Internat. J. Math. & Math. Sci.
Vol. 5 No. 4 (1982) 763-777

THE KRULL RADICAL, k-PRIMITIVE RINGS, AND CRITICAL RINGS

RALPH TUCCI

Department of Mathematics, University of New Orleans
New Orleans, Louisiana 70148 U.S.A.

(Received October 21, 1981 and in revised form June 15, 1982)

ABSTRACT. We generalize results on the Krull radical, k-primitive rings, and critical rings from rings with identity to rings which do not necessarily contain identity.

KEY WORDS AND PHRASES. Krull radical, prime radical, Jacobson radical, Krull dimension, noetherian, k-primitive, critical, co-critical.

AMS 1980 SUBJECT CLASSIFICATIONS. Primary 16A33, 16A55; secondary 16A20, 16A21, 16A34

1. INTRODUCTION. In this paper we extend some results on Krull dimension from rings with identity to rings which do not necessarily contain identity. The basic idea is to embed a ring R into the usual ring R_1 with identity, and to study the relation between the right ideals of R and of R_1.

In the first section of this paper we use Krull dimension to define the Krull radical of R, denoted K(R). The Krull radical is a generalization of the Jacobson radical, and was first defined by Deshpande and Feller [1] for rings with identity. Our main result in this section is that $K(R) = K(R_1)$. This enables us to use previous work in [1] to characterize the Krull radical as the annihilator of all critical R-modules, which in turn lets us determine the Krull radical of the n x n matrix ring over R. We then describe the relation between the Krull radical of R and that of a two-sided ideal $I \subseteq R$. Finally we derive containment relations between the Krull radical on the one hand and the Jacobson and prime radicals on the other.

In the next section we look at k-primitive rings, which are a generalization of primitive rings. We list the main properties of these rings and generalize slightly a theorem on these rings (Prop. 3.4). We finally turn our attention to critical rings, which are closely related to k-primitive rings. Necessary and sufficient conditions are given for a critical ring to be a domain, and those critical rings which are not domains are completely characterized.

In what follows the letter R denotes an associative ring which does not necessarily contain identity. An R-module M_R is a right R-module; usually we will simple call this module M.

Let Z denote the integers. We define $R_1 = \{(r, n) | r \in R, n \in Z\}$, where addition is componentwise and multiplication is given by $(r, n) \cdot (r', n') = (rr' + nr' + n'r, nn')$. This is just the usual ring with identity in which R is embedded. For notational simplicity, we identify R with the subring (R, 0) of R_1 to which R is isomorphic. All modules over R_1 are unital, so that every R-module M can be considered an R_1-module if we define $m(r, z) = mr + mz$ for all $m \in M$, $(r, z) \in R_1$. Conversely, any R_1-module M can be considered an R module with scalar multiplication defined by $mr = m(r, 0)$ for all $m \in M$, $r \in R$. Krull dimension for an R-module M is defined as in [2] and is denoted K dim M, or sometimes K dim M_R. A familiarity with the results of this paper is assumed. Note that for any R-module M, K dim M_R = K dim M_{R_1}, and M is a k-critical R-module if and only if M is a k-critical R_1-module. Finally, E(M) denotes the injective hull of this module M.

2. THE KRULL RADICAL.

As in [1] we say that a right ideal H of a ring R is <u>k-co-critical</u> if $\frac{R}{H}$ is a k-critical R-module. A right ideal H of R is <u>n-modular</u> if there exist $e \in R$, $0 \neq n \in Z$, such that $er - nr \in H$ for all $r \in R$. If $n = 1$, then we call H <u>modular</u> in accordance with the usual terminology. A right ideal which is either maximal modular, 1-co-critical and n-modular, or k-co-critical, $k \geq 2$, is called a <u>special co-critical right ideal of R</u>. The <u>Krull radical of R</u>, denoted K(R), is defined to be the intersection of all the speical co-critical right ideal of R, if any exist; if there are none, then we define K(R) = R. Note that this definition of the Krull radical coincides with that given in [1] if R has identity. In order to be able to use the results of [1], we first prove that $K(R) = K(R_1)$.

LEMMA 2.1 Let H be a right ideal of R, $H \neq R$, and let $H_1 = \{(e, -n) \in R_1 \mid er - nr \in H$ for all $r \in E\}$.

Then

(1) H_1 is the unique right ideal of R_1 which is maximal with respect to the property that $H_1 \cap R = H$;

(2) H is the n-modular if and only if $H_1 \nsubseteq R$.

