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SOME INVARIANT THEOREMS ON GEOMETRY OF EINSTEIN NON-SYMMETRIC FIELD THEORY

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ABSTRACT. This paper generalizes Einstein's theorem. It is shown that under the transformation

$$T_\Lambda : U^\ell_{ik} \to \overline{U}^\ell_{ik} \equiv U^\ell_{ik} + \delta^\ell_i \Lambda_k - \delta^\ell_k \Lambda_i ,$$

curvature tensor $S^i_{k\ell m}(U)$, Ricci tensor $S_{ik}(U)$, and scalar curvature $S(U)$ are all invariant, where $\Lambda = \Lambda_j dx^j$ is a closed 1-differential form on an n-dimensional manifold M.

It is still shown that for arbitrary U, the transformation that makes curvature tensor $S^i_{k\ell m}(U)$ (or Ricci tensor $S_{ik}(U)$) invariant

$$T_V : U^\ell_{ik} \to \overline{U}^\ell_{ik} \equiv U^\ell_{ik} + V^\ell_{ik}$$

must be T_Λ transformation, where V (its components are V^ℓ_{ik}) is a second order differentiable covariant tensor field with vector value.

KEY WORDS AND PHRASES. Einstein non-symmetric field, Einstein theorem, curvature tensor, Ricci tensor, scalar curvature, T_Λ transformation, T_V transformation.

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1. INTRODUCTION.

When A. Einstein devoted himself to research on relativism in his symmetric field [1], he regarded non-symmetric g_{ij} (or g^{ij}) and non-symmetric affine connection D (its coefficinets are Γ^ℓ_{ik} in local coordinates $\{x^i\}$) as independent variables such that the number of independent variables increased from 50 (g_{ij} and Γ^ℓ_{ik} are all symmetric for lower coordinates) to 80 (16 g_{ij} or g^{ij} and 64 Γ^ℓ_{ik}). With so many covariant variables, it was impossible to choose them according to the principle of relativism alone. To overcome this difficulty, Einstein introduced a very important concept, transposition invariance. This "transposition invariance" (or transposition symmetry) meant that when all A_{ik} were transposed ($A^{T}_{ik} = A_{ki}$), all equations were still applicable [2]. Einstein supposed that field equations were transposition invariant. He thought that in physics this hypothesis was equivalent to the law that positive and negative electricity occurred symmetrically.

As the Ricci tensor $R_{ik}(\Gamma)$ represented by connected coefficients Γ^{ℓ}_{ik} was not transposition invariant, Einstein introduced a "pseudo-tensor" U^{ℓ}_{ik} instead [3]; its definition was

$$U^{\ell}_{ik} \equiv \Gamma^{\ell}_{ik} - \Gamma^{t}_{it}\delta^{\ell}_{k}, \text{ where } \Gamma^{t}_{it} = \sum_{t=1}^{4} \Gamma^{t}_{it}. \tag{1.1}$$

Denoting Γ^{ℓ}_{ik} by U^{ℓ}_{ik}, we obtained

$$\Gamma^{\ell}_{ik} = U^{\ell}_{ik} - \frac{1}{3} U^{t}_{it}\delta^{\ell}_{k} \quad (i,\ k,\ \ell = 1,\ldots,\ 4). \tag{1.2}$$

Then the Ricci curvature denoted by U was

$$R_{ik} = U^{s}_{ik,s} - U^{s}_{it}\, U^{t}_{sk} + \frac{1}{3} U^{s}_{is}\, U^{t}_{tk} = S_{ik}(U) = S_{ik}. \tag{1.3}$$

Einstein proved that S_{ik} were transposition invariant and the following.

THEOREM (EINSTEIN). [1] Under the transformation

$$T_{\lambda} : U^{\ell}_{ik} \to \overline{U}^{\ell}_{ik} \equiv U^{\ell}_{ik} + \delta^{\ell}_{i}\lambda_{,k} - \delta^{\ell}_{k}\lambda_{,i}, \tag{1.4}$$

the Ricci tensor S_{ik} of U is invariant; i.e., under the transformation (4), there are $\overline{S}_{ik} = S_{ik}$ for arbitrary U, where $\overline{S}_{ik} \equiv S_{ik}(\overline{U})$. In (4), $\lambda_{,j} = \frac{\partial \lambda}{\partial x^{j}}$ and λ is a differentiable function on a manifold M.

REMARK. Einstein gave transformation (1.4) for n=4, but we will still call transformation (1.4) the Einstein transformation for general n (≥ 2).

One asks naturally, how about the converse of Einstein's theorem? A. Einstein and B. Kaufman did not solve the problem. It has remained unsolved.

In this paper, we generalize Einstein and Kaufman's results to an arbitrary n (≥ 2) dimensional manifold M. Objects which we discuss are not limited to the Ricci tensor S_{ik} of U. Besides S_{ik}, we discuss curvature tensor $S^{i}_{k\ell m}$ and scalar curvature S.

Then, for general n (≥ 2), we give some invariant theorems on curvature tensor $S^{i}_{k\ell m}$, Ricci tensor S_{ik} and scalar curvature S of U. For this, first we generalize Einstein's transformation. Finally, we give converse theorems of theorems for arbitrary n (≥ 2). These are the main results of this paper. In the special case n=4, we answer the problem above mentioned; that is, a converse to Einstein's theorem.

