos\theta .$$

## 3. MONOTONICITY THEOREMS.

Let $Z(k) = \{x(k),y(k)\}$ be a nontrivial solution of (2.1) and let $R(t)$ and $\theta(t)$ be its corresponding norm and phase function. Suppose $\sigma \geq 1$ and $\theta_0$ are real numbers such that $R(\sigma) = 0$, $R(\sigma^+) = 1$ and $\theta(\sigma) = \theta_0$, then we call $Z(k)$ a $(\sigma,\theta_0)$-principal solution of (2.1). It is easily seen that a $(\sigma,\theta_0)$-principal solution of (2.1) exists and is uniquely determined by the pair $(\sigma,\theta_0)$. In view of Lemma 2.1, $W(\sigma^+-1,Z) = 0$ for a $(\sigma,\theta_0)$-principal solution $Z(k)$. Furthermore, we infer from Theorem 2.3 that $W(\sigma^+,Z)$ is positive, and therefore, by Theorem 2.2, $W(k,Z)$ is positive for $k \geq \sigma^+$. From these considerations, we see that the corresponding continuous motion $Z^\circ(t)$ passes through the origin at $t = \sigma$ and if it crosses the x or y axes for $t > \sigma$, it crosses them in a counterclockwise manner.

THEOREM 3.1. Let $Z_1(k) = \{x_1(k),y_1(k)\}$ and $Z_2(k) = \{x_2(k),y_2(k)\}$ be respectively the $(\sigma,\theta_1)$- and $(\sigma,\theta_2)$-principal solutions of (2.1) with corresponding phase functions $\theta_1(t)$ and $\theta_2(t)$ respectively. If $n\pi < \theta_2-\theta_1 < (n+1)\pi$ for some integer n, then

$$n\pi < \theta_2(t)-\theta_1(t) < (n+1)\pi \tag{3.1}$$

for $t \geq \sigma$.

PROOF. Since $\theta_2(t) = \theta_2$ and $\theta_1(t) = \theta_1$ on $[\sigma,\sigma^+]$, hence (3.1) holds for $\sigma \leq t \leq \sigma^+$. Suppose there is a real number c in $(\sigma^+,\sigma^++1]$ such that $\theta_2(c) = \theta_1(c)+n\pi$ or $\theta_2(c) = \theta_1(c)+(n+1)\pi$, then the points $Z_1^\circ(c)$, $Z_2^\circ(c)$ and the origin are collinear. Consequently, by Theorem 2.5, there exists a nontrivial pair $\{\alpha,\beta\}$ of real numbers such that $Z(k) = \alpha Z_1(k)+\beta Z_2(k)$ is a solution of (2.1) satisfying $Z^\circ(c) = 0$. Since

$Z_1^\circ(\sigma) = Z_2^\circ(\sigma) = 0$, $Z^\circ(\sigma)$ is therefore equal to zero. Consequently, $W(k,Z)$ vanishes at $k = \sigma^+-1$ and at $k = c^+-1$. If $c^+-1 > \sigma^+$, Theorem 2.2 is contradicted. If $\sigma^+ = c^+-1$, Theorem 2.3 is contradicted since $Z^\circ(\sigma) = 0$. Our assertion thus holds for $\sigma^+ < t \leq \sigma^+ +1$. Finally, if there is a real mumber d in $(\sigma^+ +1,\infty)$ such that $\theta_2(d) = \theta_1(d)+n\pi$ or $\theta_2(d)+(n+1)\pi$, then similar arguments will show that (2.1) has a nontrivial solution $H(k)$ of (2.1) such that $W(\sigma^+,H) = 0$ and $W(\sigma^+-1,H) = 0$ which contradicts Theorem 2.3. Q.E.D.

THEOREM 3.2. Suppose $Z_1(k) = \{x_1(k),y_1(k)\}$ is the $(\sigma,\theta_1)$-principal solution of (2.1) with phase function $\theta_1(t)$. Let $Z_2(k) = \{x_2(k),y_2(k)\}$ be a nontrivial solution of (2.1) with phase function $\theta_2(t)$. Suppose $W(\sigma^+-1,Z_2) \geq 0$ and for $t \in [\sigma,\sigma^+]$,

$$\theta_1+n\pi < \theta_2(t) < \theta_1+(n+1)\pi, \tag{3.2}$$

then for $t > \sigma^+$

$$\theta_1(t)+n\pi < \theta_2(t) < \theta_1(t)+(n+2)\pi. \tag{3.3}$$

PROOF. In view of the Remark mentioned at the end of the previous Section, we may, by rotating the coordinate axes if necessary, assume that $n = 0$ and $\theta_1 = 0$. this is because our arguments below will involve only the values of the function $W(k)$. Suppose to the contrary that for some c in $(\sigma^+,\infty)$, (3.3) is violated. Then by Theorem 2.5, there is a nontrivial pair $\{\alpha,\beta\}$ such that $\alpha\beta < 0$ and the linear combination $Z(k) = \alpha Z_1(k)+\beta Z_2(k)$ satisfies $Z^\circ(c) = 0$. This shows $W(c^+-1, Z) = 0$. However, by substituting $k = \sigma^+-1$ into (2.3), we have

$$W(\sigma^+-1,Z) \geq \alpha\beta\{x_1(\sigma^+-1)y_2(\sigma^+)-y_2(\sigma^+-1)x_1(\sigma^+)\} .$$

Next, we assert that

$$y_2(\sigma^+-1)x_1(\sigma^+)-y_2(\sigma^+)x_1(\sigma^+-1) > 0.$$

Indeed, since $x_1(\sigma^+-1) \leq 0$, $x_1(\sigma^+) > 0$, $y_2(\sigma^+) > 0$ and $y_2(\sigma) > 0$, we would obtain a contradiction to Lemma 1.2 if the above inequality does not hold. Consequently, $W(\sigma^+-1,Z) > 0$ which, together with $W(c^+-1,Z) = 0$ and $Z^\circ(c) = 0$ contradict either Theorem 2.2 or Theorem 2.3. Q.E.D.

We remark that if (3.2) is replaced by

$$\theta_1+n\pi \leq \theta_2(t) < \theta_1+(n+1)\pi,$$

then by continuous dependence of solutions on their initial conditions, we may replace (3.3) by

$$\theta_1(t)+n\pi \leq \theta_2(t) < \theta_1(t)+(n+2)\pi.$$

As an immediate corollary of the above Theorem, we have

COROLLARY 3.3. Suppose $Z_1(k) = \{x_1(k),y_1(k)\}$ and $Z_2(k) = \{x_2(k),y_2(k)\}$ are respectively $(\sigma_1,\theta_0)$- and $(\sigma_2,\theta_0)$-principal solutions of (2.1). If $\sigma_1 < \sigma_2$ and $n\pi \leq \theta_1(\sigma_2)-\theta_2(\sigma_2) < (n+1)\pi$, then

$$n\pi \leq \theta_1(t)-\theta_2(t) < (n+2)\pi$$

for $t > \sigma_2$.

