Internat. J. Math. & Math. Sci.
Vol. 6 No. 3 (1983) 545-550

FIBONACCI POLYNOMIALS OF ORDER K, MULTINOMIAL EXPANSIONS AND PROBABILITY

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(Received October 31, 1982)

ABSTRACT. The Fibonacci polynomials of order k are introduced and two expansions of them are obtained, in terms of the multinomial and binomial coefficients, respectively. A relation between them and probability is also established. The present work generalizes results of [2] - [4] and [5].

KEY WORDS AND PHRASES. *Fibonacci polynomials of order k, expansions, multinonial and binomial coefficients, probability.*
1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. 10A40.

1. INTRODUCTION.

In the sequel, k is a fixed integer greater than or equal to 2, x is a positive and finite real number, and n is a nonnegative integer unless otherwise specified. Motivated introduced the Fibonacci polynomials of order k, to be denoted by $f_n^{(k)}(x)$, and study some of their properties. First we observe that $f_n^{(k)}(x)$ are generalized polynomials, appropriate extensions for the Fibonacci and Pell numbers of order k [3], [4], and identical to the r-bonacci polynomials $R_n(x)(n\geq-(r-2))$ of [1] for $k=r$ and $n\geq0$. Then we state and prove a theorem, which provides two expansions of $f_n^{(k)}(x)$ $(n\geq1)$ in terms of the multinomial and binomial coefficients, respectively. Hoggatt and Bicknell [1], amoung other results, give another expansion of $f_n^{(k)}(x)$, in terms of the elements of the left - justified k-nomial triangle. The latter, however, are less widely known and used than the multinomial and binomial coefficients, and on this account our expansions may be considered better. As a corollary to our theorem, we derive several results of [2]-[4] and [5]. We also obtain a relation between $f_n^{(k)}(x)$ $(n\geq 1)$ and probability.

2. THE FIBONACCI POLYNOMIALS OF ORDER K AND MULTINOMIAL COEFFICIENTS.

In this section, we introduce the Fibonacci polynomials of order k and derive two expansions of them im terms of the multinomial and binomial coefficients, respectively. The proof is along the lines of [2] and [4].

DEFINITION. The sequence of polynomials $\{f_n^{(k)}(x)\}_{n=0}^{\infty}$ is said to be the sequel of Fibonacci polynomials of order k if $f_0^{(k)}(x)=0$, $f_1^{(k)}(x)=1$, and

$$f_n^{(k)}(x) = \begin{cases} \sum_{i=1}^{n} x^{k-i} f_{n-i}^{(k)}(x) & \text{if } 2 \leq n \leq k \\ \sum_{i=1}^{k} x^{k-i} f_{n-i}^{(k)}(x) & \text{if } n \geq k+1. \end{cases} \tag{2.1}$$

If $f_n^{(r)}(x)=0$ for $-(r-2)\leq n\leq -1$, Hoggatt and Bicknell [1] call $R_n(x)=f_n^{(r)}(x)$ $(n\geq -(r-2))$ r-bonacci polynomials.

Denoting by $F_n(x)$, $f_n^{(k)}$ and $P_n^{(k)}$, respectively, the Fibonacci polynomials [5], the Fibonacci numbers of order k [3], and the Pell numbers of order k [4], it follows from (2.1) that

$$f_n^{(2)}(x) = F_n(x), \quad f_n^{(k)}(1) = f_n^{(k)} \text{ and } F_n^{(k)}(2) = P_n^{(k)}. \tag{2.2}$$

We now proceed to show the following lemma.