2. DEFINITION AND MAIN RESULTS.

To give the definitions for curvature tensor $S^{i}_{k\ell m}$ and Ricci tensor S_{ik} of U, first let us give reasonable definitions for curvature tensor $R^{i}_{k\ell m}$ and Ricci tensor R_{ik} of connection D (Γ^{ℓ}_{ik}) (order of lower coordinates is very important; what we give here differs by a minus sign from what is sometimes used, for example, in Pauli's relativism).

$$\begin{aligned} R^{i}_{k\ell m} &\equiv \Gamma^{i}_{k\ell,m} - \Gamma^{i}_{km,\ell} - \Gamma^{i}_{s\ell}\Gamma^{s}_{km} + \Gamma^{i}_{sm}\Gamma^{s}_{k\ell} \\ &= (\Gamma^{i}_{k\ell,m} - \Gamma^{i}_{s\ell}\Gamma^{s}_{km}) - (\Gamma^{i}_{km,\ell} - \Gamma^{i}_{sm}\Gamma^{s}_{k\ell}) \\ &\equiv [\ell,\ m] - [m,\ \ell], \end{aligned} \tag{2.1}$$

In (2.1), let i=m. Adding from 1 to n the curvature tensor $R^i_{k\ell m}$ is contracted to obtain the Ricci tensor

$$R_{k\ell} \equiv R^s_{k\ell s} = \Gamma^s_{k\ell,s} - \Gamma^s_{ks,\ell} - \Gamma^s_{t\ell}\Gamma^t_{ks} + \Gamma^s_{ts}\Gamma^t_{k\ell} \,. \tag{2.2}$$

To establish expressions for the curvature tensor $S^i_{k\ell m}$, Ricci tensor S_{ik}, and scalar curvature S of U for arbitrary n ($\geq$ 2), it is necessary to give transformation between U and Γ for arbitrary n ($\geq$ 2).

In (1.1), let ℓ=k, add from 1 to n obtaining

$$U^t_{it} = \Gamma^t_{it} - n\Gamma^t_{it} = -(n-1)\Gamma^t_{it}. \tag{2.3}$$

Substituting (2.3) into (1.1), we can solve

$$\Gamma^\ell_{ik} = U^\ell_{ik} - \frac{1}{n-1}\delta^\ell_k U^t_{it} \quad (i,\ k,\ \ell = 1,\ldots,\ n) \tag{2.4}$$

From (2.1) - (2.3), and definitions we obtain immediately

PROPOSITION 1. Curvature tensor $S^i_{k\ell m}$, Ricci tensor S_{ik} and scalar curvature S of U are respectively

$$(1) \quad R^i_{k\ell m} = U^i_{k\ell,m} - \frac{1}{n-1}\delta^i_\ell U^t_{kt,m} - U^i_{s\ell}U^s_{km} + \frac{1}{n-1}\delta^i_\ell U^t_{st}U^s_{km}$$
$$+ \frac{1}{n-1}U^t_{kt}U^i_{m\ell} - \frac{1}{(n-1)^2}\delta^i_\ell U^t_{mt}U^s_{ks} - U^i_{km,\ell} + \frac{1}{n-1}\delta^i_m U^t_{kt,\ell} + U^i_{sm}U^s_{k\ell}$$
$$- \frac{1}{n-1}\delta^i_m U^t_{st}U^s_{k\ell} - \frac{1}{n-1}U^t_{kt}U^i_{\ell m} + \frac{1}{(n-1)^2}U^t_{\ell t}U^s_{ks} \equiv S^i_{k\ell m}(U) \equiv S^i_{k\ell m}\,. \tag{2.5}$$

$$(2) \quad R_{ik} = U^s_{ik,s} - U^s_{it}U^t_{sk} + \frac{1}{n-1}U^s_{is}U^t_{tk} \equiv S_{ik}(U) \equiv S_{ik}. \tag{2.6}$$

$$(3) \quad S \equiv g^{ik}R_{ik} = g^{ik}U^s_{ik,s} - g^{ik}U^s_{it}U^t_{sk} + \frac{1}{n-1}g^{ik}U^s_{is}U^t_{tk} \equiv S(U) \tag{2.7}$$

When n$\geq$ 2, it is not difficult to verify that Ricci tensor S_{ik} and scalar curvature S of U are transposition invariant.

THEOREM 1. Curvature tensor $S^i_{k\ell m}$, Ricci tensor S_{ik} and scalar curvature S of U are all invariant under the following transformation

$$T_\Lambda : U^\ell_{ik} \to \overline{U}^\ell_{ik} \equiv U^\ell_{ik} + \delta^\ell_i\Lambda_k - \delta^\ell_k\Lambda_i, \tag{2.8}$$

where $\Lambda = \Lambda_j dx^j$ is a closed 1-differential form on a manifold M; i.e., $d\Lambda = 0$.

REMARK. The transformation (2.8) is a generalization of Einstein's transformation (1.4). In fact, as Λ is a closed 1-differential form on a manifold M ($d\Lambda = 0$), then by the Poincaré Lemma, there exists a coordinate neighborhood $M_1 \subset M$ and a differentiable function λ such that $\Lambda_k = \frac{\partial\lambda}{\partial x^k} = \lambda_{,k}$ (k=1,..., n). Therefore, in a local neighborhood, for example M_1, the transformation (2.8) conforms with Einstein's transformation (1.4) .

Because an exact differential form $d\lambda$ is a closed differential form ($d^2\lambda = 0$), T_λ is a transformation which makes $S^i_{k\ell m}$, S_{ik} and S invariant. When n=4, Einstein's theorem is a special case of the above theorem 1.