The next result concerns the relationship between the phase functions on $[1,m)$ of the $(m,0)$-principal solution and a solution $Z(k)$ satisfying $W(m,Z) < 0$. The proof is similar to those of Theorems 3.1 and 3.2 and is thus omitted.

THEOREM 3.4. Let $m$ be a positive integer and let $Z_1(k) = \{x_1(k),y_1(k)\}$ be the $(m,0)$-principal solution of (2.1). If $Z_2(k) = \{x_2(k),y_2(k)\}$ is a solution of (2.1) satisfying $W(m,Z_2) < 0$ and for some integer $n$,

$$n\pi < \theta_2(m) < (n+1)\pi,$$

then for $1 \leq t < m$,

$$\theta_1(t)+n\pi < \theta_2(t) < \theta_1(t)+(n+2)\pi.$$

THEOREM 3.5. Let $Z_1(k) = \{x_1(k),y_1(k)\}$ be a $(\sigma,\theta_1)$-principal solution with phase function $\theta_1(t)$. Let $Z_2(k) = \{x_2(k),y_2(k)\}$ be a nontrivial solution of (2.1) with phase function $\theta_2(t)$. Suppose $W(\sigma^+-1,Z_2) > 0$ and $\theta_2(\sigma) = \theta_1+n\pi$, then for $t > \sigma$,

$$\theta_1(t)+n\pi < \theta_2(t) < \theta_1(t)+(n+1)\pi. \tag{3.4}$$

PROOF. We shall only sketch the proof. Assume without loss of generality that $n = 0$ and $\theta_1 = 0$. It is clear from Lemma 2.1 that (3.4) holds for $\sigma < t \leq \sigma^+$. If a number $c$ exists in $(\sigma^+,\infty)$ such that (3.4) is violated, then a nontrivial combination $Z(k) = \alpha Z_1(k)+\beta Z_2(k)$ satisfies $Z°(c) = 0$ and $W(c^+-1,Z) = 0$. By substituting $W(\sigma^+-1,Z_1) = 0$, $y_1(\sigma^+-1) = y_1(\sigma^+) = 0$ into (2.3), we obtain

$$W(\sigma^+-1,Z) = \beta^2 W(\sigma^+-1,Z_2)+\alpha\beta\{x_1(\sigma^+-1)y_2(\sigma^+)-y_2(\sigma^+-1)x_1(\sigma^+)\}.$$

But since the points $(0,0)$, $(x_1(\sigma^+-1),y_2(\sigma^+-1))$ and $(x_1(\sigma^+),y_2(\sigma^+))$ are collinear, therefore by Lemma 1.2,

$$x_1(\sigma^+-1)y_2(\sigma^+)-y_2(\sigma^+-1)x_1(\sigma^+) = 0.$$

Thus $W(\sigma^+-1,Z) > 0$ which contradicts the fact that $W(c^+-1,Z) = 0$.

An important relationship exists between $(m,0)$- and $(n,0)$-principal solutions where $m, n \in I^+$.

THEOREM 3.6. Suppose $m$ and $n$ $(m < n)$ are distinct positive integers and suppose $\{x_1(k),y_1(k)\}$ and $\{x_2(k),y_2(k)\}$ denote, respectively, the $(m,0)$ and $(n,0)$-principal solutions of (2.1). Then $y_1(n) = -y_2(m)$.

PROOF. In view of (2.2),

$$x_1(k)y_2(k+1)+y_1(k)x_2(k+1)-x_2(k)y_1(k+1)-y_2(k)x_1(k+1) \equiv \text{constant},$$

consequently, by substituting $k = m$ and $n$ into the above identity respectively, we obtain $y_1(n) = -y_2(m)$.

COROLLARY 3.7. Suppose the phase function $\theta(t)$ of the $(1,0)$-principal solution satisfies $\theta(t) < n\pi$ for some integer $n$ and all $t \geq 1$. Then for any $m$ in $I^+$, the phase function $\psi(t)$ of the $(m,0)$-principal solution satisfies $\psi(t) < n\pi$ for $1 \leq t \leq m$.

The proof of the following is similar to that of Theorem 3.6.

THEOREM 3.8. Suppose $m$ and $n$ $(m+1 < n)$ are positive integers. Let $\{x(k),y(k)\}$

be the (m,0)-principal solution of (2.1). If $\{u(k),v(k)\}$ is the solution of (2.1) satisfying $u(n-1) = -1$, $v(n-1) = u(n) = v(n) = 0$, then $y(n) = -v(m)$.

4. SEPARATION THEOREMS.

For any nontrivial solution $Z(k)$ of (2.1), the corresponding phase function either strictly decreases on $[1,\infty)$ or else strictly decreases on $[1,T^+-1)$, is constant on $[T^+-1,T)$, constant on $[T,T^+]$ and strictly increases on $[T^+,\infty)$. Consequently, the continuous motion must cross the x and y axes alterantely on $[1,\infty)$ with the possible exception of a point T in $[1,\infty)$.

THEOREM 4.1. Let $\{x(k),y(k)\}$ be a nontrivial solution of (2.1). Then the nodes of $x(k)$ and $y(k)$ are simple and separate each other on $[1,\infty)$ with the possible exception of a neighborhood of a point T in $[1,\infty)$. This neighborhood is of length at most two.

THEOREM 4.2. Suppose $Z_1(k) = \{x_1(k),y_1(k)\}$ and $Z_2(k) = \{x_2(k),y_2(k)\}$ are linearly independent solutions of (2.1). Suppose $y_1(k)$ and $y_2(k)$ vanish at two consecutive integers N and N+1, then the nodes of $y_1(k)$ and $y_2(k)$-both in $[1,N-1]$ and $[N+2,\infty)$- separate each other.