PROOF (1) It is routine to verify that H_1 is a right ideal of R_1. The uniqueness of H_1 follows from the observation that $H_1 = \{x \in R_1 \mid x R \subseteq H\}$.

(2) This follows because $H_1 \nsubseteq R$ if and only if H_1 contains some $(e, -n) \in R_1$ with $n \neq 0$.

LEMMA 2.2 Let M be a trivial R-module; i.e., $mr = 0$ for every $m \in M$, $r \in R$. If K dim M = k exists, then $k \leq 1$.

PROOF Since M is a trivial R-module, its R-module structure is the same as its structure as an abelian group - i.e., as a Z-module. By [2, Cor. 4.4] K dim $M_Z \leq$ K dim $Z_Z = 1$.

THEOREM 2.3 $K(R) = K(R_1)$.

PROOF By [3, p. 11, Thm. 2] and by definition of the Krull radical, $K(R_1) \subseteq J(R_1) = J(R) \subseteq R$. Thus, $K(R_1) = \bigcap_{H_1 \in C} (H_1 \cap R)$, where C is the set of co-critical right ideals of R_1. We will show that the set of special co-critical right ideals of R coincides with the set of right ideals of the form $H_1 \cap R$, where $H_1 \in C$. Since $J(R) = J(R_1)$, we do not need to consider the case where H_1 is a maximal right ideal of R_1.

Suppose that R has some special co-critical right ideals. Let H be a special k-co-critical right ideal of R, $k > 0$, and let H_1 be as in Lemma 2.1. We first determine K dim $\frac{R_1}{H_1}$. By [2, Lemma 1.1]

$$K\dim\frac{R_1}{H_1} = \sup\left[K\dim\frac{\frac{R_1}{H_1}}{\frac{R+H_1}{H_1}},\ K\dim\frac{R+H_1}{H_1}\right]$$

$$= \sup\left[K\dim\frac{R_1}{R+H_1},\ K\dim\frac{R+H_1}{H_1}\right]$$

Now $\frac{R_1}{R+H_1}$ is a homomorphic image of $\frac{R_1}{R}$, which is a trivial R-module. Thus,

$K\dim\frac{R_1}{R+H_1} \le K\dim\frac{R_1}{R} = 1$ by lemma 2.2. Since $\frac{R+H_1}{H_1} \simeq \frac{R}{R\cap H_1} = \frac{R}{H}$ we have

$K\dim\frac{R_1}{H_1} = k$.

To show H_1 is co-critical, let K_1 be any right ideal of R_1 which properly contains H_1. Then $R \cap K_1$ properly contains H because of the way H_1 is defined. Repeating the argument we used to find $K\dim\frac{R_1}{H_1}$ gives us that $K\dim\frac{R_1}{K_1} = K\dim\frac{R}{R\cap K_1} < K\dim\frac{R}{H} = k$. Therefore H_1 is a k-co-critical right ideal of R_1.

Conversely, suppose that H_1 is a k-co-critical right ideal of R_1, $k > 0$, and assume $H_1 \nsupseteq R$ (if there is no such H_1, then R has no special co-critical right ideals contrary to our assumption). Let $H = R \cap H_1$. Since $\frac{R}{H} \simeq \frac{R+H_1}{H_1} \subseteq \frac{R_1}{H_1}$ we have that $\frac{R}{H}$ is k-critical by [2, Prop. 2.3]. If $k \ge 2$ then H is a special co-critical right ideal of R. Suppose $k = 1$. Then $H_1 \nsubseteq R$; for , if $H_1 \subseteq R$, then there is an onto map from $\frac{R_1}{H_1}$ to $\frac{R_1}{R}$. Since both modules have Krull dimension 1, and $\frac{R_1}{H_1}$ is critical, we must have $\frac{R_1}{H_1} \simeq \frac{R_1}{R}$. But then $\frac{R_1}{H_1}\cdot R = 0$, which implies $R \subseteq H_1$ and this in turn implies that $R \subseteq H_1 \cap R = H$, contradicting the fact that $K\dim\frac{R}{H} = 1$. Thus, $H_1 \nsubseteq R$. We must have, then, that H_1 contains some $(e, -n) \in R$ with $n \neq 0$, so for every $r \in R$, $(e, -n)r = er - nr \in H_1 \cap R = H$. Hence H_1 is n-modular, and therefore special.

