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ON MIXED FINITE ELEMENT TECHNIQUES FOR ELLIPTIC PROBLEMS

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ABSTRACT. The main aim of this paper is to consider the numerical approximation of mildly nonlinear elliptic problems by means of finite element methods of mixed type. The technique is based on an extended variational principle, in which the constraint of interelement continuity has been removed at the expense of introducing a Lagrange multiplier.

It is shown that the saddle point, which minimizes the energy functional over the product space, is characterized by the variational equations. The equivalence is used in deriving the error estimates for the finite element approximations. We give an example of a mildly nonlinear elliptic problem and show how the error estimates can be obtained from the general results.

KEY WORDS AND PHRASES. *Mixed finite element, Nonlinear Elliptic problems, Error analysis.*

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1. INTRODUCTION.

Recently much attention has been given to the numerical approximation of boundary value problems for elliptic equations by means of finite element methods of mixed type. The main motivation of this paper is to extend these methods for a class of mildly nonlinear elliptic problems. The mildly nonlinear problems considered in this paper are special cases of more general nonlinear problems in which the differential operator is monotone. However, their form enables the theory for the linear case to be exploited.

For simplicity, we consider the second order elliptic model problem:

$$\left.\begin{aligned} L\{u(\underline{x})\} &= f(\underline{x},u(\underline{x})), & \underline{x} \in \Omega \\ u(\underline{x}) &= 0, & \underline{x} \in \partial\Omega \end{aligned}\right\}, \tag{1.1}$$

where Ω is a bounded open subset of R^n with boundary $\partial\Omega$, $\bar{\Omega} \equiv \Omega \cup \partial\Omega$, its closure, L is a linear, self-adjoint, coercive elliptic operator, and f is a nonlinear given function of $\underline{x}$ and u on the space $L_2(\Omega)$.

The usual variational form of (1.1) (see Noor and Whiteman [1]) consists in finding $u \in H_0^1(\Omega)$, which minimizes the energy functional

$$I\{v\} = a(v,v) - 2\int_{\Omega}\int_{0}^{v} f(\underline{x},\eta)d\eta d\underline{x}$$

where the bilinear form $a(u,v)$ is associated with the operator L, and is in fact $\langle Lu,v\rangle$ after the integration by parts has been performed over the space $H_0^1(\Omega)$.

Standard finite element methods for numerically solving problems are based on this variational principle. Such methods have been extensively studied and convergence results are now classical, see for example Noor and Whiteman [1] and Noor [2].

In this paper, we shall use a more general approach in order to construct a finite element approximation of (1.1). It is based on an extended variational principle, in which the constraint of interelement continuity has been removed at the expense of introducing a lagrange multiplier. This type of methods have been introduced and analyzed by Brezzi [3], Raviart and Thomas [4], Falk and Osborn [5] and Fortin [6] for linear elliptic boundary value problems.

Section 2 contains the abstract theory for obtaining the general error estimates. In Section 3, we give an example of a mildly nonlinear elliptic problem and show how error estimates can be derived from the results in section 2. We also would like to remark that this approach can be applied to a wider class of mildly nonlinear elliptic problems having more complicated boundary conditions.

2. ABSTRACT RESULTS.

Let V, W, and H be three real Hilbert spaces with their duals V', W', and H' and norms $\|\cdot\|_V$, $\|\cdot\|_W$, and $\|\cdot\|_H$ respectively. Let $a(.,.)$ and $b(.,.)$ be continuous bilinear forms on $V\times V$ and $V\times W$ respectively and F be a continuous functional on V and $g \in W'$. We denote by $\langle .,.\rangle$, the pairing between V and V'

or W and W'.

We now consider the functional

$$I\{v,\phi\} = a(v,v) + 2b(v,\phi) - 2F(v) - 2\langle g,\phi\rangle. \tag{2.1}$$

If $a(u,v)$ is a positive symmetric bilinear form, then the problem of finding the saddle point of $I\{v,\phi\}$ on $V\times W$ for given $F\varepsilon V'$, $g\varepsilon W'$ (see Brezzi [3]) is equivalent to finding $(u,\phi)\varepsilon V\times W$ such that

$$\left.\begin{aligned} a(u,v) + b(v,\psi) &= \langle F,v\rangle, &&\text{for all } v\varepsilon V \\ b(u,\phi) &= \langle g,\phi\rangle, &&\text{for all } \phi\varepsilon W. \end{aligned}\right\} \tag{2.2}$$

We now study those conditions for a <u>Fréchet</u> <u>differentiable</u> <u>nonlinear</u> functional F on V under which the minimum of $I\{v,\phi\}$ defined by (2.1) can be characterized by the variational equations. In this case, we now state and prove the following result.