LEMMA. Let $\{f_n^{(k)}(x)\}_{n=0}^{\infty}$ be the sequence of Fibonacci polynomials of order k, and denote its generating function by $g_k(s;x)$. Then, for $|s|<x/(1+x^k)$,

$$g_k(s;x) = \frac{s(1-\frac{s}{x})}{1-\frac{s}{x}(1+x^k-s^k)} = \frac{s}{1-x^k[\frac{s}{x}+(\frac{s}{x})^2+\ldots+(\frac{s}{x})^k]}.$$

PROOF. We see from the definition that $f_2^{(k)}(x)=x^{k-1}$, $xf_n^{(k)}(X)-f_{n-1}^{(k)}(x)=x^k f_{n-1}^{(k)}(x)$ for $3\leq n\leq k+1$, and $xf_n^{(k)}(x)-f_{n-1}^{(k)}(x)=x^k f_{n-1}^{(k)}(x)-f_{n-1-k}^{(k)}(x)$ for $n\geq k+2$. Therefore,

$$f_n^{(k)}(x) = \begin{cases} \frac{1}{x}(1 + x^k) f_{n-1}^{(k)}(x), & 3 \le n \le k+1 \\ \frac{1}{x}(1 + x^k) f_{n-1}^{(k)}(x) - \frac{1}{x} f_{n-1-k}^{(k)}(x), & n \ge k+2 \end{cases}$$

$$= \begin{cases} [\frac{1}{x}(1 + x^k)]^{n-2} x^{k-1}, & 2 \le n \le k+1 \\ \frac{1}{x}(1 + x^k) f_{n-1}^{(k)}(x) - \frac{1}{x} f_{n-1-k}^{(k)}(x), & n \ge k+2. \end{cases} \tag{2.3}$$

It may be seen, by means of (2.3) and induction on n, that

$$f_n^{(k)}(x) \le [\frac{1}{x}(1+x^k)]^{n-2} x^{k-1}, \quad n\ge 2, \tag{2.4}$$

which implies the convergence of $g_k(s;x)$ for $|s|<x/(1+x^k)$. Next, by means of (2.3), we observe that

$$g_k(s;x) = \sum_{n=0}^{\infty} s^n f_n^{(k)}(x)$$

$$= s + \sum_{n=2}^{k+1} s^n [\frac{1}{x}(1 + x^k)]^{n-2} x^{k-1} + \sum_{n=k+2}^{\infty} s^n f_n^{(k)}(x), \tag{2.5}$$

and

$$\sum_{n=k+2}^{\infty} s^n f_n^{(k)}(x) = \frac{1}{x}(1+x^k) \sum_{n=k+2}^{\infty} s^n f_{n-1}^{(k)}(x) - \frac{1}{x} \sum_{n=k+2}^{\infty} s^n f_{n-1-k}^{(k)}(x)$$

$$= \frac{s}{x}(1+x^k) \left\{ \sum_{n=0}^{\infty} s^n f_n^{(k)}(x) - s - \sum_{n=2}^{\infty} s^n [\frac{1}{x}(1+x^k)]^{n-2} x^{k-1} \right\} - \frac{1}{x} s^{k+1} \sum_{n=1}^{\infty} s^n f_n^{(k)}(x)$$

$$= [\frac{s}{x}(1+x^k) - \frac{s^{k+1}}{x}] g_k(s;x) - \frac{s^2}{x} - \sum_{n=2}^{k+1} s^n [\frac{1}{x}(1+x^k)]^{n-2} x^{k-1}. \tag{2.6}$$

The last two relations give

$$g_k(s;x) = s + \frac{s}{x}(1 + x^k - s^k) g_k(s;x) - \frac{s^2}{x},$$

so that

$$g_k(s;x) = \frac{s(1-\frac{s}{x})}{1-\frac{s}{x}(1+x^k-s^k)} = \frac{s}{1-x^k[\frac{s}{x}+(\frac{s}{x})^2+\ldots+(\frac{s}{x})^k]}.$$

We will employ the above lemma to establish the following expansions of $f_n^{(k)}(x)$ $(n\ge 1)$.

