

ON WHITEHEAD'S SECOND LEMMA FOR LIE ALGEBRAS

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ABSTRACT. A proof of Whitehead's second lemma which is independent of the first Whitehead lemma is given.

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1. INTRODUCTION.

THE SECOND WHITEHEAD LEMMA FOR LIE ALGEBRAS. Let L be a finite-dimensional semi-simple Lie algebra of characteristic 0, M a finite-dimensional L -module and $(x, y) \rightarrow g(x, y)$ a bilinear mapping of $L \times L$ into M such that

$$g(x, x) = 0, \quad (1.1)$$

$$g([xy], z) + g(x, y)z + g([yz], x) + g(y, z)x + g([zx], y) + g(z, x)y = 0. \quad (1.2)$$

Then there exists a linear mapping $x \rightarrow x^\alpha$ of L into M such that

$$g(x, y) = x^\alpha y - y^\alpha x - [xy]^\alpha. \quad (1.3)$$

The standard proof of the second Whitehead lemma for Lie algebras of characteristic 0 [1, p. 89] considers the three cases: Γ , the Casimir operator, is non-singular; Γ is nilpotent; and Γ is neither non-singular nor nilpotent and; in the case where Γ is nilpotent, the proof depends on the first Whitehead lemma. The purpose of this note is to give a simplification of the proof for the case Γ is nilpotent which is independent of the first Whitehead lemma. This, of course, will simplify the proof of the Levi theorem which is one of the main uses of the lemma.

For our purposes here we will need the following lemma, the proof of which follows easily from Theorem 2 of [2] and the properties of the complementary basis.

LEMMA 1. If A is a finite-dimensional Lie algebra which has non-degenerate Killing form with basis (e_1, e_2, \dots, e_n) and complementary basis $(e'_1, e'_2, \dots, e'_n)$, then for a bilinear mapping $f(x, y)$ of $A \times A$ into A

$$\sum_k f([e'_k x], [y e_k]) = \sum_k f([e_k x], [y e'_k]).$$

2. PROOF OF THE MAIN RESULT.

We introduce the following notation: R is the kernel of the representation of L determined by M , L_1 is an ideal such that $L = R \oplus L_1$, (u_i) and (u'_i) ; $i = 1, 2, \dots, m$ are dual bases of L_1 relative to the trace form of the given representation, Γ is the Casimir operator determined by the dual bases and is the mapping $x \rightarrow \sum_i (x u_i) u'_i$ in M . Suppose that Γ is nilpotent, then $m = 0$, $R = L$, and the representation determined by M is the zero representation. Then (1.2) reduces to:

$$g([xy], z) + g([yz], x) + g([zx], y) = 0, \quad (2.1)$$

Let (e_i) and (e'_i) ; $i = 1, 2, \dots, n$ be a dual basis for L and in (2.1) set $z = [w e_i]$, $y = e'_i$ and sum over i to get

$$\sum_i \left\{ g([x e'_i], [w e_i]) + g([e'_i [w e_i]], x) + g([e_i x], [e'_i w]) \right\} = 0. \quad (2.2)$$

Since L is non-degenerate in the sense of [2], Theorem 2 of [2] implies that $\sum_i [e'_i [w e_i]] = -w$, and (2.2) reduces to

$$\sum_i \left\{ g([x e'_i], [w e_i]) - g(w, x) + g([e_i x], [e'_i w]) \right\} = 0. \quad (2.3)$$

We can verify that

$$\begin{aligned} \sum_i g([e_i [w x]], e'_i) &= - \sum_i \left\{ g([w [x e_i]], e'_i) + g([x [e_i w]], e'_i) \right\} \\ &= - \sum_i \left\{ g([x e'_i], [w e_i]) + g([x e_i], [w e'_i]) \right\} \\ &= -2 \sum_i g([x e'_i], [w e_i]), \text{ by Lemma 1.} \end{aligned}$$

Now if we substitute the above result in (2.3), we get

$$-g(w, x) - \sum_1 g([e_i [wx]], e'_i) = 0.$$

Thus, we obtain a linear mapping α of L into M such that

$$g(w, x) = -[wx]^\alpha = - \sum_1 g([e_i [wx]], e'_i).$$

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