THEOREM 1. Let F', the Fréchet differential of a nonlinear continuous functional F on V, be antimonotone, $a(.,.)$ and $b(.,.)$ be continuous bilinear forms on $V\times V$ and $V\times W$ respectively. If $a(u,v)$ is a positive symmetric bilinear form on $V\times V$, then, for given $g\varepsilon W'$, $(u,\psi)\varepsilon V\times W$ minimizes the functional $I\{v,\phi\}$, if and only if $(u,\psi)\varepsilon V\times W$ satisfies the variational equations

$$a(u,v) + b(v,\psi) = \langle F'(u),v\rangle, \qquad \text{for all } v\varepsilon V \tag{2.3}$$

$$b(u,\phi) = \langle g,\phi\rangle \qquad \text{for all } \phi\varepsilon W. \tag{2.4}$$

PROOF. Let $(u,\psi)\varepsilon V\times W$ minimize the functional $I\{v,\phi\}$; then, for all $t\varepsilon R$, and $w\varepsilon V$, $\eta\varepsilon W$,

$$I\{u,\psi\} \le I\{u+tw,\psi+t\eta\}.$$

Hence, from (2.1), it follows that

$$a(u,w)+b(w,\psi)+b(u,\eta) \le \frac{F(u+tw)-F(u)}{t} + \langle g,\eta\rangle - \frac{1}{2}ta(w,w) - \frac{1}{2}tb(w,\eta).$$

Since F is Fréchet differentiable, as $t \longrightarrow 0$, we obtain

$$a(u,w) + b(w,\psi) = \langle F'(u),w\rangle, \qquad \text{for all } w \varepsilon V$$

$$b(u,\eta) = \langle g,\eta\rangle, \qquad \text{for all } \eta \varepsilon W,$$

which are (2.3) and (2.4), the required equations.

For the converse, if $(u,\psi)\varepsilon V\times W$ satisfies (2.3) and (2.4), then, using (2.1) and the positivity and symmetry of $a(u,v)$, we obtain

$$\begin{aligned} I\{u,\psi\} - I\{v,\phi\} &= a(u,u) + 2b(u,\psi) - 2F(u) - 2\langle g,\psi\rangle - a(v,v) \\ &\quad - 2b(v,\phi) + 2F(v) + 2\langle g,\phi\rangle \\ &\leq 2a(u,u-v) - 2F(u) + 2F(v) + 2\langle g,\phi-\psi\rangle \\ &\quad + 2b(u,\psi) - 2b(v,\phi) \\ &= 2\{\langle F'(u),u-v\rangle - F(u) + F(v)\} + 2b(u,\psi) \\ &\quad - 2b(v,\phi) - 2b(u-v,\psi) + 2\langle g,\phi-\psi\rangle, \text{ by } (2.3). \end{aligned}$$

Since F' is antimonotone, so F is a concave functional, see Noor [2]. Thus it follows that

$$\langle F'(u),u-v\rangle - F(u) + F(v) \leq 0.$$

From (2.4) and the above inequality it follows that

$$I\{u,\psi\} - I\{v,\phi\} \leq 0 \qquad \text{for all } v\varepsilon V \text{ and } \phi\varepsilon W$$

REMARK 1. Note that for the case when F is a linear continuous functional, then the result of Theorem 1 is exactly the same as in [3].

REMARK 2. Since the bilinear form $a(u,v)$ is symmetric on $V\times V$, one can easily check that the solution (u,ψ) of (2.3) and (2.4) may be characterized as the unique saddle point of the quadratic functional $I\{v,\phi\}$ defined by (1.1) over the space $V\times W$, i.e., (u,ψ) satisfies

$$I\{u,\phi\} \leq I\{u,\psi\} \leq I\{v,\psi\}, \qquad \text{for all } v\varepsilon V \text{ and } \phi\varepsilon W.$$

Thus it follows that u is the unique solution of the constrained minimization problem

$$I\{u\} = \inf_{v\varepsilon Z_g} I\{v\},$$

where $I\{v\} = a(v,v) - 2F(v)$,

$$Z_g - \{v\varepsilon V; \text{ for all } \mu\varepsilon W,\ b(v,\mu) = \langle g,\mu\rangle\},$$

while ψ is the Lagrange multiplier associated with the constraint $u\varepsilon Z_g$.

