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A NOTE ON ALMOST CONTINUOUS MAPPINGS AND BAIRE SPACES

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ABSTRACT. We prove the following theorem:

THEOREM. Let Y be a second countable, infinite R_0-space. If there are countably many open sets $0_1, 0_2, \ldots, 0_n, \ldots$ in Y such that $0_1 \subsetneq 0_2 \subsetneq \ldots \subsetneq 0_n \subsetneq \ldots$, then a topological space X is a Baire space if and only if every mapping f: $X \to Y$ is almost continuous on a dense subset of X. It is an improvement of a theorem due to Lin and Lin [2].

KEY WORDS AND PHRASES. Sparation axiom R_0, almost continuous mapping, Baire space.

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1. INTRODUCTION.

This note is directed to mathematical specialists or non-specialists familiar with general topology [1].

Lin and Lin [2] proved the following theorem:

THEOREM 1. Let Y be an arbitrary infinite Hausdorff space. If X is a topological space such that every mapping f: $X \to Y$ is almost continuous on a dense subset D(f) of X, then X is a Baire space.

In the theorem above, the almost continuity is in the sense of Husain [3]. The proof of the theorem depends on the following lemma (cf. Long [1, Prob. 14, p. 147]):

LEMMA 1. Every infinite Hausdorff space contains a countably infinite discrete subspace.

In this note, we prove a lemma similar to Lemma 1 under weaker conditions, and use it to improve Theorem 1.

2. PRELIMINARIES AND RESULTS.

Before stating the result, we first recall the definition of the separation axiom R_0 (cf. [4], [5], [6, p. 49]).

DEFINITION 1. A topological space X is R_0 if and only if for each $x \in X$ and open subset U, $x \in U$ implies $\overline{\{x\}} \in U$.

It is known [1] that R_0 is weaker than T_1 and is independent of T_0, in fact $T_1 = T_0 + R_0$. A Hausdorff space is R_0.

LEMMA 2. If an infinite space X is R_0, and there are countably infinite open sets $0_1, 0_2, \dots, 0_n, \dots$ such that $0_1 \subsetneqq 0_2 \subsetneqq \dots \subsetneqq 0_n \subsetneqq \dots$, then there is a countably infinite distinct set $S = \{y_1, y_2, \dots, y_n, \dots\}$ in X such that for each n, there is an open set V_n satisfying $V_n \cap S = \{y_n\}$.

PROOF. Without loss of generality we may assume that 0_1 is not empty. Let $y_1 \in 0_1$ be an arbitrary point. Since X is R_0, $\overline{\{y_1\}} \subset 0_1$. Let $V_1 = 0_1$. From $0_1 \subsetneqq 0_2$ we can find a $y_2 \in 0_2$ such that $y_2 \notin 0_1$ and $\overline{\{y_2\}} \subset 0_2$. Let $V_2 = 0_2 \cap (0_1 \setminus \overline{\{y_1\}})$. Then V_2 is an open set and $y_2 \in V_2$. If y_{n-1} is chosen and $V_{n-1} = 0_{n-1} \cap (0_{n-2} \setminus \bigcup_{i=1}^{n-2} \overline{\{y_i\}})$ is defined, then since $0_{n-1} \subsetneqq 0_n$, we may choose $y_n \in 0_n$ such that $y_n \notin 0_{n-1}$ and $\overline{\{y_n\}} \in 0_n$. Let $V_n = 0_n \cap (0_{n-1} \setminus \bigcup_{i=1}^{n-1} \overline{\{y_i\}})$. Then $y_n \in V_n$. Thus we have a countably infinite distinct set $S = \{y_1, y_2, \dots, y_n \dots\}$ and countably infinite distinct open sets $V_1, V_2, \dots, V_n, \dots$ such that $y_n \in V_n$ $(n = 1,2,\dots)$. Since $V_n = 0_n \cap (0_{n-1} \setminus \bigcup_{i=1}^{n-1} \overline{\{y_i\}})$, we have $y_i \notin V_n$ for $i = 1,2,\dots,n-1$. Since $y_{n+m} \in 0_{n+m}$ $(m > 1)$, $y_{n+m} \notin 0_{n+m-1}$, but $0_n \subsetneqq 0_{n+m-1}$, hence $y_{n+m} \notin 0_n$, $y_{n+m} \notin V_n$. Therefore, $V_n \cap S = \{y_n\}$.

For convenience we say that a space X has an ascending chain of open sets if there are countably infinite open sets $0_1, 0_2, \dots, 0_n, \dots$ such that $0_1 \subsetneqq 0_2 \subsetneqq \dots \subsetneqq 0_n \subsetneqq \dots$.

LEMMA 3. An infinite Hausdorff space X is an R_0-space with an ascending chain of open sets.

PROOF. We need only to show that X has an ascending chain of open sets. By Lemma 1, there is a countably infinite discrete subspace $\{y_1, y_2, \dots, y_n, \dots\}$, hence there are disjoint open sets $U_1, U_2, \ldots, U_n, \ldots$ such that $y_n \in U_n$. Let $0_n = \cup_{i=1}^{n} U_i$ $(n = 1,2,\ldots)$. Then $0_1, 0_2, \ldots, 0_n, \ldots$ is an ascending chain of open sets.

The converse of Lemma 3 is not true.

EXAMPLE 1. Let $X = [0,1]$ with topology $\tau = \{X \backslash N;\ N$ is a countable set$\}$. Then X is R_0 and $0_i = X \backslash \{ \frac{1}{i}, \frac{1}{i+1}, \ldots \}$ $(i = 1,2,\ldots)$ is an ascending chain of open sets. X is not Hausdorff.

Now Theorem 1 can be improved as

THEOREM 2. Let Y be an infinite R_0-space with an ascending chain of open sets. If X is a topological space such that every mapping $f\colon X \to Y$ is almost continuous on a dense subset of X, then X is a Baire space.

The proof is all the same as the proof of Theorem 2 in [2].

Similar to Theorem 3 in [2], we have

THEOREM 3. Let Y be a second countable infinite R_0-space with an ascending chain of open sets. Then a topological space X is a Baire space if and only if every mapping $f\colon X \to Y$ is almost continuous on a dense subset of X.

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