PROOF. In local coordinates $\{x^i\}$, let $\Lambda_i = \lambda_{,i}$, then

$$\begin{aligned}
\bar S^i_{k\ell m} &= S^i_{k\ell m}(\bar U) \\
&= (U^i_{k\ell,m} + \delta^i_k\lambda_{,\ell m} - \delta^i_\ell\lambda_{,km}) - \frac{1}{n-1}\delta^i_\ell(U^t_{kt,m} + \delta^t_k\lambda_{,tm} - \delta^t_t\lambda_{,km}) \\
&- (U^i_{s\ell} + \delta^i_s\lambda_{,\ell} - \delta^i_\ell\lambda_{,s})(U^s_{km} + \delta^s_k\lambda_{,m} - \delta^s_m\lambda_{,k}) \\
&+ \frac{1}{n-1}\delta^i_\ell(U^t_{st} + \delta^t_s\lambda_{,t} - \delta^t_t\lambda_{,s})(U^s_{km} + \delta^s_k\lambda_{,m} - \delta^s_m\lambda_{,k}) \\
&+ \frac{1}{n-1}(U^t_{kt} + \delta^t_k\lambda_{,t} - \delta^t_t\lambda_{,k})(U^i_{m\ell} + \delta^i_m\lambda_{,\ell} - \delta^i_\ell\lambda_{,m}) \\
&- \frac{1}{(n-1)^2}\delta^i_\ell(U^t_{mt} + \delta^t_m\lambda_{,t} - \delta^t_t\lambda_{,m})(U^s_{ks} + \delta^s_k\lambda_{,s} - \delta^s_s\lambda_{,k}) \\
&- (U^i_{km,\ell} + \delta^i_k\lambda_{,m\ell} - \delta^i_m\lambda_{,k\ell}) + \frac{1}{n-1}\delta^i_m(U^t_{kt,\ell} + \delta^t_k\lambda_{,t\ell} - \delta^t_t\lambda_{,k\ell}) \\
&+ (U^i_{sm} + \delta^i_s\lambda_{,m} - \delta^i_m\lambda_{,s})(U^s_{k\ell} + \delta^s_k\lambda_{,\ell} - \delta^s_\ell\lambda_{,k}) \\
&- \frac{1}{n-1}\delta^i_m(U^t_{st} + \delta^t_s\lambda_{,t} - \delta^t_t\lambda_{,s})(U^s_{k\ell} + \delta^s_k\lambda_{,\ell} - \delta^s_\ell\lambda_{,k}) \\
&- \frac{1}{n-1}(U^t_{kt} + \delta^t_k\lambda_{,t} - \delta^t_t\lambda_{,k})(U^i_{\ell m} + \delta^i_\ell\lambda_{,m} - \delta^i_m\lambda_{,\ell}) \\
&+ \frac{1}{(n-1)^2}\delta^i_m(U^t_{\ell t} + \delta^t_\ell\lambda_{,t} - \delta^t_t\lambda_{,\ell})(U^s_{ks} + \delta^s_k\lambda_{,s} - \delta^s_s\lambda_{,k}) \\
&= S^i_{k\ell m}(U) - U^i_{k\ell}\lambda_{,m} + U^i_{m\ell}\lambda_{,k} - U^i_{km}\lambda_{,\ell} - \delta^i_k\lambda_{,\ell}\lambda_{,m} \\
&+ \delta^i_m\lambda_{,\ell}\lambda_{,k} + \delta^i_\ell\lambda_{,s}U^s_{km} + \delta^i_\ell\lambda_{,k}\lambda_{,m} - \delta^i_\ell\lambda_{,m}\lambda_{,k} \\
&+ \frac{1}{n-1}\delta^i_\ell\lambda_{,m}U^t_{kt} - \frac{1}{n-1}\delta^i_\ell\lambda_{,k}U^t_{mt} + \frac{1}{n-1}\delta^i_\ell\lambda_{,s}U^s_{km} + \frac{1}{n-1}\delta^i_\ell\lambda_{,k}\lambda_{,m} \\
&- \frac{1}{n-1}\delta^i_\ell\lambda_{,m}\lambda_{,k} - \frac{n}{n-1}\delta^i_\ell\lambda_{,s}U^s_{km} - \frac{n}{n-1}\delta^i_\ell\lambda_{,k}\lambda_{,m} \\
&+ \frac{n}{n-1}\delta^i_\ell\lambda_{,m}\lambda_{,k} + \frac{1}{n-1}\delta^i_m\lambda_{,\ell}U^t_{kt} - \frac{1}{n-1}\delta^i_\ell\lambda_{,m}U^t_{kt} \\
&+ \frac{1}{n-1}\lambda_{,k}U^i_{m\ell} + \frac{1}{n-1}\delta^i_m\lambda_{,k}\lambda_{,\ell} - \frac{1}{n-1}\delta^i_\ell\lambda_{,k}\lambda_{,m} \\
&- \frac{n}{n-1}\lambda_{,k}U^i_{m\ell} - \frac{n}{n-1}\delta^i_m\lambda_{,k}\lambda_{,\ell} + \frac{n}{n-1}\delta^i_\ell\lambda_{,k}\lambda_{,m} \\
&- \frac{1}{(n-1)^2}\delta^i_\ell\lambda_{,k}U^t_{mt} + \frac{n}{(n-1)^2}\delta^i_\ell\lambda_{,k}U^t_{mt} - \frac{1}{(n-1)^2}\delta^i_\ell\lambda_{,m}U^s_{ks} \\
&- \frac{1}{(n-1)^2}\delta^i_\ell\lambda_{,m}\lambda_{,k} + \frac{n}{(n-1)^2}\delta^i_\ell\lambda_{,m}\lambda_{,k} + \frac{n}{(n-1)^2}\delta^i_\ell\lambda_{,m}U^s_{ks} \\
&+ \frac{n}{(n-1)^2}\delta^i_\ell\lambda_{,m}\lambda_{,k} - \frac{n^2}{(n-1)^2}\delta^i_\ell\lambda_{,m}\lambda_{,k} - [m,\ell] = S^i_{k\ell m},
\end{aligned}$$

where $$[m,\ell] = -\lambda_{,\ell}U^i_{km} + \dots - \frac{n^2}{(n-1)^2}\delta^i_m\lambda_{,\ell}\lambda_{,k}.$$

