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A SINGULAR FUNCTIONAL-DIFFERENTIAL EQUATION

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ABSTRACT. The representation of the Hardy-Lebesque space by means of the shift operator is used to prove an existence theorem for a singular functional-differential equation which yields, as a corollary, the well known theory of Frobenius for second order differential equations.

KEY WORDS AND PHRASES. Singular functional-differential equation, Hardy-Lebesque space, Shift-operator.

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1. INTRODUCTION.

Consider the singular functional-differential equation

$$z^2y''(z) + zp(z)y'(z) + q(z)y(z) + \sum_{i=1}^{m} a_i(z)y(q^iz) = 0, \quad |q| \le 1 \tag{1.1}$$

where

$$p(z) = \sum_{n=0}^{\infty} a_n z^n, \quad q(z) = \sum_{n=0}^{\infty} b_n z^n \quad \text{and} \quad a_i(z) = \sum_{j=0}^{\infty} a_{ij} z^j, \quad i = 1,2,\dots,m$$

are analytic functions in some neighborhood of the closed unit disk $\bar{\Delta} = \{z \in ¢: |z| \le 1\}$.

We consider the problem of finding conditions for Equation (1.1) to have solutions in the space $H_2(\Delta)$, i.e. the Hilbert space of functions $f(z) = \sum_{n=1}^{\infty} a(n)z^{n-1}$ which are analytic in the open unit disk $\Delta = \{z \in ¢: |z| < 1\}$ and satisfy the condition $\sum_{n=1}^{\infty} |a(n)|^2 < +\infty$. We shall prove the following.

THEOREM. Let

$$k(k - 1) + a_0 k + b_0 = 0 \tag{1.2}$$

be the idicial equation of the unperturbed equation (1.1).

(i) If $2k + a_0 - 1 = \delta = k_1 - k_2 \neq \pm n$, $n = 1,2,\ldots$, then Equation (1.1) has two linearly independent solutions of the form:

$$y_1(z) = z^{k_1} u(z) \quad \text{and} \quad y_2(z) = z^{k_2} u(z),$$

where k_1 and k_2 are the roots of Equation (1.2) and $u(z)$ belongs to $H_2(\Delta)$.

(ii) If $2k + a_0 - 1 = \delta = k_1 - k_2 = 0$, i.e. $k_1 = k_2$, then Equation (1.1) has only one solution of the form:

$$y(z) = z^k u(z),$$

where k is the double root of Equation (1.2) and $u(z)$ belongs to $H_2(\Delta)$.

(iii) If $2k + a_0 - 1 = \delta = k_1 - k_2 = n$, $n = 1,2,\ldots$, then Equation (1.2) has always a solution of the form:

$$y(z) = z^{k_1} u(z),$$

where k_1 is the greatest root of Equation (1.2) and $u(z)$ belongs to $H_2(\Delta)$.

This theorem obviously generalizes the well known Frobenius theory [1] for the Fuchs differential equations:

$$z^2 y''(z) + zp(z)y'(z) + q(z)y(z) = 0,$$

which is a particular case of Equation (1.1).

Denote an abstract separable Hilbert space over the complex field by H, the Hardy-Lebesque space by $H_2(\Delta)$, an ortho-normal basis in H by $\{e_n\}_{n=1}^{\infty}$, and the unilateral shift operator on H(V: $Ve_n = e_{n+1}$) by V. We can easily see that the following statements hold:

(i) Every value z in the unit disk ($|z| < 1$) is an eigenvalue of V^*(V^*: $V^*e_n = e_{n-1}$, $n \neq 1$, $V^*e_1 = 0$), the adjoint of V. The eigenelements $f_z = \sum_{n=1}^{\infty} z^{n-1} e_n$ form a complete system in H, in the sense that if f is orthogonal to f_z, for every z: $|z| < 1$ then $f = 0$.

(ii) The mapping $f(z) = (f_z, f)$, $f \in H$ is an isomorphism from H onto $H_2(\Delta)$.

(iii) The diagonal operator C_0: $C_0 e_n = n e_n$, $n = 1,2,\ldots$, has a self-adjoined extension in H with a compact inverse B: $Be_n = \frac{1}{n} e_n$, $n = 1,2,\ldots,$. Moreover, if $f(z) = (f_z, f)$ then

$$z^n f(z) = (f_z, V^n f) \quad (1.3)$$

$$f^{(n)}(z) = (f_z, (C_0 V^*)^n f) \quad (1.4)$$

$$z f'(z) = (f_z, (C_0 - I) f). \quad (1.5)$$

We shall use the proposition 1 of Reference [2].

