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ON AUTOMORPHISM GROUP OF FREE QUADRATIC EXTENSIONS OVER A RING

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ABSTRACT. Let R be a ring with 1, ρ an automorphism of R of order 2. Then a normal extension of the free quadratic extension $R[x,\rho]$ with a basis $\{1,x\}$ over R with an R-automorphism group G is characterized in terms of the element $(x-(x)\alpha)$ for α in G. It is also shown by a different method from the one given by Nagahara that the order of G of a Galois extension $R[x,\rho]$ over R with Galois group G is a unit in R. When 2 is not a zero divisor, more properties of $R[x,\rho]$ are derived.

KEY WORDS AND PHRASES. Free quadratic extensions, normal extensions, Galois extensions.

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1. INTRODUCTION.

Let C be a commutative ring with 1 and with a finite automorphism group G. Then it is well known that C is Galois over C^G (= $\{r$ in C / $(r)\alpha = r$ for each α in $G\}$) with Galois group G if and only if the ideal generated by $\{(r-(r)\alpha)$ / r in C and α in $G\}$ is C ([1], or [2], Proposition 1.2, P. 81). The sufficiency of such a characterization of Galois extensions for non-commutative rings does not hold. However, for free quadratic extensions $R[x,\rho]$ (see definition below) with respect to an automorphism ρ of R (not necessarily commutative), the above characterization be-

comes that $R[x,\rho]$ is Galois over R with Galois group G if and only if $G = \{1,\alpha / \alpha^2 = 1\}$ and $(x-(x)\alpha)$ is a unit in $R[x,\rho]$ ([3], Lemma 2.3). In fact, the element $(x-(x)\alpha)$ will play an important role in the study of $R[x,\rho]$. The purposes of the present paper are to characterize a normal extension $R[x,\rho]$ over R with an R-automorphism group G in terms of the elements $(x-(x)\alpha)$ for α in G, and to show by a different method from the one given by Nagahara ([3], Lemma 2.3) that 2 is a unit in R if $R[x,\rho]$ is Galois over R. More properties of $R[x,\rho]$ are derived from the informations of $(x-(x)\alpha)$ when 2 is not a zero divisor. At the end, examples are given to demonstrate our results.

2. PRELIMINARIES.

Throughout, let R be a ring with 1, and ρ an automorphism of R of order 2. Then a free quadratic extension $R[x,\rho]$ with respect to ρ is a ring with a free basis $\{1,x\}$ over R such that $rx = x(r\rho)$, $x^2 = b$ which is an element in $U(C^\rho)$ (= the set of units in the center C such that $(b)\rho = b$) ([4], [5], [6] , and [3]). An R-automorphism α of $R[x,\rho]$ is a ring automorphism such that $(r+xt)\alpha = r+((x)\alpha)t$ for r and t in R. A ring T is called a normal extension of a subring S with respect to an automorphism group G of T if $T^G = S$, where $T^G = \{t \text{ in } T / (t)\alpha = t \text{ for each } \alpha \text{ in } G\}$. A ring T is called a Galois extension over S with a finite Galois group G if it is normal over S and if there are elements $\{a_i, b_i \text{ in } T / i = 1, \ldots, n \text{ for some integer } n\}$ such that $\sum a_i b_i = 1$ and $\sum a_i(b_i)\alpha = 0$ for each $\alpha \neq 1$ in G ([7], or [2], P. 81).

3. NORMAL AND GALOIS EXTENSIONS.

In this section, we shall characterize a normal extension $R[x,\rho]$ in terms of the elements $\{(x-(x)\alpha) / \alpha \text{ in } G\}$, where G is an R-automorphism group of $R[x,\rho]$. When G is of order 2, it is known. Hence we have a different method from Nagahara ([3], Lemma 2.3) to show that 2 is a unit in R if $R[x,\rho]$ is Galois over R with Galois group G. We begin with several lemmas.

LEMMA 3.1. The R-linear map α such that $(x)\alpha = p+xq$ for some p, q in R is an R-automorphism of $R[x,\rho]$ if and only if (1) $rp = p(r\rho)$ for each r in R, and $(p\rho) = -p$, (2) q is in U(C), the set of units in the center C, and (3) $p^2+b(q\rho)q = b$ where $x^2 = b$ in $U(C^\rho)$.

PROOF. Since $r(x\alpha) = (rx)\alpha = (x(r\rho))\alpha = (x\alpha)(r\rho)$, conditions (1) and (2) hold immediately. Using that $(x\alpha)^2 = (x^2)\alpha = b$, we have condition (3). The converse is straightforward.