Consider the transformation

$$T_\Omega : U^\ell_{ik} \to \overline{U}^\ell_{ik} \equiv U^\ell_{ik} + \delta^\ell_i\Omega_k - \delta^\ell_k\Omega_i \ , \qquad (2.9)$$

where Ω is a 1-differential form, in local coordinates $\{x^i\}$, $\Omega = \Omega_j dx^j$.

Having the above result for theorem 1, we ask naturally if the transformation which makes curvature tensor $S^i_{k\ell m}$ (or Ricci tensor S_{ik}) invariant is the transformation T_Λ? For this, although we cannot give the complete answer - it is a very difficult problem - we have the following results.

THEOREM 2. The transformation that makes curvature tensor $S^i_{k\ell m}$ (or Ricci tensor S_{ik}) of some U invariant

$$T_\Omega \ : \ U^\ell_{ik} \to \overline{U}^\ell_{ik} \equiv U^\ell_{ik} + \delta^\ell_i\Omega_k - \delta^\ell_k\Omega_i$$

must be T_Λ, where $\Omega = \Omega_j dx^j$ is a 1-differential form.

PROOF. Similar to the proof of theorem 1, we obtain $\overline{S}^i_{k\ell m} = S^i_{k\ell m}(\overline{U})$

$$\begin{aligned}
&= (U^i_{k\ell,m} + \delta^i_k\Omega_{\ell,m} - \delta^i_\ell\Omega_{k,m}) - \frac{1}{n-1}\delta^i_\ell(U^t_{kt,m} + \delta^t_k\Omega_{t,m} - \delta^t_t\Omega_{k,m}) \\
&- (U^i_{s\ell} + \delta^i_s\Omega_\ell - \delta^i_\ell\Omega_s)(U^s_{km} + \delta^s_k\Omega_m - \delta^s_m\Omega_k) \\
&+ \frac{1}{n-1}\delta^i_\ell(U^t_{st} + \delta^t_s\Omega_t - \delta^t_t\Omega_s)(U^s_{km} + \delta^s_k\Omega_m - \delta^s_m\Omega_k) \\
&+ \frac{1}{n-1}(U^t_{kt} + \delta^t_k\Omega_t - \delta^t_t\Omega_k)(U^i_{m\ell} + \delta^i_m\Omega_\ell - \delta^i_\ell\Omega_m) \\
&- \frac{1}{(n-1)^2}\delta^i_\ell(U^t_{mt} + \delta^t_m\Omega_t - \delta^t_t\Omega_m)(U^s_{ks} + \delta^s_k\Omega_s - \delta^s_s\Omega_k) - [m, \ell] \\
&= S^i_{k\ell m}(U) + (\delta^i_k\Omega_{\ell,m} - \delta^i_\ell\Omega_{k,m}) - \frac{1}{n-1}\delta^i_\ell(\delta^t_k\Omega_{t,m} - \delta^t_t\Omega_{k,m}) \\
&- U^i_{k\ell}\Omega_m + U^i_{m\ell}\Omega_k - U^i_{km}\Omega_\ell - \delta^i_k\Omega_\ell\Omega_m + \delta^i_m\Omega_\ell\Omega_k + \delta^i_\ell\Omega_s U^s_{km} \\
&+ \delta^i_\ell\Omega_k\Omega_m - \delta^i_\ell\Omega_m\Omega_k + \frac{1}{n-1}\delta^i_\ell U^t_{kt}\Omega_m - \frac{1}{n-1}\delta^i_\ell U^t_{mt}\Omega_k + \frac{1}{n-1}\delta^i_\ell\Omega_s U^s_{km} \\
&+ \frac{1}{n-1}\delta^i_\ell\Omega_k\Omega_m - \frac{1}{n-1}\delta^i_\ell\Omega_m\Omega_k - \frac{n}{n-1}\delta^i_\ell\Omega_s U^s_{km} - \frac{n}{n-1}\delta^i_\ell\Omega_k\Omega_m \\
&+ \frac{n}{n-1}\delta^i_\ell\Omega_m\Omega_k + \frac{1}{n-1}\delta^i_m U^t_{kt}\Omega_\ell - \frac{1}{n-1}\delta^i_\ell U^t_{kt}\Omega_m + \frac{1}{n-1}\Omega_k U^i_{m\ell} \\
&+ \frac{1}{n-1}\delta^i_m\Omega_k\Omega_\ell - \frac{1}{n-1}\delta^i_\ell\Omega_k\Omega_m - \frac{n}{n-1}\Omega_k U^i_{m\ell} - \frac{n}{n-1}\delta^i_m\Omega_k\Omega_\ell \\
&+ \frac{n}{n-1}\delta^i_\ell\Omega_k\Omega_m - \frac{1}{(n-1)^2}\delta^i_\ell U^t_{mt}\Omega_k + \frac{n}{(n-1)^2}\delta^i_\ell U^t_{mt}\Omega_k - \frac{1}{(n-1)^2}\delta^i_\ell\Omega_m U^s_{ks} \\
&- \frac{1}{(n-1)^2}\delta^i_\ell\Omega_m\Omega_k + \frac{n}{(n-1)^2}\delta^i_\ell\Omega_m\Omega_k + \frac{n}{(n-1)^2}\delta^i_\ell U^s_{ks}\Omega_m + \frac{n}{(n-1)^2}\delta^i_\ell\Omega_m\Omega_k \\
&- \frac{n^2}{(n-1)^2}\delta^i_\ell\Omega_m\Omega_k - [m, \ell] = S^i_{k\ell m} + \delta^i_k(\Omega_{\ell,m} - \Omega_{m,\ell}).
\end{aligned}$$

Therefore, $\bar{S}^i_{k\ell m} = S^i_{k\ell m}$ if and only if
$\Omega_{\ell,m} = \Omega_{m,\ell}$, i.e., $d\Omega=0$.