PROOF. By Theorem 2.4, the nodes of $y_1(k)$ and $y_2(k)$ in $(N+2,\infty)$ are simple. Suppose to the contrary that $y_1(k)$ has two consecutive nodes $\nu$ and $\mu$ in $[N+2,\infty)$ such that $y_2^\circ(t) \neq 0$ for $\nu \leq t \leq \mu$. Since $\mu$ is simple, $N+2 \leq \nu < \mu^- \leq \mu$. At some point $\sigma \in (\nu,\mu)$, the vectors $(x_1^\circ(\sigma),y_1^\circ(\sigma))$, (0,0) and $(x_2^\circ(\sigma),y_2^\circ(\sigma))$ must be collinear. Consequently, by Theorem 2.5, there exists a nontrivial pair $\{\alpha,\beta\}$ of real numbers such that $Z(k) = \alpha Z_1(k)+\beta Z_2(k)$ satisfies $Z^\circ(\sigma) = 0$. Thus $W(\sigma^+-1,Z) = 0$. However, by substituting $y_1(N) = y_2(N) = y_1(N+1) = y_2(N+1) = 0$ into (2.3), we obtain $W(N,Z) = 0$. This implies $W(k,Z) > 0$ for $k \geq N+2$ which contradicts $W(\sigma^+-1,Z) = 0$. Similarly, we can prove that the nodes of $y_1(k)$ and $y_2(k)$ in $[1,N-1]$ separate each other. Q.E.D.

Several other separation theorems can be proved by similar techniques. We shall give two more of these and sketch the proof of the first one.

THEOREM 4.3. Let $Z_1(k) = \{x_1(k),y_1(k)\}$ be a $(\sigma,\theta_1)$-principal solution of (2.1) with phase function $\theta_1(t)$. Let $Z_2(k) = \{x_2(k),y_2(k)\}$ be a nontrivial solution of (2.1) with phase function $\theta_2(t)$. Suppose $W(\sigma^+-1,Z_2) > 0$ and $\theta_2(\sigma) = \theta_1$, then the nodes of $y_1(k)$ and $y_2(k)$ in $[\sigma^+,\infty)$ separate each other.

PROOF. Note first that by Theorem 2.4, the nodes of $y_1(k)$ and $y_2(k)$ in $(\sigma^+,\infty)$ are simple. Next we note that both $\theta_1(t)$ and $\theta_2(t)$ are strictly increasing on $(\sigma^+,\infty)$. By Theorem 3.5, for $t > \sigma$, $\theta_1(t) < \theta_2(t) < \theta_1(t)+\pi$. Consequently, $y_1(k)$ and $y_2(k)$ cannot vanish simultaneously for $k = \sigma^+$. Furthermore, between two nodes of $y_1(k)$, $y_2(k)$ has exactly one node. This concludes the proof.

THEOREM 4.5. Let $\{x_1(k),y_1(k)\}$ and $\{x_2(k),y_2(k)\}$ be respectively the $(\sigma,\theta_1)$- and $(\sigma,\theta_2)$-principal solutions of (2.1) with corresponding phase functions $\theta_1(t)$ and $\theta_2(t)$. If $n\pi < \theta_2-\theta_1 < (n+1)\pi$ for some integer n, then the nodes of $y_1(k)$ and $y_2(k)$ in $[\sigma^+,\infty)$ separate each other.

5. NONOSCILLATION THEOREMS.

A scalar function h(k) defined on I is said to be oscillatory if it has arbitrary large nodes and nonoscillatory otherwise. Let K be a nonempty subset of the plane. A vector valued function Z(k) is said to be K-nonoscillatory on a set of consecutive integers B if the set $\{Z(k)|k \in B\}$ is contained in K. Denote the i-th open quadrant of the plane by $K_i$. Suppose Z(k) is a solution of (2.1) such that neither of its components is oscillatory. Then in view of Theorem 4.1, there exists some integer $i \in \{1,2,3,4\}$ such that Z(k) is $K_i$-nonoscillatory for large k. The converse is obviously true. Thus we may, in our subsequent discussions, identify a nonoscillatory solution of (1.1) with a $K_i$-nonoscillatory solution of (2.1) as equivalent concepts.

THEOREM 5.1. Suppose $\{x(k),y(k)\}$ is a solution of (2.1) such that either x(k) or y(k) is nonoscillatory. Then $x(k)y(k)\Delta x(k)\Delta y(k) \neq 0$ for large k. Furthermore, if $y(k) > 0$ for large k, then $\{\Delta x(k),\Delta y(k)\}$ is $K_1$-nonoscillatory for large k.

PROOF. Since both x(k) and y(k) are of one sign for large k, and since $\Delta^2 x(k) = -p(k+1)y(k+1)$ and $\Delta^2 y(k) = x(k+1)$, it then follows from Lemma 1.3 that $\Delta x(k)$ and $\Delta y(k)$ are of one sign for large k. The first part of Theorem 5.1 is thus proved. Suppose $y(k) > 0$ for large k. Assume first that $x(k) > 0$ and $y(k) > 0$ for $k \geq n$, where $n \in I^+$. We assert that $\Delta x(k) > 0$ for $k \geq n$. Indeed, if $\Delta x(m) \leq 0$ for some $m \geq n$, then from (2.1), we get

$$\Delta x(k) = \Delta x(m) - \sum_{j=m}^{k-1} p(j+1)y(j+1) < 0$$

for $k > m$. But this contradicts the fact $x(k) > 0$ and $\Delta^2 x(k) < 0$ for $k > m$. Next we assert that $\Delta y(k) > 0$ for large k. To see this, we first observe from (2.1) that $\Delta y(k) = \Delta y(n)+[x(n+1)+\ldots+x(k)]$. Since we have just shown that $\Delta x(k) > 0$ for $k \geq n$, thus $\Delta y(k) > \Delta y(n)+(k-n)x(n)$. The desired conclusion follows immediately by letting k approach infinity.

Next we assume $x(k) < 0$ and $y(k) > 0$ for $k \geq N$, where N is a positive integer. The fact that $\Delta y(k) > 0$ for $k \geq N$ can be proved in a way similar to the proof of $\Delta x(k) > 0$ above. To see that $\Delta x(k) > 0$ for $k \geq N$, assume to the contrary that $\Delta x(M) \leq 0$ for some $M \geq N$. Then from (2.1), we have

$$\Delta x(k) = \Delta x(M) - [p(M+1)y(M+1)+\ldots+p(k)y(k)] < 0$$

for $k \geq M$ and

$$0 < \Delta y(k) = \Delta y(M)+[x(M+1)+\ldots+x(k)] \leq \Delta y(M)+(k-M)x(M).$$

By letting k approach infinity, a contradiction is obtained since $x(M) < 0$. Q.E.D.

COROLLARY 5.2. Suppose $\{x(k),y(k)\}$ is a solution of (2.1) which is $K_2$-nonoscillatory for large k, then $x(k)/y(k)$ approaches zero as k approaches infinity.

PROOF. Since x(k) and y(k) are monotone increasing by Theorem 5.1 and since x(k) is bounded above by zero, it suffices to show that y(k) approaches infinity or x(k) approaches zero. If x(k) does not approach zero, then the series $x(N+1)+x(N+2)+\ldots$

diverges to negative infinity, thus from (2.1), $\Delta y(k) = \Delta y(N)+[x(N+1)+\ldots+x(k)] \to -\infty$. Consequently, y(k) cannot remain positive for large k.