Suppose now that R has no special co-critical right ideals. Then $K(R) = R$ by definition. Since $\frac{R_1}{R}$ is 1-critical, R is a co-critical right ideal of R_1. Every other co-critical right ideal H_1 of R_1 contains R; for, if $R \nsubseteq H_1$ then $H_1 \cap R$ is a special co-critical right ideal of R, contradiction. Therefore, $K(R_1) = R = K(R)$.

This completes the proof.

COROLLARY 2.4 (1) K(R) is the set of elements of R which annihilate every critical right R-module.

(2) K(R) is a two sided ideal of R.

(3) $K(\frac{R}{K(R)}) = 0$.

PROOF This follows from Thm. 2.3 and [1, Thm. 2.1].

The next result shows that $K(R_n) = (K(R))_n$, where R_n is the ring of n x n matrices over R. If R has identity, then E_{ij} denotes the matrix with 1 in the (i, j) position and zeroes elsewhere.

LEMMA 2.5. Let R be a ring with identity, and let H be a right ideal of R. Take $H^{(i)}$ to be the set of all matrices in R_n whose i^{th} row has entries from H and whose other entries are arbitrary. Then $\frac{R_n}{H^{(i)}}$ is a critical R_n-module if and only if $\frac{R}{H}$ is a critical R-module.

PROOF For simplicity, assume that i = 1. Note that $\frac{R_n}{H^{(1)}}$ consists of matrices whose only non-zero row is the first. Thus, any submodule S of $\frac{R_n}{H^{(1)}}$ can be written

$$S = \begin{bmatrix} N_1 \dots N_n \\ 0 \dots 0 \\ 0 \dots 0 \end{bmatrix}$$ where $N_1, \dots, N_n$ are subsets of R. Now $N_1 = \dots = N_n$; for, $\frac{R_n}{H^{(1)}} E_{jj} \subseteq \frac{R_n}{H^{(1)}}$ for any $1 \le j \le n$, and $\frac{R_n}{H^{(1)}} E_{jj}$ consists of matrices with nonzero entries in the (1, j) position - i.e., from N_j - and zeroes elsewhere. But then for any $1 \le k \le n$, $\frac{R_n}{H^{(1)}} E_{jj} E_{jk} \subseteq \frac{R_n}{H^{(1)}}$. This implies that $N_j \subseteq N_k$. Since j and k are arbitrary, we have that $N_1 = \dots = N_n$. Call this set N. It is routine to check that N is an R-submodule of $\frac{R}{H}$. Thus, there is a 1-1 onto order preserving map f from the R_n-submodules of $\frac{R_n}{H^{(1)}}$ to the R-submodules of $\frac{R}{H}$, given by f: $\begin{bmatrix} N \dots N \\ 0 \dots 0 \\ 0 \dots 0 \end{bmatrix} \to N$. The result follows immediately from this.

LEMMA 2.6. Let R be a ring with identity. If M is a cyclic critical R_n-module, then there is a co-critical right ideal $H^{(i)} \subseteq R_n$ as in Lemma 2.5 such that M is isomorphic to $\frac{R_n}{H^{(i)}}$.

PROOF Since M is cyclic, we can find a matrix $A \in M$ such that $M = AR_n$. We show first that there is an integer j, $1 \leq j \leq n$, such that every element in M has non-zero j^{th} row. Suppose this is not the case. Then there is a collection of elements X_1, $X_2, \ldots, X_n \in R_n$ such that $AX_k \neq 0$ and the k^{th} row of AX_k is zero. But then $(AX_1)R_n \cap \ldots \cap (AX_n)R_n = 0$, contradicting the fact that M is a uniform module by [2, Cor. 2.5 and 2.6].

We can assume without loss of generality that every non-zero element of M has non-zero first row. Let M' be the module consisting of all matrices whose first row appears as the first row of a matrix in M, and whose other entries are zero. Define a map f: $M \to M'$ as follows: If $A \in M$, then f(A) is the matrix whose first row is the same as that of A, and whose other entries are zero. This map is certainly an R_n-isomorphism and M' is of the appropriate form. This completes the proof.

