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# CONVOLUTIONS OF PRESTARLIKE FUNCTIONS

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ABSTRACT. The convolution of two functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as $(f*g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. For $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z/(1-z)^{2(1-\gamma)}$, the extremal function for the class of functions starlike of order $\gamma$, we investigate functions $h$, where $h(z) = (f*g)(z)$, which satisfy the inequality $|(zh'/h)-1|/|(zh'/h) + (1-2\alpha)| < \beta$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$, for all $z$ in the unit disk. Such functions $f$ are said to be $\gamma$-prestarlike of order $\alpha$ and type $\beta$. We characterize this family in terms of its coefficients, and then determine extreme points, distortion theorems, and radii of univalence, starlikeness, and convexity. All results are sharp.

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## 1. INTRODUCTION.

Let $S$ denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic and univalent in the unit disk $E = \{z : |z| < 1\}$. A function $f \in S$ is said to be starlike of order $\alpha$ and type $\beta$ if the inequality

$$|(zf'/f)-1|/|(zf'/f) + (1-2\alpha)| < \beta$$

holds for some $\alpha,\beta (0 \le \alpha < 1,\ 0 < \beta \le 1)$ and for all $z$ in $E$. The class of all such functions shall be denoted by $S^*(\alpha,\beta)$. Note that $S^*(\alpha,1) \equiv S^*(\alpha)$, the class of functions starlike of order $\alpha$, and that $S^*(0,\beta)$ is a subclass of starlike functions studied by Padmanabhan [1]. For $f \in S^*(\alpha,\beta)$, $0 < \beta < 1$, the values of $zf'/f$ lie in a disk centered at $(1 + (1-2\alpha)\beta^2)/(1-\beta^2)$ whose radius is $2\beta(1-\alpha)/(1-\beta^2)$.

The convolution or Hadamard product of two power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series $(f*g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. A function $f$, analytic in $E$ and normalized by $f(0) = f'(0)-1 = 0$, is said to be in the class of prestarlike functions introduced by Ruscheweyh [2] if $f*s_\gamma \in S^*(\gamma)$, where $s_\gamma(z) = z/(1-z)^{2(1-\gamma)}$ with $0 \le \gamma < 1$ is the well-known extremal function for the class $S^*(\gamma)$. We say that a normalized analytic function $f$ is $\gamma$-prestarlike of order $\alpha$ and type $\beta$ $(0 \le \alpha < 1,\ 0 < \beta \le 1)$, denoted $R_\gamma(\alpha,\beta)$, if $f*s_\gamma \in S^*(\alpha,\beta)$.

Our main interest will be with functions $f$ in $S^*(\alpha)$, $S^*(\alpha,\beta)$, or $R_\gamma(\alpha,\beta)$ that may be expressed as $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \ge 0$. We denote these classes, respectively, by $S^*[\alpha]$, $S^*[\alpha,\beta]$, and $R_\gamma[\alpha,\beta]$. The class $R_\alpha[\alpha,1) \equiv R[\alpha]$ was studied in [3] while the class $S^*[\alpha,\beta]$ was investigated in [4]. For $\gamma = 1/2$ and $\beta = 1$, the class reduces to the family $S^*[\alpha]$ studied in [5].

We begin with a characterization of the class $R_\gamma[\alpha,\beta]$, from which we determine the extreme points, distortion properties, and radii of univalence, starlikeness, and convexity.

## 2. COEFFICIENT INEQUALITIES.

In the sequel, we set

$$C(\gamma,n) = \prod_{k=2}^{n} (k-2\gamma)/(n-1)! \quad (n = 2,3,\dots), \tag{2.1}$$

so that $s_\gamma$ may be written in the form $s_\gamma(z) = z/(1-z)^{2(1-\gamma)} = z + \sum_{n=2}^{\infty} C(\gamma,n) z^n$.