THEOREM 5.3. Let y(k) be a solution of (1.1) which is positive for large k, then

$$\lim_{k\to\infty} \Delta^3 y(k) = \lim_{k\to\infty} 6y(k)/k^{(3)}.$$

Both limits are finite.

PROOF. Let $N \in I^+$ such that $y(k) > 0$ for $k \geq N$. Summing (1.1) four times, we obtain

$$y(k+4)+\sum_{j=N}^{k} (k-j+3)^{(3)}p(j+2)y(j+2)/6$$

$$= y(N+3)+\Delta y(N+2)(k-N+1)+\Delta^2 y(N+1)(k-N+2)^{(2)}/2+\Delta^3 y(N)(k-N+3)^{(3)}/6 \qquad (5.1)$$

Denoting the right hand side of the above equality by R(k),we have

$$\lim_{k\to\infty} 6R(k)/(k-N+3)^{(3)} = \Delta^3 y(N). \qquad (5.2)$$

Since $y(k) > 0$ for $k \geq N$, we have from (5.1) that

$$6R(k) \leq 6y(k+4)+(k-N+3)^{(3)} \sum_{j=N}^{k} p(j+2)y(j+2) = 6y(k+4)+(k-N+3)^{(3)}[\Delta^3 y(N)-\Delta^3 y(k+1)].$$

So that, by (5.2),

$$\lim_{k\to\infty} \Delta^3 y(k+1) \leq \liminf_{k\to\infty} 6y(k+4)/(k-N+3)^{(3)}.$$

Next choose integer m such that $N < m < k$, then from (5.1) we obtain

$$6R(k) \geq 6y(k+4)+\sum_{j=N}^{m} (k-j+3)^{(3)}p(j+2)y(j+2) \geq 6y(k+4)+(k-m+3)^{(3)}[\Delta^3 y(N)-\Delta^3 y(m+1)].$$

Dividing through by $(k-N+3)^{(3)}$ and keeping m fixed, we obtain

$$\Delta^{(3)} y(m+1) \geq \limsup_{k\to\infty} 6y(k+4)/(k-N+3)^{(3)}.$$

Since this holds for all $m > N$ and since $\Delta^4 y(k) < 0$ for $k \geq N$,we obtain

$$\lim_{k\to\infty} \Delta^3 y(k+1) \geq \limsup_{k\to\infty} 6y(k+4)/(k-N+3)^{(3)}.$$

This concludes the proof.

In view of Theorem 5.1 and that (1.1) is linear, we may classify nonoscillatory solutions of (1.1) and thus (2.1) into two classes. We say that a nonoscillatory solution y(k) of (1.1) is Class I if some constant multiple of $\{\Delta^2 y(k-1),y(k)\}$ is $K_1$-nonoscillatory for large k, and class II if some constant multiple of $\{\Delta^2 y(k-1),y(k)\}$ is $K_2$-nonoscillatory for large k. In other words, if $y(k) > 0$ for large k, then it is Class I if and only if $\Delta y(k) > 0$, $\Delta^2 y(k) > 0$ and $\Delta^3 y(k) > 0$ for large k; it is Class II if and only if $\Delta y(k) > 0$, $\Delta^2 y(k) < 0$ and $\Delta^3 y(k) > 0$ for large k.

Clearly, if $y(k) > 0$ for large k and is Class I, then $y(k) \geq A > 0$ for large k. On the other hand, if $y(k) > 0$, $\Delta y(k) > 0$, $\Delta^2 y(k) < 0$ and $\Delta^3 y(k) > 0$ for $k \geq N$ where

$N \in I^+$, then summing

$$\Delta^4 y(j) = -p(j+1)y(j+1) < 0$$

from $j = N$ to $j = k-1$ four times, we obtain,

$$y(k) \le y(N)+(k-N)\Delta y(N)+(k-N)^{(2)}\Delta^2 y(N)/2+(k-N)^{(3)}\Delta^3 y(N)/6$$

$$\le y(N)+(k-N)\Delta y(N)+(k-N)^{(2)}|\Delta^2 y(N)|/2+(k-N)^{(3)}\Delta^3 y(N)/6.$$

Thus $y(k) \le C(k-N)^{(3)}$ for large k, where C is a suitably chosen positive constant. The following is now clear.

THEOREM 5.4. Suppose y(k) is nonoscillatory solution of (1.1). Then there are positive constants $C_1$ and $C_2$ such that $C_1 \le |y(k)| \le C_2 k^{(3)}$ for large k.

We say that a nonoscillatory solution y(k) of (1.1) is asymptotically constant if there exists some constant $C_1 \neq 0$ such that $y(k) \to C_1$ as $k \to \infty$, and asymptotically cubic factorial if there exists some constant $C_2 \neq 0$ such that $y(k)/k^{(3)} \to C_2$ as $k \to \infty$. According to Theorem 5.4, we may regard asymptotically cubic factorial solutions as "maximal" and asymptotically constant solutions as "minimal". We now discuss some necessary conditions and sufficient conditions for their existence.

THEOREM 5.5. A necessary condition for (1.1) to have an asymptotically constant solution y(k) is that

$$\sum^{\infty} k^{(3)} p(k) < \infty . \tag{5.3}$$

PROOF. Let y(k) be an asymptotically constant solution of (1.1) and assume without loss of generality that $y(k) > 0$ for large k. y(k) must be a Class II solution and thus there exist positive constants $A_1, A_2$ and integer $N \ge 3$ such that $\Delta y(k) > 0$, $\Delta^2 y(k) < 0$, $\Delta^3 y(k) > 0$ and $A_1 \le y(k) \le A_2$ for $k \ge N$. Upon multiplying (1.1) by $k^{(3)}$ and summing from N to k-1, we obtain

$$0 < \sum_{j=N}^{k-1} j^{(3)} p(j+2)y(j+2) = -\sum_{j=N}^{k-1} j^{(3)} \Delta^4 y(j)$$

$$= -k^{(3)}\Delta^3 y(k)+3k^{(2)}\Delta^2 y(k+1)-6k\Delta y(k+2)+6y(k+3)+C$$

where C is a constant. But since

$$-k^{(3)}\Delta^3 y(k)+3k^{(2)}\Delta^2 y(k+1)-6k\Delta y(k+2) < 0$$

for $k \ge N$ and since y(k) is asymptotically constant, thus

$$\sum_{j=N}^{\infty} j^{(3)} p(j+2)y(j+2) < \infty.$$

Since $\Delta y(k) > 0$ for large k, thus

$$y(N+2)\sum_{N}^{\infty} j^{(3)} p(j+2) < \sum_{N}^{\infty} j^{(3)} p(j+2)y(j+2) < \infty$$

as required. Q.E.D.