THEOREM 2.7. $K(R_n) = (K(R))_n$

PROOF First assume R has identity. Let M be any cyclic critical R_n-module. By Lemma 2.6, $M \simeq \frac{R_n}{H^{(i)}}$, where $H^{(i)}$ is defined as in Lemma 2.5. By Cor. 2.4 (2), K(R) is a two-sided ideal of R. Let $X \in K(R_n)$, x the (i,j) entry of X. Then $x E_{ij} = E_{ii} X E_{jj} \in K(R_n)$ so that $\frac{R_n}{H^{(i)}} x E_{ij} = 0$. As in the proof of Lemma 2.5, this shows that x annihilates the critical R-module $\frac{R}{H}$. Since M is an arbitrary cyclic R_n module, so is $\frac{R}{H}$; thus, by Cor. 2.4 (1), $x \in K(R)$. Therefore, $K(R_n) \subseteq (K(R))_n$. The reverse inclusion follows by reversing the steps of the argument. Hence $K(R_n) = (K(R))_n$ when R has identity.

If R does not contain identity, embed R into R_1. From the previous paragraph and Thm. 2.2 we have $(K(R))_n = (K(R_1))_n = K((R_1)_n)$. However, just as a critical module over R can be considered a critical module over R_1 and vice versa, so we can identify modules over R_n and $(R_1)_n$. Therefore, by Cor. 2.4 (1), $K((R_1)_n) = K(R_n)$. This completes the proof.

We now describe the relation between the Krull radical of a ring R and that of a two-sided ideal I in R.

LEMMA 2.8. Let R be a ring such that R = K(R), let I be a two-sided ideal of R, and let M be a k-critical I-module. Then either MI = 0 or MI is a k-critical R-module.

PROOF Assume MI $\neq$ 0, and take C to be a critical R-submodule of MI. Then CR = 0 by Cor. 2.4 (1), so CI = 0. Hence K dim C_R = K dim C_I = k, which implies that K dim $MI_R \geq$ k. Since the reverse inclusion always holds, we have K dim MI_R = k. That MI is a critical R-module follows from the fact that MI is a critical I-module.

PROPOSITION 2.9. Let R be a ring such that R = K(R), and let I be a two-sided ideal of R. Then K(I) = I.

PROOF Let M be a critical right I-module. If MI $\neq$ 0, then there is some $i \in I$ for which Mi $\neq$ 0. Since MiR $\subseteq$ MIR = 0 by Lemma 2.8 and Cor. 2.4 (1), Mi is a critical R-module. Hence the map f: $M \to Mi$ defined by f(m) = mi for all $m \in M$ is actually an I-isomorphism. But then for any $m \in M$, f(mi) = f(m)i = mi^2 = 0 and hence Mi = 0, contradiction. Therefore MI = 0, and by Cor. 2.4 (1), I = K(I).

EXAMPLE 2.10. Prop. 2.9 is true if we substitute the Jacobson radical for the Krull radical. By [4, Thm. 48], this is equivalent to the fact that J(I) = I $\cap$ J(R) for any ideal I of a ring R. Unfortunately, this does not hold for the Krull radical. Let $R = \begin{bmatrix} F & F[x] \\ 0 & F[x] \end{bmatrix}$ where F is any field, x is a commuting indeterminate over F, and the ring operations are the usual matrix addition and multiplication. By [1, Ex. 4 , K(R) = 0. However, if we take $I = \begin{bmatrix} 0 & F[x] \\ 0 & 0 \end{bmatrix}$, then K(I) = I because I has no special co-critical right ideals; for, since I_I is isomorphic to a direct sum of copies of F, any special co-critical right ideal would have to be maximal modular. Certainly I has no such right ideal.

We now describe the containment relations between K(R) on the one hand and J(R) and P(R) (the prime radical of R) on the other.

PROPOSITION 2.11. (1) For any ring R, $K(R) \subseteq J(R)$.

(2) If R is a ring with Krull dimension, then $K(R) \subseteq P(R)$.

(3) If R is a commutative ring, then $P(R) \subseteq K(R)$.

PROOF If we embed R into R_1, then $P(R) = P(R_1)$, $J(R) = J(R_1)$, and $K(R) = K(R_1)$ by [4, Cor. after Thm. 59], [3, p. 11 Thm. 2], and Thm. 2.3 of this paper respectively. Hence we may assume that R has identity. Now (1) follows from the definitions of K(R) and J(R), while (2) and (3) are mentioned in [1, p. 188] for rings with identity.

EXAMPLE 2.12. (1) The containments in Prop. 2.11 (1) and 2.11 (2) are both proper. Let R be as in Ex. 2.10. Then $K(R) = 0$, but $P(R) \neq 0$.