Note that $C(\gamma,n)$ is a decreasing function of $\gamma$, $0 \le \gamma < 1$, with

$$\lim_{n\to\infty} C(\gamma,n) = \begin{cases} \infty, & \gamma<1/2 \\ 1, & \gamma=1/2 \\ 0, & \gamma>1/2 \end{cases}.$$

THEOREM 1. *A function* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$, *is in the class* $R_\gamma[\alpha,\beta]$ *if and only if*

$$\sum_{n=2}^{\infty} \frac{[(n-1)+\beta(n+1-2\alpha)]\,C(\gamma,n)a_n}{2\beta(1-\alpha)} \leq 1 . \tag{2.2}$$

PROOF. If $f \in R_\gamma[\alpha,\beta]$, then $g(z) = (f*s_\gamma)(z) = z - \sum_{n=2}^{\infty} C(\gamma,n)a_n z^n \in S^*[\alpha,\beta]$, so that

$$\frac{|(zg'/g)-1|}{|(zg'/g)+(1-2\alpha)|} = \left| \frac{\sum_{n=2}^{\infty} (n-1)C(\gamma,n)a_n z^{n-1}}{2(1-\alpha) - \sum_{n=2}^{\infty} (n+1-2\alpha)C(\gamma,n)a_n z^{n-1}} \right| < \beta \tag{2.3}$$

for all $z \in E$. Since the denominator in (2.3) is positive for small positive values of $z$ and, consequently, for all $z$, $0 < z < 1$, we let $z \to 1^-$ to obtain

$$\sum_{n=2}^{\infty} (n-1)C(\gamma,n)a_n \leq \beta[2(1-\alpha) - \sum_{n=2}^{\infty} (n+1-2\alpha)C(\gamma,n)a_n] ,$$

which is equivalent to (2.2).

Conversely, if (2.2) holds, we wish to show that $g = f*s_\gamma$ is in $S^*[\alpha,\beta]$. For $|z| = r < 1$, we have

$$\left|\frac{(zg'/g)-1}{(zg'/g)+(1-2\alpha)}\right| = \left| \frac{\sum_{n=2}^{\infty} (n-1)C(\gamma,n)a_n z^{n-1}}{2(1-\alpha) - \sum_{n=2}^{\infty} (n+1-2\alpha)C(\gamma,n)a_n z^{n-1}} \right|$$

$$\leq \frac{\sum_{n=2}^{\infty} (n-1)C(\gamma,n)a_n}{2(1-\alpha) - \sum_{n=2}^{\infty} (n+1-2\alpha)C(\gamma,n)a_n} .$$

The function $g$ is in $S^*[\alpha,\beta]$ if the last expression is $\leq \beta$, which is equivalent to (2.2). Hence, $f \in R_\gamma[\alpha,\beta]$ and the theorem is proved.

COROLLARY. *If* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in R_\gamma[\alpha,\beta]$, *then* $a_n \leq 2\beta(1-\alpha)/[(n-1) + \beta(n+1)-2\alpha)]C(\gamma,n)$ , $n \geq 2$, *with equality for functions of the form*

$$f_n(z) = z-2\beta(1-\alpha)z^n/[(n-1) + \beta(n+1-2\alpha)]C(\gamma,n) .$$

It follows from Theorem 1 that $R_\gamma[\alpha,\beta]$ is a closed, convex family. We shall now show that the extreme points of the closed convex hull are those that maximize the coefficients.

THEOREM 2. *Set*

$$f_1(z) = z \text{ and } f_n(z) = z-2\beta(1-\alpha)z^n/[(n-1)+\beta(n+1-2\alpha)]C(\gamma,n) , \tag{2.4}$$

$n = 2,3,\ldots$ . *Then* $f \in R_\gamma[\alpha,\beta]$, $0 \leq \alpha, \gamma < 1$, $0 < \beta \leq 1$, *if and only if it can be expressed as* $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$ , *where* $\lambda_n \geq 0$ *and* $\sum_{n=1}^{\infty} \lambda_n = 1$ .

PROOF. If $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$ , then

$$\sum_{n=2}^{\infty} \frac{[(n-1)+\beta(n+1-2\alpha)]C(\gamma,n)}{2\beta(1-\alpha)} \cdot \frac{\lambda_n(2\beta)(1-\alpha)}{[(n-1)+\beta(n+1-2\alpha)]C(\gamma,n)} = \sum_{n=2}^{\infty} \lambda_n = 1-\lambda_1 \leq 1$$

and $f \in R_\gamma[\alpha,\beta]$ .