THEOREM. Let $\{f_n^{(k)}(x)\}_{n=0}^{\infty}$ be the Fibonacci polynomials of order k. Then

(a) $$f_{n+1}^{(k)}(x) = \sum_{n_1,\ldots,n_k} \binom{n_1+\ldots+n_k}{n_1,\ldots,n_k} x^{k(n_1+\ldots+n_k)-n}, \quad n\ge 0,$$

where the summation is over all non-negative integers $n_1,\dots,n_k$ such that $n_1+2n_2+\dots+kn_k=n$;

$$\text{(b)}\quad f_{n+1}^{(k)}(x) = \left(\frac{1+x^k}{x}\right)^n \sum_{i=0}^{[n/(k+1)]} (-1)^i \binom{n-ki}{i} x^{ki}(1+x^k)^{-(k+1)i}$$

$$- \frac{1}{x}\left(\frac{1+x^k}{x}\right)^{n-1} \sum_{i=0}^{[(n-1)/k+1)]} (-1)^i \binom{n-1-ki}{i} x^{ki}(1+x^k)^{-(k+1)i},\ n\geq 1,$$

$$n\geq 1,$$

where, as usual, $[x]$ denotes the greatest interger in x.

PROOF. First we show (a). Let $|s| < x/(1+x^k)$, so that $|x^k[\frac{s}{x}+(\frac{s}{x})^2+\dots+(\frac{s}{x})^k]|<1$. Let $n_i\,(1\leq i\leq k)$ be non-negative integers as specified below. Then, using the lemma and the multinomial theorem, and replacing n by $n-\sum_{i=1}^{k}(i-1)n_i$, we get,

$$\sum_{n=0}^{\infty} s^n f_{n+1}^{(k)}(x) = \{1-x^k[\frac{s}{x} + (\frac{s}{x})^2+\dots+(\frac{s}{x})^k]\}^{-1}$$

$$= \sum_{n=0}^{\infty} \{x^k[\frac{s}{x} + (\frac{s}{x})^2+\dots+(\frac{s}{x})^k]\}^n$$

$$= \sum_{n=0}^{\infty} x^{kn} \sum_{\substack{n_1,\dots,n_k \ni \\ n_1+\dots+n_k=n}} \binom{n}{n_1,\dots,n_k} (\frac{s}{x})^{n_1+2n_2+\dots+kn_k}$$

$$= \sum_{n=0}^{\infty} s^n \sum_{\substack{n_1,\dots,n_k \ni \\ n_1+2n_2+\dots+kn_k=n}} \binom{n_1+\dots+n_k}{n_1,\dots,n_k} n^{k(n_1+\dots+n_k)-n}, \qquad (2.8)$$

from which (a) follows.

We now proceed to establish (b). Let $0 < s < x/(1 + x^k)$, so that $|\frac{s}{x}(1+x^k-s^k)| < 1$. Then, using the lemma and the binomial theorem, replacing n by n-ki, and setting

$$B_n^{(k)}(x) = (\frac{1+x^k}{x})^n \sum_{i=0}^{[n/(k+1)]} (-1)^i \binom{n-ki}{i} x^{ki}(1+x^k)^{-(k+1)i},\ n\geq 0, \qquad (2.9)$$

we get

$$\sum_{n=0}^{\infty} s^n f_{n+1}^{(k)}(x) = (1-\frac{s}{x})[1-\frac{s}{x}(1+x^k-s^k)]^{-1}$$

$$= (1-\frac{s}{x}) \sum_{n=0}^{\infty} [\frac{s}{x}(1+x^k-s^k)]^n$$

$$= (1-\frac{s}{x})^n \sum_{i=0}^{\infty} (\frac{s}{x})^n \sum_{i=0}^{\infty} (-1)^i \binom{n}{i} (1+x^k)^{n-i} s^{ki}$$

$$= (1-\frac{s}{x}) \sum_{n=0}^{\infty} s^n \sum_{i=0}^{[n/(k+1)]} (-1)^i \binom{n-ki}{i} (1+x^k)^{n-(k+1)i} x^{-(n-ki)}$$

$$= (1-\frac{s}{x}) \sum_{n=0}^{\infty} s^n B_n^{(k)}(x), \text{ by (2.9)}$$

$$= 1+ \sum_{n=1}^{\infty} s^n [B_n^{(k)}(x) - \frac{1}{x} B_{n-1}^{(k)}(x)], \tag{2.10}$$

since $B_0^{(k)}(x) = 1$ from (2.9). The last two relations show part (b) of the theorem.