THEOREM 5.6. A necessary condition for (1.1) to have an asymptotically cubic factorial solution is that (5.3) holds.

PROOF. Let $y(k)$ be an asymptotically cubic factorial solution of (1.1) and assume without loss of generality that $y(k) > 0$ for large $k$. Then $y(k)$ is Class I and thus there exist positive numbers $A_1$, $A_2$ and integer $N \geq 3$ such that $\Delta y(k) > 0$, $\Delta^2 y(k) > 0$, $\Delta^3 y(k) > 0$ and $A_1 k^{(3)} \leq y(k) \leq A_2 k^{(3)}$ for $k \geq N$. Upon summing (1.1) from $N$ to $k-1$, we obtain

$$\infty > \Delta^3 y(N) = \Delta^3 y(k) + \sum_{j=N}^{k-1} p(j+2)y(j+2)$$

$$> \sum_{j=N}^{k-1} p(j+2)y(j+2) \geq A_1 \sum_{j=N}^{k-1} p(k+2)k^{(3)}$$

as required. Q.E.D.

THEOREM 5.7. A sufficient condition for (1.1) to have an asymptotically cubic factorial solution $y(k)$ is that (5.3) holds.

PROOF. Choose $N$ so large that $p(N+2) < 10$, $p(N+2)+p(N+3) < 5$ and the series $N^{(3)}p(N)+\ldots \leq 5$. Consider the solution of (1.1) determined by the initial conditions $y(N-1) = y(N) = y(N+1) = 0$ and $y(N+2) = 1$. From (1.1), we can calculate successively that $y(N+3) = 4$, $y(N+4) = 10-p(N+2)$ and $y(N+5) = 20-4[p(N+2)+p(N+3)]$. For $k \geq N+6$, we sum (1.1) four times to obtain

$$6y(k) = (k-N)^{(3)} - \sum_{j=N+2}^{k-2} (k-j+1)^{(3)} p(j)y(j). \tag{5.4}$$

Clearly, $y(k) > 0$ for $N+3 \leq k \leq N+5$. Assume $y(k) > 0$ holds for $N+5 \leq k \leq m$, we shall prove that $y(m+1) > 0$. Note first that our induction hypothesis implies $6y(k) \leq (k-N)^{(3)}$ for $N+2 \leq k \leq m+2$. Since

$$\sum_{j=N+2}^{m-1} (m-j+2)^{(3)} p(j)y(j) \leq (m-N)^{(3)} \sum_{j=N+2}^{m-1} p(j)(j-N)^{(3)}/6 \leq (m-N)^{(3)} \leq (m-N+1)^{(3)}$$

thus

$$6y(m+1) = (m-N+1)^{(3)} - \sum_{j=N+2}^{m-1} (m-j+2)^{(3)} p(j)y(j) > 0.$$

Consequently, $y(k) > 0$ for all $k \geq N+2$. In view of (5.4), $y(k) \leq (k-N)^{(3)}/6$ for $k \geq N+2$. Furthermore, from (1.1),

$$(k-N)^{(3)} \leq 6y(k) + (k-N-1)^{(3)} \sum_{j=N+2}^{k-4} (j-N)^{(3)} p(j)/6,$$

so that $1/6 \leq y(k)/(k-N)^{(3)}$. It follows now from Theorem 5.3 that the limit of $y(k)/(k-N)^{(3)}$ exists. This concludes the proof.

We conclude this section by the next result which shows that (5.3) is also sufficient for (1.1) to have an asymptotically constant solution.

THEOREM 5.8. A sufficient condition for (1.1) to have an asymptotically constant solution is that (5.3) holds.

PROOF. The required solution of (1.1) will be obtained with the aid of the following equation:

$$y(k) = 1+\sum_{j=N}^{k} j^{(3)}p(j+2)y(j+2)+\sum_{j=k}^{\infty}[k(j-k+1)^{(2)}/2+k^{(2)}(j-k)/2+k^{(3)}/6]p(j+2)y(j+2), \quad (5.5)$$

where we choose N so large that

$$\sum_{j=N}^{\infty}(2j)^{(3)}p(j+2) < 1/2.$$

As can be verified directly,

$$y(k) = \sum_{j=k}^{\infty}(j-k+2)^{(2)}p(j+2)y(j+2)/2,$$

$$\Delta^2 y(k) = \sum_{j=k}^{\infty}(k-j-1)p(j+2)y(j+2),$$

$$\Delta^3 y(k) = \sum_{j=k}^{\infty}p(j+2)y(j+2)$$

and $\Delta^4 y(k) = -p(k+2)y(k+2)$. We shall find a solution to (5.5) which is asymptotically constant. For this purpose, we define a sequence of sequences as follows:

$$y_0(k) = 1$$

$y_{m+1}(k)$ is obtained by substituting $y_m(k)$ into the right hand side of (5.5).

Clearly, $1 \le y_0 \le 2$. Moreover,

$$1 \le y_1(k) \le 1+1/12+\sum_{j=k}^{\infty}[kj^{(2)}/2+k^{(2)}j/2+k^{(3)}/6+j^{(3)}/6]p(j+2) \le 2.$$

Similarly, we may show by induction that $1 \le y_m(k) \le 2$ for all $m \ge 0$ and $k \ge N$. Next we observe that

$$|y_1(k)-y_0(k)| = \left|\sum_{j=N}^{k} j^{(3)}p(j+2)/6+\sum_{j=k}^{\infty}[k(j-k+1)^{(2)}/2+k^{(2)}(j-k)/2+k(3)/6]p(j+2)\right| \le 1/6,$$

and inductively, $|y_m(k)-y_{m-1}(k)| \le 1/6^m$ for all $k \ge N$. Consequently, $y_m(k)$ converges to some limiting function y(k) uniformly. This function is a solution of (1.1), furthermore, since it is bounded between 1 and 2 for large k, it is an asymptotically constant solution.