Conversely, if $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in R_\gamma[\alpha,\beta]$, then set

$\lambda_n = [(n-1) + \beta(n+1-2\alpha)]C(\gamma,n)a_n/2\beta(1-\alpha)$ , $n = 2,3,\ldots,$ and set $\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n$ .

We see from Theorem 1 that $\lambda_1 \geq 0$ . Since $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$ , the proof is complete.

## 3. DISTORTION THEOREMS.

We may now find bounds on the modulus of $f$ and $f'$ for $f \in R_\gamma[\alpha,\beta]$ .

THEOREM 3. *If* $f \in R_\gamma[\alpha,\beta]$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$, *and either* $0 \leq \gamma \leq (2+3\beta-\alpha\beta)/(2+4\beta-2\alpha\beta)$ *or* $r \leq (1+2\beta-\alpha\beta)/(1+3\beta-2\alpha\beta)$, *then, for* $|z| \leq r$ ,

$\max\{0,r-\beta(1-\alpha)r^2/[(1+\beta(3-2\alpha)](1-\gamma)\} \leq |f(z)| \leq r+\beta(1-\alpha)r^2/[1+\beta(3-2\alpha)](1-\gamma)$ . *The bounds are sharp, with extremal function* $f_2(z) = z-\beta(1-\alpha)z^2/[1+\beta(3-2\alpha)](1-\gamma)$ .

$$\max\{0, r-\max_n \frac{2\beta(1-\alpha)r^n}{[(n-1)+\beta(n+1-2\alpha)]C(\gamma,n)}\} \le |f(z)| \le r+\max_n \frac{2\beta(1-\alpha)r^n}{[(n-1)+\beta(n+1-2\alpha)]C(\gamma,n)} .$$

Under the constraints for $\gamma$ and $r$, it suffices to show that

$$\Psi(\alpha,\beta,\gamma,r,n) = 2\beta(1-\alpha)r^n/[(n-1)+\beta(n+1-2\alpha)]C(\gamma,n) \qquad (3.1)$$

is a decreasing function of $n$ for $n\geq 2$. From (2.1) we see that $C(\gamma,n+1) = [(n+1-2\gamma)/n]C(\gamma,n)$ so that $\Psi(\alpha,\beta,\gamma,r,n) \geq \Psi(\alpha,\beta,\gamma,r,n+1)$ if and only if

$$h(\alpha,\beta,\gamma,r,n) = (n+1-2\gamma)[n+\beta(n+2-2\alpha)]-rn[n-1+\beta(n+1-2\alpha)] \geq 0 . \qquad (3.2)$$

For $\alpha$ and $\beta$ fixed, the function $h$ is decreasing in $\gamma$ and $r$ and increasing in $n$. Hence, $h(\alpha,\beta,\gamma,r,n) \geq h(\alpha,\beta,(2+3\beta-\alpha\beta)/(2+4\beta-2\alpha\beta), 1,2) = 0$ for $0\leq\gamma\leq(2+3\beta-\alpha\beta)/(2+4\beta-2\alpha\beta)$, $r < 1$, and $n \geq 2$. Similarly, $h(\alpha,\beta,\gamma,r,n) \geq h(\alpha,\beta,1,(1+2\beta-\alpha\beta)/(1+3\beta-2\alpha\beta), 2) = 0$ for $0\leq\gamma<1$, $r\leq(1+2\beta-\alpha\beta)/(1+3\beta-2\alpha\beta)$, and $n\geq 2$. Thus $\max_{n\geq 2} \Psi(\alpha,\beta,\gamma,r,n)$ is attained at $n=2$, and the proof is complete.

As a special case of Theorem 3, we get the result in [3] as a

COROLLARY. *If* $f \in R_\alpha[\alpha,1]$, $0 \leq \alpha < 1$, *then*

$$r-r^2/2(2-\alpha) \leq |f(z)| \leq r+r^2/2(2-\alpha) \qquad (|z|=r) .$$

PROOF. When $\beta = 1$, we have $\gamma = \alpha \leq (5-\alpha)/(6-2\alpha)$ , so that the first condition in Theorem 3 is satisfied.