Theorem 1 shows that the variational problem (2.1) and the weak formulations (2.3) and (2.4) are equivalent. We use the weak formulation for deriving the error estimates for the finite element method of the mixed type.

In many applications, we are led to the problems (2.3) and (2 4) by the following procedure. Let V_o and V be real Hilbert spaces, with V_o a closed subspace of V, and let $a(u,v)$ be a continuous coercive bilinear form on $V\times V$. We want to solve the problem:

$$\left.\begin{array}{l}\text{Find } u\varepsilon V_o \text{ such that} \\ a(u,v) = \langle F'(u),v\rangle, \quad \text{for all } v\varepsilon V_o,\end{array}\right\} \tag{2.5}$$

where $F'(u)\varepsilon V'$ is the Fréchet differential of the nonlinear functional $F(u) = \int_\Omega\int_0^u f(\eta)d\eta d\Omega$ at u. For this, we consider the space $W = V_o^o$ (the polar space of V^o), which is a closed subspace of V'. Thus in this setting, problem (2.5) is equivalent to:

$$\left.\begin{array}{ll}\text{Find } (u,\psi)\varepsilon V\times W \text{ such that} & \\ a(u,v) + \langle v,\psi\rangle = \langle F'(u),v\rangle , & \text{for all } v\varepsilon V, \\ \langle u,\phi\rangle = 0, & \text{for all } \phi\varepsilon W.\end{array}\right\} \tag{2.6}$$

Taking

$$b(v,\phi) = \langle \phi,v\rangle , \quad \text{for all } v\varepsilon V, \quad \phi\varepsilon W \subseteq H' ,$$

it is obvious that (2.6) is of the form (2.3) and (2.4).

We consider the following problem, which we refer to as problem P:

Find $(u,\psi)\ \varepsilon\ V\times W$ satisfying

$$a(u,v) + b(v,\psi) = \langle F'(u),v\rangle, \quad \text{for all } v\varepsilon V \tag{2.7}$$

$$b(u,\phi) = \langle g,\phi\rangle \quad \text{for all } \phi\varepsilon W . \tag{2.8}$$

We consider this problem for a subclass of data, i.e., for $(F'(u),g)\varepsilon D$, where D is a subclass of $V'\times W'$. We assume that;

H1. For $(F'(u),g)\varepsilon D$, P has a unique solution.

In the analysis of P, we shall also consider the adjoint problem:

Given $d\varepsilon G'$, where G is a Hilbert space satisfying $W \leq G$ with a continuous embedding, find $(y,\lambda) = (y_d,\lambda_d)\varepsilon V\times W$ satisfying

$$a(v,y) + b(v,\lambda) = 0 , \quad \text{for all } v\varepsilon V \tag{2.9}$$

$$v(y,\phi) = \langle d,\phi\rangle, \quad \text{for all } \phi\varepsilon W \tag{2.10}$$

We shall assume that:

H2. Problem (2.9) and (2.10) has a unique solution for $d\varepsilon G'$.

Let V_h and W_h be given finite dimensional subspaces of V and W respectively. We now consider the following approximate problem P_h:

$$\text{Find } (u_h,\psi_h)\varepsilon V_h\times W_h \text{ such that}$$

$$a(u_h,v_h) + b(v_h,\psi_h) = \langle F'(u_h),v_h\rangle, \quad \text{for all } v_h\varepsilon V_h, \quad (2.11)$$

$$b(u_h,\phi_h) = \langle g,\phi_h\rangle , \quad \text{for all } \phi_h\varepsilon W_h . \quad (2.12)$$

We are interested in obtaining error estimates for $u - u_h$ and $\psi - \psi_h$. We now state several assumptions which are required in the proof of our main results.