(2) The containment in Prop. 2.11 (3) also is proper. Let $S = Z_2[x_1, x_2, \ldots, x_n, \ldots]$ where $\{x_1, x_2, \ldots, x_n, \ldots\}$ is a countably infinite set of commuting indeterminates. Take I to be the ideal generated by the polynomials $x_{2j-1}x_{2j} + x_{2j+1}x_{2j+2}$, $j = 1, 2, \ldots$ and let $R = \frac{S}{I}$. Say that $\bar{x}_{2j-1}\bar{x}_{2j} = x$ in R for all j. Then $x \notin P(R)$, but $x \in K(R)$ by [5, Ex. 4.17].

(3) In general, P(R) and K(R) are incomparable. Let

$$R = \begin{bmatrix} Z_2 & \frac{S}{I} \\ 0 & \frac{S}{I} \end{bmatrix}, \quad a = \begin{bmatrix} 0 & \bar{x}_1 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}$$ where these symbols are defined in the previous paragraph. Then $a \in P(R)$, but $a \notin K(R)$; for, if I' is the ideal of S generated by all $\bar{x}_j$, $j > 1$, then $C = \begin{bmatrix} Z_2 & \frac{S}{(I' + I)} \\ 0 & 0 \end{bmatrix}$ is critical but $C a \neq 0$. Now $b \notin P(R)$, but $b \in K(R)$, because a map $f\colon \frac{S}{I} \to M$, where M has Krull dimension, has kernel containing almost all the $\bar{x}_j$'s, and hence x.

3. CO-PRIMITIVE IDEALS

Just as J(R) can be expressed as the intersection of certain two-sided ideals of R, so can K(R). Let $H_1, H_2, \ldots, H_n$ be a finite collection of special co-critical right ideals of R_1, and suppose that $E(\frac{R}{H_j}) \simeq E(\frac{R}{H_k})$ for all $1 \le j, k \le n$. If $K \dim \frac{R}{H_j} = k$ for all $1 \le j \le n$, then the largest two-sided ideal $D \subseteq \bigcap_{j=1}^{n} H_j$ is called a <u>k-co-primitive ideal of R</u>. An ideal which is k-co-primitive for some ordinal k is called co-primitive. It is not hard to see that $D = \{r \in R \mid \frac{R_1}{(H_j)_1} r = 0$ for all $1 \le j \le n\}$. Here $(H_j)_1$ is the extension of H_j to a co-critical right ideal of R_1 as in Lemma 2.1.

THEOREM 3.1. K(R) is the intersection of all the co-primitive right ideals of R.

PROOF From Cor. 2.4 (2), K(R) is a two-sided ideal of R. Since $K(R) \subseteq H$ for every special co-critical right ideal $H \subseteq R$, then $K(R) \subseteq D$ for every co-primitive ideal of R. Thus, $K(R) \subseteq \cap D$. Conversely, if $r \in \cap D$, then by the observation previous to this theorem we have $M \cdot r = 0$ for any critical R-module M. Thus, $\cap D \subseteq K(R)$ by Cor. 2.4 (1), so $K(R) = \cap D$.

If 0 is a k-co-primitive ideal of a ring R with Krull dimension k, then R is said to be k-primitive. This definition coincides with that given in [6].

PROPOSITION 3.2. Let R be a ring with Krull dimension k. Then R is k-primitive if and only if R has a faithful critical finitely generated module C with K dim R = K dim C.

PROOF Suppose that R is k-primitive. Then there is a finite collection of special k-co-critical right ideals $H_1, \ldots, H_n$ whose intersection is 0 and such that $E(\frac{R}{H_j}) \simeq E(\frac{R}{H_k})$ for all $1 \le j, k \le n$. But then $E(\frac{R_1}{(H_j)_1}) \simeq E(\frac{R_1}{(H_k)_1})$ so we may assume that each $\frac{R_1}{(H_j)_1}$ lies in the same injective hull. The module $C = \frac{R_1}{(H_1)_1} + \ldots + \frac{R_1}{(H_n)_1}$ is critical by [6, Lemma 3.1], finitely generated, and faithful, and K dim R = k = K dim C. The converse follows by reversing the steps of this argument.

The main properties of k-primitive rings have been investigated in [6]. We list some of these properties here. Recall that the assassinator of a uniform module C over a ring R with Krull dimension is that ideal P which is maximal among the annihilators of submodules of C.

THEOREM 3.3. Let R be a k-primitive ring with faithful critical module C, and let P be the assassinator of C.