REMARK. The function $f_2(z) = 0$ in Theorem 3 when $z = [1+\beta(3-2\alpha)](1-\gamma)/\beta(1-\alpha)$ . Letting $z \to 1^-$ , we thus have $|f(z)| \geq r - \beta(1-\alpha)r^2/[1+\beta(3-2\alpha)](1-\gamma)$ for all $z$ in $E$ if and only if $0\leq\gamma\leq[1+\beta(2-\alpha)]/[1+\beta(3-2\alpha)]$.

Theorem 3 leaves open the question of an upper bound for $|f|$ when $\gamma>(2+3\beta-\alpha\beta)/(2+4\beta-2\alpha\beta)$ and $r>(1+2\beta-\alpha\beta)/(1+3\beta-2\alpha\beta)$. We resolve this with

THEOREM 4. *Set* $r_{n_0}(\alpha,\beta,\gamma)=(n_0+1-2\gamma)[n_0+\beta(n_0+2-2\alpha)]/n_0[n_0-1+\beta(n_0+1-2\alpha)]$ .

*If* $f \in R_\gamma[\alpha,\beta]$, $0\leq\alpha<1$, $0<\beta\leq 1$ ,

$$\gamma_0 = \frac{(1+\beta)n_0+\beta(1-\alpha)}{n_0+\beta(n_0+2-2\alpha)} < \gamma \leq \frac{1+(1+\beta)n_0+\beta(2-\alpha)}{1+(1+\beta)n_0+\beta(3-2\alpha)} = \gamma_1 \quad (n_0=2,3,\ldots)$$

*and* $r_{n_0}(\alpha,\beta,\gamma) < r < 1$, *then*

$$|f(z)| \leq r + 2\beta(1-\alpha)r^{n_0+1}/[n_0+\beta(n_0+2-2\alpha)]C(\gamma,n_0+1) \qquad (|z|=r) \quad ,$$

*with equality for* $f_{n_0+1}$ *given in* (2.4).

PROOF. It suffices to determine when $\Psi(\alpha,\beta,\gamma,r,n)$, defined in (3.1), is maximized for $n = n_0 + 1 > 2$. The function $\Psi$ attains its maximum value at $n = n_0 +1$ if the function $h$, defined in (3.2), is negative for $n = n_0$ and positive for $n = n_0 +1$, which occurs for $r_{n_0}(\alpha,\beta,\gamma) < r < r_{n_0+1}(\alpha,\beta,\gamma)$ ; however, $r_{n_0}(\alpha,\beta,\gamma) < 1$ if and only if $\gamma \geq \gamma_0$ and $r_{n_0+1}(\alpha,\beta,\gamma) \geq 1$ for $\gamma \leq \gamma_1$. Therefore, $\max_n \psi(\alpha,\beta,\gamma,r,n)$ occurs at $n = n_0+1$ for $r_{n_0}(\alpha,\beta,\gamma) < r < 1$ and $\gamma_0 \leq \gamma \leq \gamma_1$, and the proof is complete.

We use similar methods to determine a distortion theorem for $f'$.

THEOREM 5. *If* $f \in R_\gamma[\alpha,\beta]$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$, *and either* $0 \leq \gamma \leq 1/2$ *or* $r \leq (2+4\beta-2\alpha\beta)/(3+9\beta-6\alpha\beta) = r_0$, then

$1-2\beta(1-\alpha)r/[1+\beta(3-2\alpha)](1-\gamma) \leq |f'(z)| \leq 1+2\beta(1-\alpha)r/[1+\beta(3-2\alpha)](1-\gamma)$ *for* $|z| = r$, *with equality when* $f_2(z) = z-2\beta(1-\alpha)z^2/[1+\beta(3-2\alpha)](1-\gamma)$.

PROOF. For $A(\alpha,\beta,\gamma, r,n) = 2\beta(1-\alpha)nr^{n-1}/[(n-1)+\beta(n+1-2\alpha)]C(\gamma,n)$ we have, according to Theorem 2,

$1 - \max_{n\geq 2} A(\alpha,\beta,\gamma,r,n) \leq |f'(z)| \leq 1 + \max_{n\geq 2} A(\alpha,\beta,\gamma,r,n)$. But $A$ is a decreasing function of $n$ if and only if

$$h_1(\alpha,\beta,\gamma,r,n) = (n+1-2\gamma)[n+\beta(n+2-2\alpha)] - (n+1)r[(n-1)+\beta(n+1-2\alpha)] \geq 0.$$

Since $h_1$ is decreasing in $r$ and $\gamma$ for $\gamma \leq 1/2$ and increasing in $n$, we have

$$h_1(\alpha,\beta,\gamma,r,n) \geq h_1(\alpha,\beta,1/2,1,2) = 1-\beta(1-2\alpha) \geq 0$$

for $0 \leq \gamma \leq 1/2$, and

$$h_1(\alpha,\beta,\gamma,r,n) \geq h_1(\alpha,\beta,1,r_0,2) = 0 \quad \text{for} \quad r \leq r_0 \ .$$

This completes the proof.