From $\bar{S}_{k\ell} = \bar{S}^i_{k\ell i} = S^i_{k\ell i} + \delta^i_k(\Omega_{\ell,i} - \Omega_{i,\ell}) = S_{k\ell} + (\Omega_{\ell,k} - \Omega_{k,\ell})$ it follows that $\bar{S}_{k\ell} = S_{k\ell}$ if and only if

$$\Omega_{\ell,k} = \Omega_{k,\ell},$$

i.e., $d\Omega = 0$.

THEOREM 3. A necessary and sufficient condition that transformation T_Ω makes scalar curvature S of some U invariant is

$$g^{ik}\Omega_{k,i} - g^{ik}\Omega_{i,k} = 0.$$

where $\Omega_{k,i} = \dfrac{\partial\Omega_k}{\partial x^i}$.

PROOF. From $\bar{S} = g^{ik}\bar{S}_{ik} = g^{ik}S_{ik} + g^{ik}(\Omega_{k,i} - \Omega_{i,k}) = S + (g^{ik}\Omega_{k,i} - g^{ik}\Omega_{i,k})$ it follows that $\bar{S} = S$ if and only if

$$g^{ik}\Omega_{k,i} - g^{ik}\Omega_{i,k} = 0.$$

REMARK. If g^{ik} are symmetric, then $g^{ik}\Omega_{k,i} - g^{ik}\Omega_{ik} = 0$. If g^{ik} are not symmetric, for example $g^{12} \neq g^{21}$, let

$$\Omega_k = \begin{cases} x^2, & k = 1, \\ 0, & k = 2, \ldots, n, \end{cases}$$

then $g^{ik}\Omega_{k,i} - g^{ik}\Omega_{i,k} = g^{21} - g^{12} \neq 0$.

Now we give the converse of theorem 1. For this, what we must emphasize is that because of theorem 1, the transformation T_Λ makes curvature tensor $S^i_{k\ell m}$ and Ricci tensor S_{ik} of every U invariant.

The following theorems, 4 and 5 respectively, are the converses of theorem 1 on curvature tensor $S^i_{k\ell m}$ and Ricci tensor S_{ik}.

THEOREM 4. Let V be a second order differentiable covariant tensor field with vector value and its components be V^ℓ_{ik} in local coordinates $\{x^i\}$. If the transformation

$$T_V : U^\ell_{ik} \to \bar{U}^\ell_{ik} \equiv U^\ell_{ik} + V^\ell_{ik} \quad (i, k, \ell = 1, \ldots, n) \tag{2.10}$$

makes curvature tensor $S^i_{k\ell m}$ of every U invariant, then it implies

$$V^\ell_{ik} = \delta^\ell_i\Lambda_k - \delta^\ell_k\Lambda_i,$$

where $\Lambda = \Lambda_j dx^j$ is a closed 1-differential form; i.e., T_V must be T_Λ.

PROOF. By (2.10),

$$\bar{S}^i_{k\ell m} = S^i_{k\ell m}(\bar{U}) = (U^i_{k\ell,m} + V^i_{k\ell,m}) - \frac{1}{n-1}\delta^i_\ell(U^t_{kt,m} + V^t_{kt,m})$$
$$- (U^i_{s\ell} + V^i_{s\ell})(U^s_{km} + V^s_{km}) + \frac{1}{n-1}\delta^i_\ell(U^t_{st} + V^t_{st})(U^s_{km} + V^s_{km})$$

$$+ \frac{1}{n-1}(U^t_{kt} + V^t_{kt})(U^i_{m\ell} + V^i_{m\ell}) - \frac{1}{(n-1)^2}\delta^i_\ell (U^t_{mt} + V^t_{mt})(U^s_{ks} + V^s_{ks})$$

$$- [m, \ell] = S^i_{k\ell m} + V^i_{k\ell,m} - \frac{1}{n-1} V^t_{kt,m}\delta^i_\ell - V^i_{s\ell}U^s_{km} - U^i_{s\ell}V^s_{km}$$

$$- V^i_{s\ell}V^s_{km} + \frac{1}{n-1}\delta^i_\ell V^t_{st}U^s_{km} + \frac{1}{n-1}\delta^i_\ell V^s_{km}U^t_{st} + \frac{1}{n-1}\delta^i_\ell V^t_{st}V^s_{km} + \frac{1}{n-1}U^t_{kt}V^i_{m\ell}$$

$$+ \frac{1}{n-1}V^t_{kt}U^i_{m\ell} + \frac{1}{n-1}V^i_{m\ell}V^t_{kt} - \frac{1}{(n-1)^2}\delta^i_\ell U^t_{mt}V^s_{ks} - \frac{1}{(n-1)^2}\delta^i_\ell U^s_{ks}V^t_{mt} - \frac{1}{(n-1)^2}\delta^i_\ell V^t_{mt}V^s_{ks}$$