H3. There is a constant $\alpha > 0$ such that

$$a(v_h,v_h) \geq \alpha\|v_h\|^2 \qquad \text{for all } v_h\varepsilon Z_h ,$$

where $$Z_h = \{v_h\varepsilon V_h\colon b(v_h,\phi_h) = 0 \text{ for all } {}_h\varepsilon W_h\}.$$

H4. There exists a number $S(h)$ satisfying

$$\|v_h\|_V \leq S(h)\|v_h\|_H, \quad \text{for all } v_h\varepsilon V_h .$$

H5. There is an operator $\P_h\colon Y\longrightarrow V_h$ satisfying $b(y-\P_h y,\phi_h) = 0$, for all $y\varepsilon Y$ and $\phi_h\varepsilon W_h$, where $Y = \text{span}(\{y_d\}_{d\varepsilon G'},u)$, (u,ψ) is the solution of P and (y_d,λ_d) is the solution of (2.9) and (2.10) corresponding to $d\varepsilon G'$.

H6. $F'(u)$ is antimonotone on V, i.e., $\langle F'(u) - F'(v),u-v\rangle \leq 0$, for all $u,v\varepsilon V$.

H7. $F'(u)$ is required to satisfy the Lipschitz condition, that is there exists a constant $\gamma > 0$ such that

$$\|F'(u) - F'(v)\| \leq \gamma\|u-v\| , \quad \text{for all } u,v\varepsilon \ .$$

We would like to remark that hypothesis (H1)-(H5) are due to Falk and Osborn [5] and (H6)-(H7) are due to Noor [2].

From (2.7), we observe that

$$a(\P_h u,v_h) + b(v_h,\psi) = a(u,v_h) + b(v_h,\psi) + a(\P_h u-u,v_h)$$
$$= \langle F'(u),v_h\rangle + a(\P_h u-u,v_h), \quad \text{for all } v\varepsilon V_h. \quad (2.13)$$

Substracting (2.11) from (2.13), we get for all $v_h\varepsilon V_h$,

$$a(\P_h u-u_n,v_h) + b(v_h,\psi-\psi_h) = \langle F'(u) - F'(u_n),v_h\rangle + a(\P_h u-u,v_h). \quad (2.14)$$

In order to derive the error estimate for $u-u_h$, we define $\bar{u}_h \varepsilon V_h$ by the conditions

$$a(\P_h u-\bar{u}_h, vh) + b(v_h, \psi-\psi_h) = a(\P_h u-u, v_h), \quad \text{for all } v_h \in V_h \tag{2.15}$$

$$b(\P_h u-\bar{u}_h, \phi_h) = 0, \quad \text{for all } \phi_h \in W_h. \tag{2.16}$$

Furthermore, let a(.,.) and b(.,.) be continuous bilinear forms on H×H and V×W respectively, that is

$$a(u,v) \leq \alpha_1 ||u||_H ||v||_H, \quad \text{for all } u, v \in H \tag{2.17}$$

$$b(u,\psi) \leq \beta_1 ||u||_V ||\psi||_W, \quad \text{for all } u \in V \text{ and } \psi \in W. \tag{2.18}$$

We now derive the main results of this section, which are the abstract error bounds for $u-u_h$ and $\psi-\psi_h$.

THEOREM 2. Suppose that the hypothesis (H1)-(H7) hold. If (u,ψ) and (u_h,ψ_h) are solutions of problems P and P_h respectively, then

$$||u-u_h||_H \leq C_1 ||u-\P_h u||_H + C_2 ||\psi-\phi_h||_W, \text{ for all } \phi_h \in W_h \tag{2.19}$$

$$||u-u_h||_V \leq ||u-\P_h u||_V + C_3 ||\P_h u-u||_H + C_4 ||\psi-\phi_h||_W. \tag{2.20}$$

If in addition, $Z_h \subset Z = \{v \in V, b(v,\phi) = 0, \text{ for all } \phi \in W\}$, then

$$||u-u_h||_H \leq C_5 ||\P_h u-u||_H \tag{2.21}$$

and

$$||u-u_h||_V \leq ||u-\P_h u||_V + C_6 ||u - \P_h u||_H, \tag{2.21}$$

where C's are constants independent of u and ψ.