REMARK. The theorem is the best possible in that $h_1(\alpha,\beta,1/2,r,2) < 0$ for

$r > r_0$ and $A(\alpha,\beta,\gamma,1,n) > A(\alpha,\beta,\gamma,1,2)$ for each fixed $\gamma > 1/2$ and $n = n(\gamma)$ sufficiently large.

4. RADII OF UNIVALENCE, STARLIKENESS, AND CONVEXITY.

As we have seen in Theorem 3, it is possible to have $f(z_0) = 0$, $0 < |z_0| < 1$ for $f$ in $R_\gamma[\alpha,\beta]$, which means that $f$ need not be univalent. We now determine when the family contains only univalent functions.

THEOREM 6. $R_\gamma[\alpha,\beta] \subseteq S$ <u>if and only if</u> $\gamma \leq 1/2$.

PROOF. Since $z + \sum_{n=2}^{\infty} a_n z^n \in S$ if $\sum_{n=2}^{\infty} n|a_n| \leq 1$, if suffices to show for $\gamma \leq 1/2$ -- according to Theorem 1 -- that

$$[(n-1)+\beta(n+1-2\alpha)]C(\gamma,n)/2\beta(1-\alpha) \geq n \quad \text{for} \quad n=2,3,\ldots \quad . \tag{4.1}$$

But $C(\gamma,n) \geq C(1/2,n) = 1$ for $\gamma \leq 1/2$, so we need only prove (4.1) for $\gamma = 1/2$, which is equivalent to $n[1+\beta-2\beta(1-\alpha)] \geq 1-\beta(1-2\alpha)$. This last inequality is true for $n=2$, and consequently for all $n \geq 2$.

Conversely, since $C(\gamma,n) \to 0$ for $\gamma > 1/2$, we take $f_n(z)$ defined by (2.4), and note that

$$f_n'(z) = 1 - \frac{2\beta(1-\alpha)nz^{n-1}}{[(n-1)+\beta(n+1-2\alpha)]C(\gamma,n)} = 0$$

for

$$z^{n-1} = [(n-1)+\beta(n+1-2\alpha)]C(\gamma,n)/2\beta(1-\alpha)n ,$$

which is less than 1 for n sufficiently large. Thus, $f_n(z)$ is not univalent for $\gamma > 1/2$ and $n = n(\gamma)$ sufficiently large.

Since functions of the form $z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$, are starlike if and only if they are univalent [5], we have shown that functions in $R_\gamma[\alpha,\beta]$, $0 \leq \gamma \leq 1/2$, are all starlike. We now determine the largest disk in which such functions are starlike of order $\delta$, $0 \leq \delta < 1$.

THEOREM 7. <u>If</u> $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in R_\gamma[\alpha,\beta]$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $0 \leq \gamma \leq 1/2$, <u>then</u> $f$ <u>is starlike of order</u> $\delta$, $0 \leq \delta < 1$, <u>in the disk</u>

$|z| < r_0$ , *where*

$$r_0 = \inf_n \left[\frac{(1-\delta)[(n-1)+\beta(n+1-2\alpha)]C(\gamma,n)}{2\beta(1-\alpha)(n-\delta)}\right]^{1/(n-1)}$$

*with equality for a function of the form* (2.4).