$$- [m, \ell] = S^i_{k\ell m} + F^i_{k\ell m}.$$

Since $\bar{S}^i_{k\ell m} = S^i_{k\ell m}$ (for every U), now $F^i_{k\ell m}(U) = 0$ (i, k, ℓ, m = 1, ..., n and for every U). Therefore,

$$0 = \frac{\partial F^i_{k\ell m}}{\partial U^\gamma_{\alpha\beta}} = -V^i_{\gamma\ell}\delta^\alpha_k\delta^\beta_m - V^\alpha_{km}\delta^i_\gamma\delta^\beta_\ell + \frac{1}{n-1}\delta^i_\ell V^t_{\gamma t}\delta^\alpha_k\delta^\beta_m$$

$$+ \frac{1}{n-1}\delta^i_\ell V^\alpha_{km}\delta^\beta_\gamma + \frac{1}{n-1}V^i_{m\ell}\delta^\alpha_k\delta^\beta_\gamma + \frac{1}{n-1}V^t_{kt}\delta^i_\gamma\delta^\alpha_m\delta^\beta_\ell - \frac{1}{(n-1)^2}\delta^i_\ell V^s_{ks}\delta^\alpha_m\delta^\beta_\gamma$$

$$- \frac{1}{(n-1)^2}\delta^i_\ell V^t_{mt}\delta^\alpha_k\delta^\beta_\gamma + V^i_{\gamma m}\delta^\alpha_k\delta^\beta_\ell + \delta^i_\gamma\delta^\beta_m V^\alpha_{k\ell} - \frac{1}{n-1}\delta^i_m V^t_{\gamma t}\delta^\alpha_k\delta^\beta_\ell$$

$$- \frac{1}{n-1}\delta^i_m V^\alpha_{k\ell}\delta^\beta_\gamma - \frac{1}{n-1}V^i_{\ell m}\delta^k_\alpha\delta^\beta_\gamma - \frac{1}{n-1}V^t_{kt}\delta^i_\gamma\delta^\alpha_\ell\delta^\beta_m + \frac{1}{(n-1)^2}\delta^i_m V^s_{ks}\delta^\alpha_\ell\delta^\beta_\gamma$$

$$+ \frac{1}{(n-1)^2}\delta^i_m V^t_{\ell t}\delta^\alpha_k\delta^\beta_\gamma.$$

In the above formula, let i = ℓ. Adding from 1 to n we obtain

$$0 = \frac{\partial F^\ell_{k\ell m}}{\partial U^\gamma_{\alpha\beta}} = -V^\ell_{\gamma\ell}\delta^\alpha_k\delta^\beta_m - \delta^\beta_\gamma V^\alpha_{km} + \frac{n}{n-1}V^t_{\gamma t}\delta^\alpha_k\delta^\beta_m + \frac{n}{n-1}V^\alpha_{km}\delta^\beta_\gamma$$

$$+ \frac{1}{n-1}V^\ell_{m\ell}\delta^\alpha_k\delta^\beta_\gamma + \frac{1}{n-1}V^t_{kt}\delta^\beta_\gamma\delta^\alpha_m - \frac{n}{(n-1)^2}V^s_{ks}\delta^\alpha_m\delta^\beta_\gamma - \frac{n}{(n-1)^2}V^t_{mt}\delta^\alpha_k\delta^\beta_\gamma$$

$$+ V^\beta_{\gamma m}\delta^\alpha_k + \delta^\beta_m V^\alpha_{k\gamma} - \frac{1}{n-1}V^t_{\gamma t}\delta^\alpha_k\delta^\beta_m - \frac{1}{n-1}V^\alpha_{km}\delta^\beta_\gamma - \frac{1}{n-1}V^\ell_{\ell m}\delta^\alpha_k\delta^\beta_\gamma$$

$$- \frac{1}{n-1}V^t_{kt}\delta^\alpha_\gamma\delta^\beta_m + \frac{1}{(n-1)^2}V^s_{ks}\delta^\alpha_m\delta^\beta_\gamma + \frac{1}{(n-1)^2}V^t_{mt}\delta^\alpha_k\delta^\beta_\gamma$$

$$= \delta^\alpha_k V^\beta_{\gamma m} + \delta^\beta_m V^\alpha_{k\gamma} - \frac{1}{n-1}V^\ell_{\ell m}\delta^\alpha_k\delta^\beta_\gamma - \frac{1}{n-1}V^t_{kt}\delta^\alpha_\gamma\delta^\beta_m.$$

Let α = k. Adding from 1 to n we obtain

$$0 = nV^\beta_{\gamma m} + \delta^\beta_m V^\alpha_{\alpha\gamma} - \frac{n}{n-1}V^\ell_{\ell m}\delta^\beta_\gamma - \frac{1}{n-1}V^t_{\gamma t}\delta^\beta_m. \tag{2.11}$$

In (2.11) let β = γ again. Adding from 1 to n we obtain

$$nV^{\beta}_{\beta m} + V^{\alpha}_{\alpha m} - \frac{n^2}{n-1} V^{\ell}_{\ell m} - \frac{1}{n-1} V^{t}_{mt} = - \frac{V^{\beta}_{\beta m}}{n-1} - \frac{V^{t}_{mt}}{n-1} = 0,$$

$$-V^{t}_{mt} = V^{t}_{tm} \;(= V_m).$$

Substituting the above formula into (2.11) to obtain

$$V^{\beta}_{\gamma m} = \delta^{\beta}_{\gamma} \frac{V_m}{n-1} - \delta^{\beta}_{m} \frac{V_{\gamma}}{n-1} = \delta^{\beta}_{\gamma} \Omega_m - \delta^{\beta}_{m} \Omega_{\gamma},$$

where $\Omega_{\gamma} = \dfrac{V_{\gamma}}{n-1} = \dfrac{V^{t}_{t\gamma}}{n-1}$.