PROOF. From (2.8) and (H5), we see that

$$b(\P_h u, \phi_h) = b(u, \phi_h) = \langle g, \phi_h \rangle, \quad \text{for all } \phi_h \in W_h. \tag{2.23}$$

Subtracting (2.12) from (2.23), we obtain

$$b(\P_h u-u_h, \phi_h) = 0, \quad \text{for all } \phi_h \in W_h. \tag{2.24}$$

Now taking $v_h = \P_h u - \bar{u}_h$ in (2.15), we get

$$\begin{aligned} a(\P_h u-\bar{u}_h, \P_h u-\bar{u}_h) &= a(\P_h u-u, \P_h u-\bar{u}_h) + b(\P_h u-\bar{u}_h, \psi_h-\psi) \\ &= a(\P_h u-u, \P_h u-\bar{u}_h) + b(\P_h u-\bar{u}_h, \phi_h-\psi) \\ &\quad + b(\P_h u-\bar{u}_h, \psi_h-\phi_h) . \end{aligned} \tag{2.25}$$

Thus from (2.16) and (2.25), we have

$$a(\P_h u-\bar{u}_h, \P_h u-\bar{u}_h) = a(\P_h u-u, \P_h u-\bar{u}_h) + b(\bar{u}_h - \P_h u, \psi-\phi_h),$$

from which, using (H3), (H5), (2.17) and (2.18), it follows that

$$\begin{aligned} \alpha ||\P_h u-\bar{u}_h||_H^2 &\leq a_1 ||\P_h u-u||_H ||\P_h u-\bar{u}_h||_H + \beta_1 ||\P_h u-\bar{u}_h||_V ||\psi-\phi_h||_W \\ &\leq \alpha_1 ||\P_h u-u||_H ||\P_h u-\bar{u}_h||_H + \beta_1 S(h) ||\P_h u-\bar{u}_h||_H ||\psi-\phi_h||_W . \end{aligned}$$

Hence, we have

$$||\P_h u-u|| \le \frac{1}{\alpha}\{\alpha_1||\P_h u-u||_H + \beta_1 S(h)||\psi-\phi_h||W, \text{ for all } \phi_h \in W_h. \tag{2.26}$$

Consider now

$$\begin{aligned} a(\bar{u}_h-u_h,\bar{u}_h-u_h) &\le a(\bar{u}_h-u_h,\bar{u}_h-u_h) - \langle F'(\bar{u})-F'(u_h),\bar{u}_h-u_h\rangle, \text{ using (H6).} \\ &= a(\bar{u}_h-\P_h u,\bar{u}_h-u_h) + a(\P_h u-u_h,\bar{u}_h-u_h) \\ &\quad -\langle F'(\bar{u}_h)-F'(u),\bar{u}_h-u_h\rangle - \langle F'(u)-F'(u_h),\bar{u}_h-u_h\rangle. \end{aligned}$$

Taking $v_h = \bar{u}_h-u_h$ in (2.15), we have

$$a(\bar{u}_h-\P_h u,\bar{u}_h-u_h) = b(\bar{u}_h-u_h,\psi-\psi_h) - a(\P_h u-u,\bar{u}_h-u_h).$$

Thus

$$\begin{aligned} a(\bar{u}_h-u_h,\bar{u}_h-u_h) &\le b(\bar{u}_h-u_h,\psi-\psi_h) - a(\P_h u-u,\bar{u}_h-u_h) + a(\P_h u-u_h,\bar{u}_h-u_h) \\ &\quad - \langle F'(u)-F'(u_h),\bar{u}_h-u_h\rangle + \langle F'(u)-F'(\bar{u}_h),\bar{u}_h-u_h\rangle. \end{aligned} \tag{2.27}$$

Using (2.14), we have

$$\begin{aligned} a(\P_h u-u_h,\bar{u}_h-u_h) + b(\bar{u}_h-u_h,\psi-\psi_h)-\langle F'(u)-F'(u_h),\bar{u}_h-u_h\rangle \\ -a(\P_h u-u,\bar{u}_h-u_h) = 0. \end{aligned} \tag{2.28}$$

From (2.27), (2.28) and (H7), it follows that

$$\begin{aligned} a(\bar{u}_h-u_h,\bar{u}_h-u_h) &\le \langle F'(u)-F'(\bar{u}_h),\bar{u}_h-u_h\rangle \\ &\le \gamma||u-\bar{u}_h||_H||\bar{u}_h-u_h||_H, \end{aligned}$$

from which it follows that

$$||\bar{u}_h-u_h||_H \le \frac{\gamma}{\alpha}||u-\bar{u}_h||_H \le \gamma/\alpha\{||u-\P_h u||_H + ||\P_h u-\bar{u}_h||_H\}. \tag{2.29}$$