(1) If A, B = 0 for two right ideals A and B, then either A = 0 or $B \subseteq P$ (i.e., R is P-primary);

(2) P is the only prime ideal of R which is not a large right ideal;

(3) if H is any non-zero right ideal of R, then K dim H = K dim R;

(4) R and C are nonsingular;

(5) if H is a large right ideal of R, then K dim $\frac{R}{H}$ < K dim R;

(6) the injective hull of R is a simple artinian ring.

In [7, Thm. 3.4], Boyle, Deshpande and Feller characterize a k-primitive piecewise domain (PWD) which contains a faithful critical right ideal. (We shall refer to this type of ring as a BDF ring after the authors.) This result can be used to describe a slightly broader class of rings. Recall that a PWD R is a ring with identity which contains a complete set of orthogonal idempotents $e_1, \ldots, e_n$ such that if $x \in e_i R e_j$, $y \in e_j R e_k$, then x y = 0 implies x = 0 or y = 0. In what follows, we assume that R is written as an n x n upper triangular matrix ring; see [8]. Recall also that a ring S is a quotient ring of R if R is a large R-submodule of S.

In the next result, we assume that R is a noetherian k-primitive ring with identity which is a direct sum of non-isomorphic critical right ideals (and hence is a PWD by [8]). Since E(R) is a matrix ring over a division ring D with identity 1, we can define the matrix $M = E_{11} + \ldots + E_{1n}$ where E_{1j} is the matrix with 1 in the (1,j) position and 0's elsewhere, $1 \leq j \leq n$.

PROPOSITION 3.4 Let R and M be as above. Then R has a quotient ring S = R + RMR which is a noetherian BDF ring if and only if $(RMR)^2 \subseteq RMR + R$ and RMR is a finitely generated R-module.

PROOF Note that R is an upper triangular matrix ring with $e_jRe_k = 0$ for j > k. Also, each e_jRe_j is noetherian, for if $I = \sum_{k \neq j} e_kR + \sum_{\ell > j} e_jRe_\ell$ then $e_jRe_j \simeq \frac{R}{I}$. Finally, note that $RMR = e_1S$.

Let S = R + RMR. Assume that RMR is a finitely generated R module and that $(RMR)^2 \subseteq R + RMR$. Since $S \subseteq E(R)$, S is a quotient ring of R. Also, S is a finitely generated R-module, which implies that S is a noetherian ring and that e_1S is a finitely generated R-module. Now $e_1S \subseteq E(e_1R)$ which is uniform, so e_1S is a critical R-module by [9, Cor. 2.4]. Let $0 \neq H$ be an S-submodule of e_1S. Then

$$K\dim\left(\frac{e_1S}{H}\right)_S \le K\dim\left(\frac{e_1S}{H}\right)_R < K\dim(e_1S)_R . \tag{3.1}$$

But $K\dim(e_1S)_S = K\dim(e_1S)_R$; for, since R is a PWD, e_1Se_n is merely a sum of copies of e_nRe_n. Further,

$$(e_1Se_n)_S = (e_1Se_n)_{e_nRe_n} = (e_1Se_n)_R \tag{3.2}$$

Thus, $K\dim(e_1S)_R = K\dim(e_1Se_n)_R \le K\dim(e_1S)_S$ so that $K\dim(e_1S)_R = K\dim(e_1S)_S$. This together with (3.1) shows that e_1S is a critical S-module. Now e_1S is faithful; for, if $e_1S\, s = 0$ for some $s \in S$, then for any idempotents e_j, $e_k \in R$ we have $e_1Se_j\, e_jse_k = 0$. Since S is a PWD, $e_j\, se_k = 0$. Therefore, $s = 0$.

Conversely, let $S = R + RMR$ be a noetherian BDF ring. Since S_S is noetherian $(e_1Se_n)_S$ is noetherian. But by (3.2), $(e_1Se_n)_R$ is noetherian. Let

$S' = \dfrac{e_1Se_{n-1} + e_1Se_n}{e_1Se_n}$. Again, S'_S being noetherian implies S'_R is noetherian, because $S'_S = S'_{e_{n-1}Re_{n-1}} = S'_R$, so that $(e_1Se_{n-1} + e_1Se_n)_R$ is noetherian. Continuing in this manner, we have e_1S_R noetherian, and hence RMR_R is finitely generated. Finally, since S is a ring, $(RMR)^2 \subseteq R + RMR$.

Prop. 3.4 applies more generally to a ring R with identity which is a direct sum of non-singular non-isomorphic critical right ideals; such a ring is a direct sum of ideals, each of which is a k-primitive ring by [10, Prop. 5.3].