## 6. OSCILLATION THEOREMS.

In this section we consider several results concerning with the oscillatory solutions of (1.1). According to the considerations in Section 2, it is clear that y(k) is an oscillatory solution of (1.1) if and only if $\{\Delta^2 y(k-1), y(k)\}$ is a solution of

(2.1) and $|\theta(t)| \to \infty$ as $t \to \infty$. We say that a solution $\{x(k),y(k)\}$ is rotary if $\lim|\theta(t)| = \infty$ as $t \to \infty$. In view of Theorem 2.2, we expect that the behavior of a rotary solution depends on whether $W(k)$ has a node in $[1,\infty)$ or $W(k) < 0$ for all $k \geq 1$. In order to simplify the statements of our latter results, we make the following definitions. A nontrivial rotary solution $Z(k)$ of (2.1) is said to be Type I if $W(k,Z)$ has a node in $[1,\infty)$, otherwise it is said to be Type II. We shall first characterize the behavior of Type I and Type II solutions. Suppose $\{x(k),y(k)\}$ is a nontrivial solution of (2.1). Suppose further that $\alpha$ is a node of $x(k)$ such that $W(\alpha^+-1) > 0$ and $y°(\alpha) > 0$. In view of Theorem 2.2, $W(k) > 0$ for $k \geq \alpha^+$ so that by Theorem 2.4, $R^2(t) > 0$ for $t \geq \alpha^+-1$. Furthermore, if $\beta$ is the first node of $x(k)y(k)$ in $(\alpha,\infty)$, then $\beta$ must be the first node of $y(k)$ in $(\alpha,\infty)$ so that $(x°(t),y°(t))$ lies in $K_2$ for each $t \in (\alpha,\beta)$. We assert that if $\beta \in I^+$, then $\Delta R^2(\beta-1) > 0$ and $\Delta R^2(\beta) > 0$. To see these, we first note that $\beta-1 = \alpha^+-1$, we easily infer from $W(\beta-1) > 0$ and $W(\beta) > 0$ that $\Delta y(\beta-1) < 0$, $\Delta y(\beta) < 0$, $\Delta x(\beta-1) < 0$ and $\Delta x(\beta) < 0$. If $\beta > \alpha^+$, $W(\beta-1) > 0$ and $W(\beta) > 0$ again imply $\Delta y(\beta-1) < 0$ and $\Delta y(\beta) < 0$. Furthermore, from (2.1), we have

$$\Delta x(\beta) = \Delta x(\beta-1) = \Delta x(\alpha^+-1) - \sum_{k=\alpha^+-1}^{\beta-2} p(k+1)y(k+1) < 0.$$

Since $\Delta R^2(k) = \Delta x^2(k) + \Delta y^2(k)$, consequently, $\Delta R^2(\beta-1) > 0$ and $\Delta R^2(\beta) > 0$ as required. Next, suppose $\beta$ is not an integer, we assert that $\Delta R^2(\beta^+-1) > 0$. The assertion is easily verified if $\beta^+-1 = \alpha^+-1$. Otherwise, $y°(\beta) = 0$, $y(\beta^+-1) > 0$ and $W(\beta^+-1) > 0$ imply the chain of conclusions $R^2(\beta) \neq 0$, $\Delta y(\beta^+-1) < 0$, $y(\beta^+-1)\Delta x(\beta^+-1) = -W(\beta^+-1) < 0$ and $\Delta x(\beta^+-1) < 0$. The assertion is thus proved. We summarize these as follows.

LEMMA 6.1. Suppose $Z(k) = \{x(k),y(k)\}$ is a nontrivial solution of (2.1). If $\alpha$ and $\beta$ ($\alpha < \beta$) are consecutive nodes of $x(k)y(k)$ such that $W(\alpha^+-1) > 0$ and $R^2(\alpha) \neq 0$, then $W(\beta^+-1) > 0$, $R^2(\beta) \neq 0$, $(R^2)'(\beta+) > 0$ and $(R^2)'(\beta-) > 0$.

THEOREM 6.2. Suppose $Z(k) = \{x(k),y(k)\}$ is a nontrivial solution of (2.1) such that $W(\alpha^+-1) \geq 0$ for some $\alpha \in [1,\infty)$. Then $W(\beta^+-1) > 0$, $R^2(\beta) \neq 0$, $(R^2)'(\beta-) > 0$ and $(R^2)'(\beta+) > 0$ at any node $\beta$ (except possibly the first one) of the function $x(k)y(k)$ in $(\alpha^+,\infty)$.

It follows from the above Theorem that if $\{x(k),y(k)\}$ is a Type I solution of (2.1), then for large t, its corresponding continuous motion is a "positive polygonal spiral" and at any time it crosses the x or y axes, $R^2(t) \neq 0$ and $(R^2)'(t-) > 0$, $(R^2)'(t+) > 0$.

We now turn our attention to Type II solutions.

THEOREM 6.3. Suppose $Z(k) = \{x(k),y(k)\}$ is a solution of (2.1) such that $W(k) < 0$ for $1 \leq k \leq N$. Suppose $\alpha$ and $\beta$ ($\alpha < \beta < N$) are nodes of the function $x(k)y(k)$ such that $R^2(\alpha) \neq 0$ and $(R^2)'(\alpha-) < 0$, $(R^2)'(\alpha+) < 0$, then $R^2(\beta) \neq 0$ and $(R^2)'(\beta-) < 0$, $(R^2)'(\beta+) < 0$, except possibly when $\beta$ is the last node of $x(k)y(k)$ in $(\alpha,N)$.

PROOF. Suppose without loss of generality that $x°(\alpha) = 0$ and $y°(\alpha) > 0$. To prove the Theorem we assume that $\beta$ and $\mu$ are the first and the second nodes, respectively, of the function $x(k)y(k)$ in $(\alpha,N)$. Clearly, our assumptios imply that the corresponding motion of $Z(k)$ is contained in $K_1$ for $\alpha < t < \beta$. Note that $R^2(\beta) \neq 0$ since $W(k) < 0$ for $k \geq 1$. Consequently, if $\beta$ is not an integer, then $\Delta y(\beta^+-1) < 0$; and if $\beta$ is an integer, $\Delta y(\beta-1) < 0$ and $\Delta y(\beta) < 0$. To complete the proof, we shall show that $\Delta x(\beta^+-1) < 0$ if $\beta$ is not an integer and $\Delta x(\beta-1) < 0$, $\Delta x(\beta) < 0$ if $\beta \in I^+$. In the first case, if $\Delta x(\beta^+-1) \geq 0$, then $\mu$ is the first node of $x(k)$ in $(\beta,\infty)$ so that $\Delta x(\mu^+-1) \leq 0$ and $y(k) < 0$ for $\beta^+ \leq k \leq \mu^+-1$. From (2.1), we obtain

$$\Delta x(\mu^+-1) = \Delta x(\beta^+-1) - \sum_{j=\beta^+-1}^{\mu^+-2} p(j+1)y(j+1) > 0$$

which is a contradication. In the latter case, we obtain from (2.1) that

$$\Delta x(\beta) = \Delta x(\beta-1) = \Delta x(\alpha^+-1) - \sum_{j=\alpha^+-1}^{\beta-2} p(j+1)y(j+1) < 0.$$

This concludes the proof.