Now

$$\begin{aligned} ||u-u_h||_H &\le ||u-\P_h u||_H + ||\P_h u-\bar{u}_h||_H + ||\bar{u}_h-u_h||_H \\ &\le (1+\tfrac{\gamma}{\alpha})||u-\P_h u||_H + (1+\tfrac{\gamma}{\alpha})||\P_h u-\bar{u}_h||_H, \text{ by (2.29)} \\ &\le (\tfrac{\alpha+\gamma}{\alpha})(\tfrac{\alpha+\alpha_1}{\alpha})||u-\P_h u||_H + \frac{\beta_1(\alpha+\gamma)}{\alpha^2} S(h)||\psi-\phi_h||_W, \text{ by (2.26)} \\ &= C_1||u-\P_h u||_H + C_2||\psi-\phi_h||_W, \end{aligned}$$

which is the required (2.20) with $C_1 = (\alpha+\gamma)(\alpha+\alpha_1)/\alpha^2$, and $C_2 = S(h)(\alpha+\gamma)\beta_1/\alpha^2$ are constants .

In order to prove (2.19), we first note that

$$\begin{aligned} ||u-u_h||_V &\le ||u-\P_h u||_V + ||\P_h u-u_h||_V \\ &\le ||u-\P_h u||_V + S(h)||\P_h u-u_h||_H, \text{ by (H4).} \end{aligned} \tag{2.30}$$

But

$$\begin{aligned}||\P_h u-u_h||_H &\le ||\P_h u-\bar{u}_h||_H + ||\bar{u}_h-u_h||_H \\ &\le (\alpha+\gamma)/\alpha\{||\P_h u-\bar{u}_h||_H\} + \frac{\gamma}{\alpha}||u-\P_h u||_H, \text{ by (2.29)} \\ &\le \{\frac{\gamma}{\alpha} + \frac{\alpha+\gamma}{\alpha^2}\alpha_1\}||u-\P_h u||_H + (\alpha+\gamma)\beta_1 S(h)/\alpha^2||\psi-\phi_h||_W,\end{aligned}$$

by (2.26), (2.31)

From (2.30) and (2.31), we obtain (2.20), the required result with $C_3 = S(h)\{\gamma/\alpha + (\alpha+\gamma)\alpha_1/\alpha^2$ and $C_4 = \frac{\alpha+\gamma}{\alpha^2}\beta_1\{S(h)\}^2$.

To prove (2.21), we observe that (2.24) and (2.16) together with $Z_h \subset Z$ implies that

$$h(\P_h u-u_h,\phi) = 0, \qquad \text{for all } \phi \in W \tag{2.32}$$

and

$$a(\P_h u-\bar{u}_h, \P_h u-\bar{u}_h) = a(\P_h u-u,\P_h u-\bar{u}_h). \tag{2.33}$$

Hence from (H3) and (2.33), we get

$$||\P_h u-\bar{u}_h||_H \le \frac{\alpha_1}{\alpha}||\P_h u-u_h||_H \tag{2.34}$$

Now from (2.34), (2.29) and the triangle inequality, we obtain

$$||u-u_h||_H \le \{(\alpha+\gamma)/\alpha\}\{(\alpha+\alpha_1)/\alpha\}||\P_h u-u||_H,$$

which is (2.21) with $C_5 = (\frac{\alpha+\gamma}{\alpha})\ (\frac{\alpha+\alpha_1}{\alpha})$.

To establish (2.22), we note that

$$\begin{aligned}||u-u_h||_V &\le ||u-\P_h u||_V + ||\P_h u-u_h||_V \\ &\le ||u-\P_h u||_V + S(h)||\P_h u-u_h||_V, \text{ by (H4)}. \\ &\le ||u-\P_h u||_V + S(h)\{\frac{\gamma}{\alpha} + \frac{(\alpha+\gamma)}{\alpha^2}\alpha_1\}||u-\P_h u||_H,\end{aligned}$$

by (2.34) and (2.30) which is the required (2.22).

REMARK 3. If F is independent of u, that is F(u) = f (say), then the Lipschitz constant γ is zero. Consequently theorem 2 is exactly the same as one proved by Falk and Osborn [5] for linear elliptic problems. It is obvious that our results include their results as special cases.