This proves that the transformation T_V must be T_{Ω}; then, according to theorem 2, $\Omega = \Omega_j dx^j$ must be a closed differential form; i.e., $\Omega = \Lambda$.

The analogous result for S_{ik} is stronger.

THEOREM 5. The transformation T_V that makes the Ricci tensor S_{ik} of arbitrary U invariant must be the transformation T_{Λ}.

PROOF. In fact,

$$\overline{S}_{ik} = S_{ik}(\overline{U}) \equiv \overline{U}^{s}_{ik,s} - \overline{U}^{s}_{it}\overline{U}^{t}_{sk} + \frac{1}{n-1} \overline{U}^{s}_{is} \overline{U}^{t}_{tk}$$

$$= (U^{s}_{ik,s} + V^{s}_{ik,s}) - (U^{s}_{it} + V^{s}_{it}) (U^{t}_{sk} + V^{t}_{sk}) + \frac{1}{n-1} (U^{s}_{is} + V^{s}_{is}) (U^{t}_{tk} + V^{t}_{tk})$$

$$= S_{ik} + F_{ik},$$

where,

$$F_{ik} = V^{s}_{ik,s} - U^{s}_{it} V^{t}_{sk} - V^{s}_{it} U^{t}_{sk} - V^{s}_{it} V^{t}_{sk} + \frac{1}{n-1} U^{s}_{is} V^{t}_{tk}$$

$$+ \frac{1}{n-1} V^{s}_{is} U^{t}_{tk} + \frac{1}{n-1} V^{s}_{is} V^{t}_{tk}.$$

According to the condition of the theorem, $\overline{S}_{ik} = S_{ik}$, and we obtain $F_{ik} = 0$ (for arbitrary U). Therefore,

$$0 = \frac{\partial F_{ik}}{\partial U^{\gamma}_{\alpha\beta}} = -\delta^{s}_{\gamma}\delta^{\alpha}_{i}\delta^{\beta}_{t} V^{t}_{sk} - \delta^{t}_{\gamma}\delta^{\alpha}_{s}\delta^{\beta}_{k} V^{s}_{it} + \frac{1}{n-1} \delta^{s}_{\gamma}\delta^{\alpha}_{i}\delta^{\beta}_{s} V^{t}_{tk}$$

$$+ \frac{1}{n-1}\delta^{t}_{\gamma}\delta^{\alpha}_{t}\delta^{\beta}_{k} V^{s}_{is} = - \delta^{\alpha}_{i} V^{\beta}_{\gamma k} - \delta^{\beta}_{k} V^{\alpha}_{i\gamma} + \frac{1}{n-1} \delta^{\alpha}_{i} \delta^{\beta}_{\gamma} V^{t}_{tk} + \frac{1}{n-1} \delta^{\alpha}_{\gamma}\delta^{\beta}_{k} V^{s}_{is}.$$

In the above formula, let $\alpha = \gamma$. Adding from 1 to n we obtain

$$-V^{\beta}_{ik} - \delta^{\beta}_{k} V^{t}_{it} + \frac{1}{n-1} \delta^{\beta}_{i} V^{t}_{tk} + \frac{n}{n-1} \delta^{\beta}_{k} V^{s}_{is} = 0,$$

$$-V^{\beta}_{ik} + \frac{1}{n-1} \delta^{\beta}_{i} V^{t}_{tk} + \frac{1}{n-1} \delta^{\beta}_{k} V^{t}_{it} = 0,$$

$$V^{\beta}_{ik} = \delta^{\beta}_{i} \left(\frac{1}{n-1} V^{t}_{tk}\right) + \delta^{\beta}_{k} \left(\frac{1}{n-1} V^{t}_{it}\right). \tag{2.12}$$

In the above formula, let $\beta = i$; again, adding from 1 to n we obtain

$$V_k = V^t_{tk} = \frac{n}{n-1} V^t_{tk} + \frac{1}{n-1} V^t_{kt} = \frac{n}{n-1} V_k + \frac{1}{n-1} V^t_{kt},$$

$$V_k = -V^t_{kt} = V^t_{tk}.$$

Substituting the above formula into (2.12) we obtain (note $\Omega_k = \frac{1}{n-1} V_k$)

$$V_{ik} = \delta^\beta_i \Omega_k - \delta^\beta_i \Omega_i.$$

From theorem 2, it follows that $\Omega = \Omega_j dx^j$ is a closed differential form; namely T_V must be T_Λ.

Moreover, we have the following

THEOREM 6. The transformation T_V that makes scalar curvature $S = g^{ik} S_{ik}$ of arbitrary U invariant must satisfy the following system of equations

$$g^{\alpha k}(V^\beta_{\gamma k} - \frac{1}{n-1} \delta^\beta_\gamma V_k) + g^{i\beta}(V^\alpha_{i\gamma} + \frac{1}{n-1} \delta^\alpha_\gamma V_i) = 0, \tag{2.13}$$

where $V_k = V^t_{tk} = -V^t_{kt}$.