PROOF. It suffices to show that $|(zf'/f) - 1| < 1-\delta$ for $|z| < r_0$ . But

$$|(zf'/f) - 1| \le \frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n|z|^{n-1}} \le 1 - \delta \quad (|z| = r)$$

if and only if

$$\sum_{n=2}^{\infty} \frac{n-\delta}{1-\delta} a_n r^{n-1} \le 1 . \qquad (4.2)$$

In view of Theorem 1, we need only find values of $r$ for which

$$\left(\frac{n-\delta}{1-\delta}\right) r^{n-1} \le \frac{[(n-1)+\beta(n+1-2\alpha)]C(\gamma,n)}{2\beta(1-\alpha)} \qquad (n=2,3,\ldots) \quad ,$$

which will be true when $r \le r_0$ , and the theorem is proved.

COROLLARY 1. *If* $f \in R_\gamma[\alpha,\beta]$, $0 \le \alpha < 1$, $0 < \beta \le 1$, $0 \le \gamma \le 1/2$, *then* $f$ *is convex of order* $\delta$, $0 \le \delta < 1$ *in the disk* $|z| < r_1$ , *where*

$$r_1 = \inf_n \left[\frac{(1-\delta)[(n-1)+\beta(n+1-2\alpha)]C(\gamma,n)}{2\beta(1-\alpha)n(n-\delta)}\right]^{1(n-1)} .$$

PROOF. Since $z + \sum_{n=2}^{\infty} a_n z^n$ is convex of order $\delta$ if and only if $z + \sum_{n=2}^{\infty} na_n z^n$ is starlike of order $\delta$, the proof follows that of Theorem 7, with $a_n$ replaced by $na_n$ .

By taking $\delta = 0$ in Theorem 7, we may determine the radius of univalence (and starlikeness) of $R_\gamma[\alpha,\beta]$ when $\gamma > 1/2$.

COROLLARY 2. *If* $f \in R_\gamma[\alpha,\beta]$, $0 \le \alpha < 1$, $0 < \beta \le 1$, $1/2 < \gamma < 1$, *then* $f$ *is univalent and starlike for* $|z| < r_2$, *where*

$$r_2 = \inf_n \left[\frac{((n-1)+\beta(n+1-2\alpha))C(\gamma,n)}{2\beta n(1-\alpha)}\right]^{1/(n-1)}$$

## 5. ORDER OF STARLIKENESS

Since functions in $R_\gamma[\alpha,\beta]$, $0 \le \gamma \le 1/2$, are starlike, it is of interest to determine the order of starlikeness. We do this in

THEOREM 8. *If* $f \in R_\gamma[\alpha,\beta]$, $0 \le \alpha < 1$, $0 \le \beta < 1$, $0 \le \gamma \le 1/2$, *then* $f$ *is starlike of order*

$$\lambda = \frac{[1+\beta(3-2\alpha)](1-\gamma)-2\beta(1-\alpha)}{[1+\beta(3-2\alpha)](1-\gamma)-\beta(1-\alpha)} ,$$

*with equality for* $f(z) = z - \beta(1-\alpha)z^2/[1+\beta(3-2\alpha)](1-\gamma)$ .

PROOF. From Theorem 1 and [5], it suffices to show, for $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in R_\gamma[\alpha,\beta]$, that $\sum_{n=2}^{\infty} [(n-1)+\beta(n+1-2\alpha)]C(\gamma,n)a_n/2\beta(1-\alpha) \le 1$ implies $\sum_{n=2}^{\infty} [(n-\lambda)/(1-\lambda)]a_n \le 1$. This will be true if

$$g(\alpha,\beta,\gamma,n) = \frac{[(n-1)+\beta(n+1-2\alpha)]C(\gamma,n)(1-\lambda)}{2\beta(1-\alpha)(n-\lambda)} \ge 1 \quad (n=2,3,\dots) .$$

For $\alpha$ and $\beta$ fixed, $g$ can be shown to be an increasing function of $\gamma$, $0 \le \gamma \le 1/2$, and an increasing function of $n$, $n \ge 2$, so that $g(\alpha,\beta,\gamma,n) \ge g(\alpha,\beta, 1/2,2) = 1$ for $0 \le \gamma \le 1/2$ and $n \ge 2$. This completes the proof.

Choosing $\beta = 1$ and $\gamma = \alpha$ in Theorem 8, we get the following result proved in [3] as a

COROLLARY. *If* $f \in R_\alpha[\alpha,1]$, $0 \le \alpha \le 1/2$, then $f$ is starlike of order $(2-2\alpha)/(3-2\alpha)$.

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