THEOREM 3. Assume that (H1), (H2), (H3) and (H5) hold and that (υ,ψ) and (υ_h,ψ_h) are solutions of problems (P) and (P_h), then

$$\begin{aligned}||\psi-\psi_h||_G = \sup_{d\in G'}\{&b(y_d-\P_h y_d,\psi-\phi_h)+a(u_h-u,\P_h y_d-y_d)+b(u-u_h,\lambda_d-\eta) \\ &+ \langle F'(u)-F'(u_h),\vec{u}_d\rangle\}/||d||_{G'}, \text{ for all } \phi_h,\eta\in W_h. \end{aligned} \tag{2.35}$$

PROOF. Substracting (2.11) from (2.7) and (2.12) from (2.8), we have

$$a(u-u_h,v_h)+b(v_h,\psi-\psi_h) = \langle F'(u)-F'(u_h),v_h\rangle, \text{ for all } v_h \varepsilon V_h. \tag{2.36}$$

and

$$b(u-u_h,\eta_h) = 0, \quad \text{for all } \eta_h \ \varepsilon\ W_h. \tag{2.37}$$

Combining (2.9), (H5), (2.36) and (2.37), we obtain

$$\begin{aligned} b(y_d,\psi-\psi_h) &= b(y_d-\P_h y_d,\psi-\psi_h) + b(\P_h y_d,\psi-\psi_h) \\ &= b(y_d-\P_h y_d,\psi-\phi_h)+b(y_d-\P_h y_d,\phi_h-\psi_h)+a(u_h-u,\P_h y_d) \\ &\qquad +\langle F'(u)-F'(u_h),\P_h y_d\rangle, \text{ by (37).} \\ &= b(y_d-\P_h y_d,\psi-\phi_h)+a(u_h-u,\P_h y_d)+\langle F'(u)-F'(u_h),\P_h y_d\rangle \\ &= b(y_d-\P_h y_d,\psi-\phi_h)+a(u_h-u,\P_h y_d-y_d)+a(u_h-u,y_d) \\ &\qquad + \langle F'(u)-F'(u_h),\P_h y_d\rangle \\ &= b(y_d-\P_h y_d,\psi-\phi_h)+a(u_h-u,\P_h y_d)+b(u-u_h,\lambda_d) \\ &\qquad +\langle F'(u)-F'(u_h),\P_h y_d\rangle . \end{aligned}$$

Thus by using (2.37), we get

$$\begin{aligned} b(y_d,\psi-\psi_h) &= b(y_d-\P_h y_d,\psi-\phi_h)+a(u_h-u,\P_h y_d-y_d)+b(u-u_h,\lambda_d-\eta) \\ &\qquad + \langle F'(u)-F'(u_h),\P_h y_d\rangle . \end{aligned} \tag{2.38}$$

Since from (2.10), we have

$$||\psi-\psi_h||_G = \sup_{d\varepsilon G'} \frac{\langle d,\psi-\psi_h\rangle}{||d||_{G'}} = \sup_{d\varepsilon G'} \frac{b(y_d,\psi-\psi_h)}{||d||_{G'}} . \tag{2.39}$$

Thus (2.35) follows from (2.38) and (2.39).

Note that for $F(u) = f$, (say), the term $\langle F'(u)-F'(u_h),\P_h y_d\rangle$ drops out, we have the same estimate for $\psi-\psi_h$ as proved in [5]. Theorem 2 and 3 allow us to consider the mildly nonlinear elliptic boundary value problems.

3. APPLICATIONS.

We consider the following problem:

$$\left.\begin{aligned} -\Delta u(\underline{x}) &= f(\underline{x},u(\underline{x})), & \underline{x} &\ \varepsilon\ \Omega \\ u(\underline{x}) &= 0, & \underline{x} &\ \varepsilon\ \partial\Omega, \end{aligned}\right\} \tag{3.1}$$

where Δ is the Laplacian operator, the given function $f\varepsilon C(\bar{\Omega})$ is Lipschitz continuous and antimonotone. Throughout, we shall use the classical Sobolev spaces, see [7] for notations and definitions.