PROOF. From (2.7), we have

$$\overline{S} = g^{ik}\overline{S}_{ik} = g^{ik}\overline{U}^s_{ik,s} - g^{ik}\overline{U}^s_{it}\overline{U}^t_{sk} + \frac{1}{n-1} g^{ik}\overline{U}^s_{is}\overline{U}^t_{tk}$$

$$= g^{ik} (U^s_{ik,s} + V^s_{ik,s}) - g^{ik} (U^s_{it} + V^s_{it}) (U^t_{sk} + V^t_{sk})$$

$$+ \frac{1}{n-1} g^{ik} (U^s_{is} + V^s_{is}) (U^t_{tk} + V^t_{tk}) = S + F,$$

where,

$$F = g^{ik}V^s_{ik,s} - g^{ik}U^s_{it} V^t_{sk} - g^{ik}V^s_{it} U^t_{sk} - g^{ik}V^s_{it} V^t_{sk}$$

$$+ \frac{1}{n-1} g^{ik}U^s_{is} V^t_{tk} + \frac{1}{n-1} g^{ik}V^s_{is}U^t_{tk} + \frac{1}{n-1} g^{ik} V^s_{is} V^t_{tk}.$$

From the condition of the theorem, $\overline{S} = S$, it follows that $F = 0$ (for arbitrary U). Therefore,

$$0 = \frac{\partial F}{\partial U^\gamma_{\alpha\beta}} = - g^{ik}\delta^\alpha_i V^\beta_{\gamma k} - g^{ik}V^\alpha_{i\gamma}\delta^\beta_k + \frac{1}{n-1} g^{ik}\delta^\beta_\gamma \delta^\alpha_i V^t_{tk}$$

$$+ \frac{1}{n-1} g^{ik} V^s_{is} \delta^\alpha_\gamma \delta^\beta_k = - g^{\alpha k}V^\beta_{\gamma k} - g^{i\beta} V^\alpha_{i\gamma} + \frac{1}{n-1} g^{\alpha k}\delta^\beta_\gamma V^t_{tk} + \frac{1}{n-1}g^{i\beta}\delta^\alpha_\gamma V^s_{is} \tag{2.14}$$

Multiply the above formula by $g_{\alpha\beta}$, adding from 1 to n for α and β to obtain

$$-g_{\alpha\beta} g^{\alpha k} V^\beta_{\gamma k} - g_{\alpha\beta} g^{i\beta} V^\alpha_{i\gamma} + \frac{1}{n-1} g_{\alpha\beta} g^{\alpha k}\delta^\beta_\gamma V^t_{tk} + \frac{1}{n-1} g_{\alpha\beta}g^{i\beta}V^s_{is} \delta^\alpha_\gamma = 0,$$

$$-V^k_{\gamma k} - V^\alpha_{\alpha\gamma} + \frac{1}{n-1} V^t_{t\gamma} + \frac{1}{n-1} V^s_{\gamma s} = 0.$$

From this we obtain,

$$V_\gamma = V^t_{t\gamma} = -V^t_{\gamma t}\ .$$

Substituting the above formula into (2.14), we have

$$g^{\alpha k}\,(V^\beta_{\gamma k} - \frac{1}{n-1}\,\delta^\beta_\gamma\, V_k) + g^{i\beta}\,(V^\alpha_{i\gamma} + \frac{1}{n-1}\,\delta^\alpha_\gamma\, V_i) = 0.$$

REMARK. Let $\Omega = \Omega_j dx^j$ be a 1-differential form. It is easy to prove that $V^\ell_{ik} = \delta^\ell_i\Omega_k - \delta^\ell_k\Omega_i$ is a second order differentiable covariant tensor field with vector value. By the following computation, we know it satisfies (2.13).

$$V^\beta_{\gamma k} - \frac{1}{n-1}\,\delta^\beta_\gamma V_k = (\delta^\beta_\gamma\,\Omega_k - \delta^\beta_k\Omega_\gamma) - \frac{1}{n-1}\,\delta^\beta_\gamma\,(\delta^t_t\,\Omega_k - \delta^t_k\,\Omega_t)$$

$$= \delta^\beta_\gamma\,\Omega_k - \delta^\beta_k\,\Omega_\gamma - \frac{n}{n-1}\,\delta^\beta_\gamma\,\Omega_k + \frac{1}{n-1}\,\delta^\beta_\gamma\,\Omega_k = -\delta^\beta_k\,\Omega_\gamma,$$

$$V^\alpha_{i\gamma} + \frac{1}{n-1}\,\delta^\alpha_\gamma\, V_i = (\delta^\alpha_i\,\Omega_\gamma - \delta^\alpha_\gamma\,\Omega_i) + \frac{1}{n-1}\,\delta^\alpha_\gamma\,(\delta^t_t\,\Omega_i - \delta^t_i\,\Omega_t)$$

$$= \delta^\alpha_i\,\Omega_\gamma - \delta^\alpha_\gamma\,\Omega_i + \frac{n}{n-1}\,\delta^\alpha_\gamma\,\Omega_i - \frac{1}{n-1}\,\delta^\alpha_\gamma\,\Omega_i = \delta^\alpha_i\,\Omega_\gamma\ ,$$

$$g^{\alpha k}\,(V^\beta_{\gamma k} - \frac{1}{n-1}\,\delta^\beta_\gamma\, V_k) + g^{i\beta}\,(V^\alpha_{i\gamma} + \frac{1}{n-1}\,\delta^\alpha_\gamma\, V_i)$$

$$= g^{\alpha k}\,(-\delta^\beta_k\Omega_\gamma) + g^{i\beta}(\delta^\alpha_i\,\Omega_\gamma) = -g^{\alpha\beta}\Omega_\gamma + g^{\alpha\beta}\,\Omega_\gamma = 0.$$

From the remark of theorem 3, it follows that although T_Ω satisfies (2.13), perhaps it does not make scalar curvature S invariant. Therefore, (2.13) is only a necessary condition under which the transformation T_V makes scalar curvature S invariant.

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