THEOREM 6.4. Suppose $Z(k) = \{x(k),y(k)\}$ is a nontrivial solution of (2.1). If $\alpha$ and $\beta$ $(\alpha < \beta)$ are consecutive nodes of $x(k)y(k)$ such that $W(\alpha^+-1) < 0$, $R^2(\alpha) \neq 0$ and $(R^2)'(\alpha-) > 0$, $(R^2)'(\alpha+) > 0$ then $R^2(\beta) \neq 0$, $W(\beta^+-1) > 0$ and $(R^2)'(\beta-) > 0$, $(R^2)'(\beta+) > 0$.

PROOF. Assume without loss of generality that $x^\circ(\alpha) = 0$ and $y^\circ(\alpha) > 0$. We assert that $\beta$ must be the first node of $x(k)$ in $(\alpha,\infty)$. Otherwise, $\beta$ would be the first node of $y(k)$ in $(\alpha,\infty)$ so that $\Delta y(\beta^+-1) \leq 0$. But since our assumptions imply that $\Delta y(\alpha^+-1) > 0$, from (2.1), we obtain

$$\Delta y(\beta^+-1) = \Delta y(\alpha^+-1) + \sum_{j=\alpha^+-1}^{\beta^+-2} x(j+1) > 0$$

which is a contradiction. Now that $\beta$ is the first node of $x(k)$ in $(\alpha,\infty)$, $\Delta x(\beta^+-1) < 0$. Moreover, from (2.1),

$$\Delta y(k) = \Delta y(\alpha^+-1) + \sum_{j=\alpha^+-1}^{k-1} x(j+1) > 0$$

for $\alpha^+-1 \leq k \leq \beta^+-1$ so that $R^2(\beta) = (y^\circ)^2(\beta) > (y^\circ)^2(\alpha) = 0$. Q.E.D.

From the above two Theorems, it is clear that the continuous motion of a Type II solution is a "negative polygonal spiral" and at any time it crosses the x or y axis, $R^2(t) \neq 0$, $(R^2)'(t-) < 0$ and $(R^2)'(t+) < 0$.

We now consider some necessary conditions and sufficient conditions for the existence of Type I, Type II and rotary solutions.

LEMMA 6.5. Suppose (2.1) has a Type I solution, then all its principal solutions are rotary.

PROOF. Let $Z_2(k) = \{x_2(k),y_2(k)\}$ be the Type I solution of (2.1) and suppose $W(k,Z_2)$ has a node $\sigma$ in $[1,\infty)$ so that $W(\sigma^+-1,Z_2) \geq 0$. Let $\theta_1$ satisfies

$$\theta_1 < \theta_2(t) < \theta_1 + \pi$$

for $t \in [\sigma,\sigma^+]$ and consider the $(\sigma,\theta_1)$-principal solution $Z_1(k) = \{x_1(k),y_1(k)\}$. By Theorem 3.2, $\theta_1(t) < \theta_2(t) < \theta_1(t)+2\pi$ for $t > \sigma^+$. Thus $Z_1(k)$ is a Type I solution of (2.1). Next consider the $(\beta,\theta_1)$-principal solutions of (2.1) where $1 \leq \beta < \infty$. It is clear from Corollary 3.3 that they are Type I. Finally we see from Theorem 3.1 that an arbitrary $(\beta,\theta)$-principal solution is Type I. Q.E.D.

LEMMA 6.6. Suppose (2.1) has a Type II solution, then all principal solutions are rotary.

PROOF. In view of Lemma 6.5, it suffices to show that one principal solution of (2.1) is rotary. Suppose to the contrary that the (1,0)-principal solution $Z_1(k)$ satisfies $\theta_1(t) \uparrow \theta_0$ as $t \to \infty$ where $\theta_0 < n\pi$ for some integer n. By Corollary 3.7, the phase function $\theta_2(t)$ of the (m,0)-principal solution $Z_2(k)$ satisfies $\theta_2(t) < n\pi$ for $1 \leq t \leq m$. Choose m sufficiently large so that the phase function $\theta(t)$ of the Type II solution $Z(k)$ satisfies $\theta(1)-\theta(m) > (n+2)\pi$. Suppose $N\pi \leq \theta(m) < (N+1)\pi$ for some integer N, then by Theorem 3.4, $\theta_2(1)+N\pi \leq \theta(1) < \theta_2(1)+(N+2)\pi$. Consequently,

$$\theta(1)-\theta(m) < \theta_2(1)+(N+2)\pi-N\pi = \theta_2(1)+2\pi < (n+2)\pi,$$

which contradicts our assumption. Q.E.D.

THEOREM 6.7. Suppose (2.1) has a Type I solution. If $Z(k) = \{x(k),y(k)\}$ is a solution such that $W(k,Z)$ has a node in $[1,\infty)$, then $Z(k)$ is Type I.

PROOF. Suppose $W(k,Z)$ has a node $\mu-1$ in $[1,\infty)$, then $W(\mu^+-1) \geq 0$. The phase function $\theta(t)$ of $Z(k)$ satisfies $\theta(\mu) \leq \theta(t) < \theta(\mu)+\pi$ for $t \in [\mu,\mu^+]$ of course. If we let $\theta_1(t)$ be the phase function of the $(\mu,\theta(\mu))$-principal solution, then by Theorem 3.2, $\theta_1(t) \leq \theta(t) < \theta_1(t)+2\pi$ for $t \geq \mu$. Thus $Z(k)$ is Type I. Q.E.D.

THEOREM 6.8. Suppose all principal solutions of (2.1) are rotary, then any solution $Z(k)$ of (2.1) satisfying $W(k,Z) < 0$ for $k \geq 1$ is rotary.

PROOF. Assume to the contrary that the phase function $\theta(t)$ of $Z(k) = \{x(k),y(k)\}$ is continuously decreasing to a limit $\theta_0$. By rotating the coordinate axes if necessary, we may assume $\theta_0 \in [0,\pi)$. Choose M large enough so that $\theta(k) \in (\theta_0,\pi)$ for all $k \geq M$. Consider the (m,0) principal solution $\{x_m(k),y_m(k)\}$ where $m > M$. Since it is rotary, $y_m^\circ(t)$ has a node $\lambda$ in $(m+1,\infty)$ and $y_m^\circ(t) > 0$ for $m+1 < t < \lambda$. Let $F(k) = \{u(k),v(k)\}$ be the solution of (2.1) satisfying $u(\lambda^+-1) = -1$ and $v(\lambda^+-1) = u(\lambda^+) = v(\lambda^+) = 0$. By Theorem 3.8, $v(m) = -y(\lambda^+) > 0$. Consequently, there exists a number $\sigma$ in $(m,\lambda^+)$ such that the points $(x^\circ(\sigma),y^\circ(\sigma))$, $(0,0)$ and $(u^\circ(\sigma),v^\circ(\sigma))$ are collinear and that $(x^\circ(\sigma),y^\circ(\sigma))$ and $(u^\circ(\sigma),v^\circ(\sigma))$ lie on opposite sides of $(0,0)$. It follows from Theorem 2.5 that there exists a nontrivial pair $\{\alpha,\beta\}$ such that $\alpha\beta > 0$ and that the linear combination $H(k) = \alpha Z(k)+\beta W(k)$ satisfies $H^\circ(\sigma) = 0$. By Lemma 2.1, $W(\sigma^+-1,H) = 0$. But from (2.3) and the previous assumptions, we have

$$W(\lambda^+-1,H) = \alpha^2 W(\lambda^+-1,Z)-\alpha\beta y(\lambda^+-1) < 0,$$

which is a contradiction. Q.E.D.