Following Raviart and Thomas [4], let

$$\underline{H}(\text{div};\Omega) = \{v\varepsilon\{L_2(\Omega)\}^2 : \text{div } \underline{v}\varepsilon L_2(\Omega)\} \quad \text{with the norm}$$
$$||v||_{\underline{H}(\text{div};\Omega)} = \{||v||_o^2 + ||\text{div } v||_o^2\}^{\frac{1}{2}}.$$

In this setting, the energy functional associated with (3.1) can be written in the form:

$$I\{v,\varphi\} = \int_\Omega (v)^2 dx + \int_\Omega \phi \text{div } v \, dx - 2\int_\Omega\int_0^v f(\eta)d\eta dx. \tag{3.2}$$

It follows from theorem 1 that $(u,\psi)\varepsilon\underline{H}(\text{div};\Omega)xL_2(\Omega)$ minimizes the functional $I[v,\phi]$, defined by (3.2) if and only if $(u,\psi)\varepsilon\underline{H}(\text{div};\Omega)xL_2(\Omega)$ such that

$$\int_\Omega u.vdx + \int_\Omega \psi\text{div } vdx = \int_\Omega f(u).vdx, \text{ for all } v\varepsilon\underline{H}(\text{div};\Omega) \tag{3.3}$$

and

$$\int_\Omega \phi\text{div } u \, dx = 0, \quad \text{for all } \phi\varepsilon L_2(\Omega) . \tag{3.4}$$

One can easily see that (3.3) and (3.4) is an example of problem P with $V = \underline{H}(\text{div};\Omega)$, $W = L_2(\Omega)$, $H = \{L_2(\Omega)\}^2$, $a(u,v) = \int_\Omega u.vdx$, $b(u,\psi) = \int_\Omega \psi\text{div } udx$, $g = 0$, $\langle F'(u),v\rangle = \int_\Omega f(u)vdx$, see [1,2].

The subclass D of data for which (H1) is satisfied is given by D = OxW'. Since the bilinear form a(u,v) is symmetric, it can be easily shown that H(2) and (H3) are satisfied where $\alpha = 1$.

Let $\{T_h\}_{h>0}$ be a regular family (see[7]) of triangulations of Ω and define

$$V_h = \{v_h\varepsilon\underline{H}(\text{div};\Omega), \text{ for all } T\varepsilon T_h, \ v_h|_T\varepsilon\underline{Q}_T\},$$

where

$$\underline{Q}_T = \{v\varepsilon\underline{H}(\text{div};T):\hat{v}\varepsilon\hat{\underline{Q}}\} \text{ , see [4,5].}$$

and
$$W_h = \{\phi_h\varepsilon L_2(\Omega), \text{ for all } T\varepsilon T_h, \ \phi_h|_T\varepsilon P_k\},$$

where P_k is a polynomial of degree k.

To apply the results of section 2, we must verify that the appropriate hypothesis are satisfied. Actually (H4) and (H5) are shown in [5] to hold in this case. Furthermore, it has been shown that for $v\varepsilon\{H^{r-1}(\Omega)\}^2$, $r\geq 2$, the following results hold.

$$||v-\P_h v||_0 \leq Ch^\ell ||v||_\ell, \ 1 \leq \ell \leq \min(r-1,k+1) \tag{3.5}$$

and

$$||\text{div}(v-\P_h v)||_0 \leq Ch^m||\text{div } v||_m, \quad 0\leq m\leq \min(r-2,k+1). \tag{3.6}$$

From the definitions of V_h and W_h, it follows that for all $v_h\varepsilon V_h$, div $v_h|_T\varepsilon P_k$. Thus $v_h\varepsilon Z_h$ implies that div $v_h = 0$ and so $v_h \varepsilon Z$. Thus $Z_h \subset Z$ and so we are in special cases of theorem 2.

In order to apply our results, it remains to show that (H6) and (H7) are satisfied. Actually these have been proved in [1,2]. Thus from the above discussion we gather that all the hypotheses (H1) - (H7) are satisfied for our model problem (3.1) and so we can apply the results of theorem 2 and 3.

Furthermore, we now derive the error estimates. Assuming $\psi \in H^r(\Omega)$, $r \geq 2$, from (2.21) and (3.5), we obtain

$$||u-u_h||_0 \leq C||u - \Pi_h u||_0 , \quad \text{for } k \geq 0.$$
$$\leq Ch^t||u||_t \leq Ch^t||\psi||_{t+1}, \text{ if } u = \text{grad}\psi, \qquad (3.7)$$

where $t = \min(r-1,k+1)$.

Again using the technique and results of Falk and Osborn [5], we can show that in our case the following estimates are true.

$$||\psi-\psi_h||_0 \leq h^\mu||\psi||_\mu \quad \mu = \min(r,k+1)$$

and

$$||\psi-\psi_h||_0 \leq \quad ||\psi||_2, \quad k=2.$$

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