Let α be an R-automorphism of $R[x,\rho]$, and $A^r(x-(x)\alpha)$ the right annihilator of $(x-(x)\alpha)$ in R. Then we have:

LEMMA 3.2. $R[x,\rho]$ is normal over R with respect to an automorphism group G if and only if $\bigcap_{\alpha} A^r(x-(x)\alpha) = \{0\}$ for all α in G.

PROOF. Since $(x-(x)\alpha)r = xr-(xr)\alpha$ for r in R and α in G, $(x-(x)\alpha)r = 0$ if and only if $xr = (xr)\alpha$. But (xr) is in R if and only if $r = 0$, so the lemma follows.

LEMMA 3.3. Assume 2 is not a zero divisor in R. Let $(x)\alpha = p+xq$, for some α in G. Then $p^2 = 0$ and $(q)\rho = q^{-1}$.

PROOF. Since $(p)\rho = -p$ and $rp = p(r\rho)$ by condition (1) of Lemma 3.1, $p^2 = -p^2$ (for $r = p$). Hence $2p^2 = 0$. But 2 is not a zero divisor by hypothesis, so $p^2 = 0$. By condition (3) of Lemma 3.1, $p^2+b(q\rho)q = b$, so $(q\rho)q = 1$. Thus $(q\rho) = q^{-1}$.

Lemma 3.2 gives a different method from the one given by Nagahara ([3], Lemma 1.1) to show the normality of $R[x,\rho]$ over R with respect to an R-automorphism group G.

THEOREM 3.4. (T. Nagahara) If $(x-(x)\alpha)$ is not a zero divisor in $R[x,\rho]$, then $R[x,\rho]$ is a normal extension over R with respect to the cyclic R-automorphism group $\langle\alpha\rangle$.

PROOF. Assume $R[x,\rho]$ is not normal over R with respect to $\langle\alpha\rangle$. Then $A^r(p)\cap A^r(1-q) \neq \{0\}$ by Lemma 3.2. Let $u \neq 0$ be in the intersection. Then $(x-(x)\alpha)u = (-p+x(1-q))u = 0$. This contradicts to that $(x-(x)\alpha)$ is not a zero divisor. Thus $R[x,\rho]$ is normal.

The converse of Theorem 3.4 holds in case 2 is not a zero divisor.

THEOREM 3.5. Assume 2 is not a zero divisor in R. If $R[x,\rho]$ is a normal extension over R with respect to a cyclic R-automorphism group $\langle\alpha\rangle$, then $(x-(x)\alpha)$ is not a zero divisor in $R[x,\rho]$.

PROOF. Let $(x)\alpha = p+xq$ for some p, q in R. Then $A^r(x-(x)\alpha) = A^r(p)\cap A^r(1-q)$. Now let $R[x,\rho]$ be a normal extension over R with respect to $\langle\alpha\rangle$. Then

$A^r(p) \cap A^r(1-q) = \{0\}$ by Lemma 3.2. For $u+xv$ in $R[x,\rho]$ such that $(x-(x)\alpha)(u+xv) = 0$, we have that $-pu+b((1-q)\rho)v = 0$ and $pv+(1-q)u = 0$ ... (*). By multiplying p in equation (*), Lemma 3.3 implies that $p((1-q)\rho)v = 0$ and $p(1-q)u = 0$ ... (**) (for b is in $U(C^{\rho})$). By Lemma 3.3 again, $(q)\rho = q^{-1}$ which is in $U(C)$, so $p(1-q)v = 0$ and $p(1-q)u = 0$ from equations (**). Hence pv and pu are in $A^r(1-q)$. But $p^2 = 0$, so pv and pu are also in $A^r(p)$. Thus pv and pu are in $A^r(p) \cap A^r(1-q)$ which is 0, and so pv and pu are 0. This implies that v and u are in $A^r(p)$. But then equations (*) become that $(1-q)\rho \cdot v = 0$ and $(1-q)u = 0$. Noting that $(q)\rho = q^{-1}$, we have that $(1-q)v = 0$ and $(1-q)u = 0$. Thus v and u are also in $A^r(1-q)$. Therefore, v and u are in $A^r(p) \cap A^r(1-q)$ which is 0; and so $v = 0$ and $u = 0$. Similarly, we can show that $(u+xv)(x-(x)\alpha) = 0$ implies that $v = 0$ and $u = 0$. Thus $(x-(x)\alpha)$ is not a zero divisor.

Next, we determine all R-automorphism groups of $R[x,\rho]$ of order 2 such that $R[x,\rho]$ is a normal extension over R.