EXAMPLE 3.5. (1) Let F be a field, x a commuting indeterminate over F, and let

$$R = \begin{bmatrix} F & 0 & F[x] \\ 0 & F & F[x] \\ 0 & 0 & F[x] \end{bmatrix}$$ with the usual matrix operations. Then R satisfies the

conditions of Prop. 3.4. If

$$RMR = \begin{bmatrix} F & F & F[x] \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ then } S = R + RMR = \begin{bmatrix} F & F & F[x] \\ 0 & F & F[x] \\ 0 & 0 & F[x] \end{bmatrix} \text{ is a BDF ring.}$$

(2) Let x and y be commuting indeterminates over F, and let

$$R = \begin{bmatrix} F[x] & 0 & F[x, y] \\ 0 & F[y] & F[x, y] \\ 0 & 0 & F[x, y] \end{bmatrix} \text{ and } RMR = \begin{bmatrix} F[x] & F[x, y] & F[x, y] \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $S = \begin{bmatrix} F[x] & F[x, y] & F[x, y] \\ 0 & F[y] & F[x, y] \\ 0 & 0 & F[x, y] \end{bmatrix}$ has no Krull dimension, since F[x, y] does not have finite uniform dimension as an F[y] module. In this case RMR is not finitely generated.

4. CRITICAL RINGS

A ring R is critical if R_R is a critical R-module. If R has identity, then R_R is faithful, and hence R is k-primitive. Thus, we could describe the structure of this ring using Thm. 3.3. However, it is possible to prove more about R, even if we do not assume that R has identity.

PROPOSITION 4.1. If R is a domain with Krull dimension, then R is critical.

PROOF Let C be a critical right ideal of R, $0 \neq c \in C$. The map $f: R \to C$ given by $f(r) = cr$ is 1-1, proving R is critical.

The converse of Prop. 4.1 is true if R has identity. To examine this converse for k-critical rings which do not possess identity, we need to consider separately the cases $k > 1$ and $k = 1$. Recall that a module M is monoform if, for any submodule $N \subseteq M$, a homomorphism $f: N \to M$ is either zero or 1-1. Any critical module is monoform by [2, Cor. 2.5].

PROPOSITION 4.2. If R is a k-critical ring, $k > 1$, then R is a domain.

PROOF Let $T = \{r \in R \mid rR = 0\}$. By Lemma 2.2, K dim T = 1 < K dim R, contradiction. Hence T = 0. Now if a, b $\in$ R with ab = 0, then either b = 0 or a $\in$ T because R is monoform, so a = 0.

To examine 1-critical rings, we need the following notation: Q is the set of rational numbers, $G(p) = \{\frac{a}{p^k} \mid \frac{a}{p^k} \in Q$, p a fixed prime$\}$, and $Z_{p^\infty} = \frac{G(p)}{Z}$.

THEOREM 4.3. Let R be a 1-critical ring. Then the following are equivalent:

(1) R is a domain;

(2) $R^2 \neq 0$;

(3) 0 is an n-modular right ideal of R.

PROOF (1) ⇒ (2) Trivial.

(2) ⇒ (3) Assume $R^2 \neq 0$. If there is $0 \neq n \in Z$, $0 \neq r \in R$ such that nr = 0, then nr = 0 for all r $\in$ R because R is monoform and so 0 is n-modular. Otherwise, since $R^2 \neq 0$, we can pick x $\in$ R such that xr $\neq$ 0 for any $0 \neq r \in R$. Now the module $\frac{xR + xZ}{xZ} \cong \frac{xR}{xZ \cap xR}$, being a proper homomorphic image of the 1-critical module xR + xZ, is artinian. Hence $xZ \cap xR \neq 0$. In particular, there exist e $\in$ R, n $\in$ Z such that $0 \neq x e = nx$. Multiply on the right by any r $\in$ R and cancel the element x to show that 0 is n-modular.

(3) ⇒ (1) In this part of the proof we use the argument from [11, Prop. 4.1]. Let 0 be an n-modular ideal of R; i.e., there are e $\in$ R, $0 \neq n \in Z$ such that er - nr = 0 for all r $\in$ R. Let $T = \{r \in R \mid r R = 0\}$. As in Prop. 4.2, we show that T = 0. If nr = 0 for some $0 \neq r \in R$, then nR = 0; in particular, any element t $\in$ T generates a finite, hence artinian, right ideal of R. Hence t = 0, so that T = 0. Now suppose that nr $\neq$ 0 for any $0 \neq r \in R$. We note first that $\frac{R}{T}$ is a domain; for ; let a, b $\in$ R with ab $\in$ T. If b $\notin$ T, then bR $\neq$ 0. Since abR = 0, the fact that R is monoform implies that aR = 0 and so a $\in$ T. Hence $\frac{R}{T}$ is a domain. Because R is 1-critical, $\frac{R}{T}$ is an artinian domain, and hence is a division ring D.