We have the following obvious corollary to the theorem, by means of relation (2.2).

COROLLARY 2.1. Let $F_n(x)$, $f_n^{(k)}$ and $P_n^{(k)}$ denote the Fibonacci polynomials, the Fibonacci numbers of order k and the Pell numbers of order k, respectively. Then,

(a) $$F_{n+1}(x) = \sum_{i=0}^{[n/2]} \binom{n-i}{i} x^{n-2i}, \quad n\geq 0;$$

(b)(i) $$f_{n+1}^{(k)} = \sum_{\substack{n_1,\ldots,n_k \ni \\ n_1+2n_2+\ldots+kn_k=n}} \binom{n_1+\ldots+n_k}{n_1,\ldots,n_k}, \quad n\geq 0;$$

(b)(ii) $$f_{n+1}^{(k)} = 2^n \sum_{i=0}^{[n/(k+1)]} (-1)^i \binom{n-ki}{i} 2^{-(k+1)i}$$

$$-2^{n-1} \sum_{i=0}^{[(n-1)/(k+1)]} (-1)^i \binom{n-1-ki}{i} 2^{-(k+1)i}, \quad n\geq 1;$$

(c)(i) $$P_{n=1}^{(k)} = \sum_{\substack{n_1,\ldots,n_k \ni \\ n_1+2n_2+\ldots+kn_k=n}} \binom{n_1+\ldots+n_k}{n_1,\ldots,n_k} 2^{k(n_1+\ldots+n_k)-n}, \quad n\geq 0;$$

(c)(ii) $$P_{n+1}^{(k)} = (\frac{1+2^k}{2})^n \sum_{i=0}^{[n/(k+1)]} (-1)^i \binom{n-ki}{i} 2^{ki} (1+2^k)^{-(k+1)i}$$

$$-\frac{1}{2}(\frac{1+2^k}{2})^{n-1} \sum_{i=0}^{[(n-1)/(k+1)]} (-1)^i \binom{n-1-ki}{i} 2^{ki} (1+2^k)^{-(k+1)i}, \quad n\geq 1.$$

REMARK. Part (a) of Corollary 2.1 was proposed by Swamy [5], who appears to be the first to introduce the Fibonacci polynomials. Part (b)(i) was first shown in [3], while (b) and (c), respectively, were later proved by a different method in [2] and [4].

The following corollary relates the Fibonacci polynomials of order k to probability.

COROLLARY 2.2. Let $\{f_n^{(k)}(x)\}_{n=0}^{\infty}$ be the sequence of Fibonacci polynomials of order k, and denote by N_k the number of trials until the occurrence of the kth consecutive success in independent trials with success probability p $(0<p<1)$. Then,

$$P(N_k=n+k) = p^{n+k}\left(\frac{1-p}{p}\right)^{n/k} f_{n+1}^{(k)}\left(\left(\frac{1-p}{p}\right)^{1/k}\right), \quad n\geq 0.$$

PROOF. It follows directly from Theorem 3.1 of [3] and part (a) of the present theorem.

In particular, Corollary 2.2. reduces to the following results of [2] and [4], respectively, by means of (2.2).

Let N_k be as above, and set $p=(1+2^k)^{-1}$. Then,

$$P(N_k=n+k) = \frac{2^n}{(1+2^k)^{n+k}} P_{n+1}^{(k)}, \quad n\geq 0. \tag{2.11}$$

Let N_k be as above, and set $p=1/2$. Then,

$$P(N_k=n+k) = \frac{1}{2^{n+k}} f_{n+1}^{(k)}, \quad n\geq 0. \tag{2.12}$$

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