THEOREM 3.6. Let $R[x,\rho]$ be a normal extension over R with respect to a cyclic automorphism group $\langle\alpha\rangle$. Then, the order of α is 2 if and only if $(x)\alpha = p-x$ such that $rp = p((r)\rho)$, $(p)\rho = -p$, and $p^2 = 0$.

PROOF. Let $(x)\alpha = p+xq$ for p, q in R satisfying three conditions of Lemma 3.1 and $\alpha^2 = 1$. Then $(x)\alpha^2 = (x\alpha)\alpha = x$. This implies that $p+pq = 0$ and $q^2 = 1$. Hence $p(1+q) = 0$ and $(1-q)(1+q) = 0$. But then $(1+q)$ is in $A^r(p) \cap A^r(1-q)$. By hypothesis, $R[x,\rho]$ is normal over R with respect to $\langle\alpha\rangle$, so $q = -1$ by Lemma 3.2. Also, by condition 3 of Lemma 3.1, $p^2+b((-1)\rho)(-1) = b$, so $p^2 = 0$. Thus, $(x)\alpha = p-x$ such that $rp = p(r\rho)$, $p\rho = -p$ (Lemma 3.1), and $p^2 = 0$. The converse is easy to verify.

We are going to show that 2 is a unit in R if $R[x,\rho]$ is Galois over R by a different method from Lemma 2.3 in [3].

COROLLARY 3.7. If $R[x,\rho]$ is Galois over R with Galois group G, then 2 is a unit in R.

PROOF. By Lemma 1.2 in [3], the Galois group G of $R[x,\rho]$ is of order 2 and $(x-(x)\alpha)$ is a unit, where $G = \langle\alpha\rangle$. But $R[x,\rho]$ is normal over R with respect to $\langle\alpha\rangle$. This implies that $2bv-pu = 1$ ... (1), and $2u+pv = 0$ ... (2). Multiplying p in (2),

we have $2pu = 0$ (for $p^2 = 0$ by Theorem 3.6). But $p(pu) = 0$, so pu is in $A^r(2) \cap A^r(p)$ which is $\{0\}$ by Lemma 3.2. Thus $pu = 0$. Thus equation (1) becomes $2bv = 1$. Therefore, 2 is a unit in R.

The following are more properties of $A^r(x-(x)\alpha)$.

THEOREM 3.8. Assume 2 is not a zero divisor. Then (1) $A^r(x-(x)\alpha)$ is an invariant ideal I of R under ρ (that is, $(I)\rho = I$), and (2) $A^r(x-(x)\alpha) = A^l(x-(x)\alpha)$, the left annihilator of $(x-(x)\alpha)$ in R.

PROOF. (1) For r, s in I and t in R, clearly, (r-s) and (rt) are in I. Since $(x-(x)\alpha)tr = (t\rho)(x-(x)\alpha)r = 0$, (tr) is in I. Hence I is an ideal of R. Also, let $(x)\alpha = p+xq$ for p, q satisfying 3 conditions in Lemma 3.1. Then $(x-(x)\alpha)r = 0$ if and only if $-pr = 0$ and $(1-q)r = 0$. Hence $(-pr)\rho = 0$ and $((1-q)r)\rho = 0$; that is, $p(r\rho) = 0$ and $(1-q)(r\rho) = 0$ (for $(p)\rho = -p$ and $(q)\rho = q^{-1}$ by Lemma 3.3). Thus $(r\rho)$ is in I.

(2) Let r be in I. Then $(r\rho)$ is in I by Part (1). Since $r(x-(x)\alpha) = (x-(x)\alpha)(r\rho) = 0$, r is in $A^l(x-(x)\alpha)$. Conversely, let r be in $A^l(x-(x)\alpha)$. Then $r(x-(x)\alpha) = (x-(x)\alpha)(r\rho) = 0$. Hence $(r\rho)$ is in I. Thus $r\ (= (r\rho)\rho)$ is in I.

4. EXAMPLES.

We conclude the paper with three examples to demonstrate the results in Section 3. 1. Let Z be the ring of integers, and ρ an automorphism of R $(= Z[\sqrt{3}])$ defined by $(n+m\sqrt{3})\rho = n-m\sqrt{3}$ in R. Then $R[i,\rho]$ is normal over R, where $i^2 = -1$, with respect to an R-automorphism group $\langle\alpha\rangle$ such that $(i)\alpha = -i$.

2. By replacing Z with Z/(4), Example 1 becomes a non-normal extension.

3. Let Q be the rational field. By replacing Z with Q, Example 1 becomes a Galois extension over R with Galois group $\langle\alpha\rangle$.

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