Define a group homomorphism f: $\frac{R}{T} \to T$ by f(r + T) = rt for all r $\in$ R, where $0 \neq t \in T$ is fixed but arbitrary. This map is 1-1; for if rt = 0 for some r $\in$ R, r $\notin$ T, then rR = 0 because R is monoform, and hence r $\in$ T, contradiction. Hence T contains a subgroup isomorphic to D. Now R, and hence D, has no elements of finite order by assumption. This implies that D, and hence T, has a subgroup which is isomorphic to Q. Without loss of generality we write $Q \subseteq T$. Thus, Q is a trivial R-module, which implies that K dim Q_R = K dim Q_Z. However, K dim Q_Z does not exist, contradiction. It follows that T = 0, and R is a domain. This completes the proof.

The case when 0 is a maximal modular right ideal is handled similarly. Hence we may summarize:

COROLLARY 4.4. Let R be a critical ring. Then R is a domain if and only if 0 is a special co-critical right ideal of R; otherwise, $R^2 = 0$.

THEOREM 4.5. Let R be a 1-critical ring satisfying $R^2 = 0$. Then as a group R is isomorphic to a finite sum of Z and G(p)'s for various primes p.

PROOF Since the injective hull of R is isomorphic to Q, identify R with some subgroup of Q. If G is finitely generated, then G is isomorphic to Z by [13, Thm. 9.24]. If G is not finitely generated, then $\frac{G+Z}{Z}$ is isomorphic to a direct sum of Z_{p^∞}'s and Z_{p^n}'s, where there is a distinct summand for every prime p which divides b for some $\frac{a}{b} \in G$.

EXAMPLE 4.6. Let $R = \{\frac{a}{2^k} \mid a, k \in Z\}$ where the product of any two elements is 0. Then R is 1-critical but not right noetherian.

We note that Hein [12] has recently generalized Thm. 4.5.

ACKNOWLEDGEMENT Much of this paper is taken from the author's doctorial thesis. The author would like to thank his advisor, E. H. Feller, for his kindness, encouragement, and boundless patience. The author would also like to thank the referee for his helpful comments which have led to substantial improvements in this paper.

REFERENCES

1. DESHPANDE, M.G., and FELLER, E.H., The Krull Radical, Comm Algebra 3(2) (1975), 185-193.
2. GORDON, R., and ROBSON, J.C., Krull Dimension, Memoirs of the Amer. Math. Soc. No. 133, (1973).

3. JACOBSON, N., Structure of Rings, Amer. Math. Soc. Colloquium Publications, Vol. XXXVIII, (1968).

4. DIVINSKY, N., Rings and Radicals, University of Toronto Press, (1965).

5. TUCCI, R., Krull Dimension And The Krull Radical In Arbitrary Rings, Ph.D. Thesis, University of Wisconsin, Milwaukee, (1976).

6. BOYLE, A.K., and FELLER, E.H., Semicritical Modules And k-Primitive Rings, in Module Theory, Springer-Verlag Lecture Notes No. 700, (1979), 57-74.

7. BOYLE, A.K., DESHPANDE, M.G., and FELLER, E.H., On Nonsingularly k-Primitive Rings, Pacific J. Math, Vol. 68, (1977), No. 2, 303-311.

8. GORDON, R., and SMALL, L., Piecewise Domains, J. Alg. 23 (1972) 553-564.

9. BOYLE, A.K., The Large Condition, Proc. Amer. Math. Soc. 27 (1978), 27-32.

10. BOYLE, A.K., and TUCCI, R.P., When Semicritical Rings Are Semiprime, Comm Algebra, 9(17), (1981), 1747-1761.

11. HANSEN, F., On One-Sided Prime Ideals, Pacific J. Math, 58 (1975) 79-85.

12. HEIN, J., Almost Artinian Modules, Math. Scand. 45 (1979) 198-204.

13. ROTMAN, J., The Theory of Groups, An Introduction, Allyn and Bacon, (1965).