THEOREM 6.9. Suppose all solutions of (2.1) are rotary, then (2.1) has a Type II solution.

PROOF. For each integer $n \geq 4$, let $Z_n(k) = \{x_n(k), y_n(k)\}$ be solutions of (2.1) determined by $x_n(n-1) = -1$, $y_n(n-1) = 0$, $x_n(n) = 0$ and $y_n(n) = 0$. Since $W(k,Z_n)$ vanishes at $k = n-1$ and $n$, thus $W(k,Z_n) < 0$ for $1 \leq k \leq n-2$. Now let $W_1(k)$, $W_2(k)$, $W_3(k)$ and $W_4(k)$ be four linearly independent solutions of (2.1). Then for each n, there exist constants $A_n$, $B_n$, $C_n$ and $D_n$ such that

$$Z_n(k) = A_nW_1(k)+B_nW_2(k)+C_nW_3(k)+D_nW_4(k), \qquad A_n^2+B_n^2+C_n^2+D_n^2 = 1.$$

Since the sequence of vectors $\{(A_n, B_n, C_n, D_n)\}$ is bounded, there is a subsequence $\{(A_{n(i)}, B_{n(i)}, C_{n(i)}, D_{n(i)})\}$ convergent, say, to (A, B, C, D). Since $A^2+B^2+C^2+D^2$ is equal to 1, the solution

$$Z(k) = AW_1(k)+BW_2(k)+CW_3(k)+DW_4(k)$$

of (2.1) is nontrivial. Furthermore, the sequence $Z_{n(i)}(k)$ converges to Z(k) uniformly on any finite subset $\{1,2,3,\ldots,N\}$ of $I^+$. Since all solutions of (2.1) are rotary, Z(k) is either Type I or Type II. We assert that it is Type II. Otherwise, $W(k,Z) > 0$ at some integer M. But this would mean that for i large enough, $W(k,Z_{n(i)})$ is positive at $k = M$. This contradiction concludes the proof.

The following important result is a direct consequence of Lemmas 6.5, 6.6 and Theorems 6.7, 6.8.

THEOREM 6.10. If some solution of (2.1) is rotary, then every nontrivial solution of (2 1) is rotary.

Our final result in this paper is the following

THEOREM 6.11. All nontrivial solutions of (2.1) are rotary if

$$\sum^{\infty} k^{(2)}p(k) = \infty.$$

PROOF. Assume to the contrary that (2.1) has a nonrotary solution $Z(k) = \{x(k),y(k)\}$. Suppose first that it is Class I. Without loss of generality we further assume that $y(k) > 0$ for large k. By Theorem 4.2, there exists some positive integer N such that $x(k) > 0$, $y(K) > 0$, $\Delta x(k) > 0$ and $\Delta y(k) > 0$ for $k \geq N$. Since $x(k+1)$ is equal to $\Delta^2 y(k)$, it follows that

$$\Delta y(k)-\Delta y(N) = \sum_{j=N}^{k-1} x(j+1) \geq x(N+1)(k-N)$$

and

$$y(k) \geq y(N)+\Delta y(N)(k-N)+x(N+1)(k-N)^{(2)}/2 \geq x(N+1)(k-N)^{(2)}/2.$$

On the other hand, since

$$\Delta x(N) = \Delta x(k)+\sum_{j=N}^{k-1} p(j+1)y(j+1) \geq \sum_{j=N}^{k-1} p(j+1)y(j+1),$$

thus

$$\Delta x(N) \geq x(N+1)\sum_{j=N}^{k-1} (j-N+1)^{(2)}p(j+1)/2.$$

Since the left hand side is independent of k, we conclude that

$$\sum_{j=N}^{k-1} (j-N+1)^{(2)} p(j+1)/2$$

is finite, contrary to hypothesis.

Next we suppose Z(k) is Class II and that $y(k) > 0$ for large k. By Theorem 5.1, there exists some integer n such that $y(k) > 0$, $\Delta y(k) > 0$, $\Delta^2 y(k) < 0$ and $\Delta^3 y(k) > 0$ for $k \geq n$. After multiplying (1.1) by $(k-n)^{(2)}/2$ and summing by parts, we obtain

$$\Delta y(n+2) = (k-n)^{(2)} \Delta^3 y(k) - (k-n)\Delta^2 y(k+1) + \Delta y(k+2) + \sum_{j=n}^{k-1} (j-n)^{(2)} p(j+2) y(j+2)/2$$
$$\geq y(n+2) \sum_{j=n}^{k-1} (j-n)^{(2)} p(j+2)/2.$$

Again, we conclude from the above inequality that

$$\sum_{j=n}^{\infty} (j-n)^{(2)} p(j+2)/2 < \infty,$$

contrary to hypothesis. Q.E.D.

## 7. CONCLUDING REMARKS.

Either equation (1.1) or system (2.1) can be programmed easily to calculate exact of close to exact values of x(k), y(k), $\Delta x(k)$, $\Delta y(k)$, etc. Numerical demonstrations of our results in the previous Sections can therefore be left to the readers. Trial functions p(k) such as 1, $k^{(2)}$, $e^{-k}$, etc. have been tested and most of our results verified. For instance, we expect from Theorem 6.11 that when p(k) is identically one, the sequence that satisfies (1.1) and initiated by the sequence of values 0, 0, 0, 1 will show oscillating behavior. Indeed, the computer output is as follows: 0, 0, 0, 1, 4, 9, 12, 0, -52, -169, -324, -321, 360, 2560, 6760, 10881, 6084, -27351, -115228, -256000, -332892, 26871,.... We remark, however, that numerical demonstrations of nontrivial asymptotically constant solutions and Type II solutions have not been successful; neither do the proofs of our Theorems 5.8 and 6.9 give any clue to their constructions.

Numerical experiments suggest that stability of oscillatory solution can be investigated. For instance, we see from the above sequence that the successive maximum and minimum increase and exhibit an "unstable" oscillatory behavior.

There are other aspects of the recurrence equation (1.1) that can be investigated. Among these we mention the comparison of oscillatory behavior between equations of the same form. For related material the reader may consult the references cited below.

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