2. PROOF OF THE THEOREM.

The transformation $y(z) = z^k u(z)$, reduces Equation (1.1) in the following:

$$z u''(z) + (h_0 + h_1 z + h_2 z^2 + \ldots) u'(z) + (\rho_0 + \rho_1 z + \rho_2 z^2 + \ldots) u(z) + \sum_{i=1}^{m} q^{ik} a_i(z) u(q^i z) = 0, \quad (2.1)$$

where $k(k - 1) + ka_0 + b_0 = 0$, $2k + a_0 = h_0$, $a_1 = h_1$, $a_2 = h_2$, $a_3 = h_3, \ldots$ and $ka_1 + b_1 = \rho_0$, $ka_2 + b_2 = \rho_1$, $ka_3 + b_3 = \rho_2, \ldots$. Following Reference [2], we define the operators $R_1, R_2, \ldots, R_m$ on $H_2(\Delta)$ as

$$R_1 u(z) = u(qz),\ |q| \le 1,\ R_2 u(z) = u(q^2 z) = R_1^2(u(z)) \ldots R_m u(z) = u(q_m z) = R_1^m u(z).$$

Thus the operator R: $Ru(z) = \sum_{i=1}^{m} q^{ik} a_i(z) u(q^i z)$, $|q| \le 1$, on $H_2(\Delta)$ is represented in the space H by the operator

$$\tilde{R}: \ \tilde{R} u = \sum_{i=1}^{m} q^{ik} a_i^*(V) (\tilde{R}_1^*)^i u$$

where $\tilde{R}_1$ is defined on H as $R_1 e_n = q^{n-1} e_n$, $n = 1,2,\ldots$. The equation (2.1) has a solution in $H_2(\Delta)$ if and only if the operator equation

$$[V(C_0 V^*)^2 + \phi_1(V) C_0 V^* + \phi_2(V) + \tilde{R}] u = 0 \quad (2.2)$$

has a solution u in the abstract separable Hilbert space H.

Here $u = \sum_{n=1}^{\infty} \overline{(u,e_n)} e_n$, $\phi_1(V) = (2k + a_0) I + h_1 V + h_2 V^2 + \ldots$,

$$\phi_2(V) = \rho_0 I + \rho_1 V + \rho_2 V^2 + \ldots ,$$

where the bar denotes complex conjugation.

Taking into account the relations

$$V^2 C_0 V^* = V(C_0 - I) \quad \text{and} \quad VC_0 - C_0 = -V,$$

Equation (2.2) can be written as

$$\left[[C_0 + (2k + a_0 - 1)I + B\phi(V) - B^2V\phi_1'(V)]\ V^* + B\phi_2(V) + B\tilde{R}\right] u = 0, \quad (2.3)$$

where

$$\phi(V) = h_1V + h_2V^2 + h_3V^3 + \dots \quad \text{and} \quad \phi_1'(V) = h_1 + 2h_2V + 3h_3V^2 + \dots .$$

Also, if we put $2k + a_0 - 1 = \delta$ in Equation (2.3), we have

$$V^*[I + VK]\ u = 0 \quad (2.4)$$

where the operator

$$K = \delta BV^* + B^2\phi(V)C_0V^* + B^2\phi_2(V) + B^2\tilde{R}$$

is compact. Relation (2.4) implies that

$$(I + VK)\ u = ce_1, \quad c = \text{const.} \quad (2.5)$$

Now it follows that the operator $(I + VK)^{-1}$ exists. In fact, $(I + VK)u = 0 \Rightarrow u = -VKu \Rightarrow (u,e_1) = -(Ku,V^*e_1) = 0$. Also,

$$(u,e_2) = -(u,K^*e_1) \Rightarrow (u,e_2)(1 + \delta) = 0 \quad (2.6)$$

Relation (2.6) if $\delta \neq -1 \Rightarrow (u,e_2) = 0$. Similarly,

$$(u,e_3) = -(uK^*e_2) \Rightarrow (u,e_3)(1 + \frac{\delta}{2}) = 0. \quad (2.7)$$

Relation (2.7) if $\delta \neq -2 \Rightarrow (u,e_3) = 0$. By the same way and if $\delta \neq -n$, $n = 1,2,\dots,$ we find

$$u = \sum_{n=1}^{\infty} (\overline{u,e_n})e_n = 0.$$

Since also the operator VK is compact Fredholm alternative implies that the operator $(I + VK)^{-1}$ is defined every where. Thus from Equation (2.5), we have

$$u = c \cdot (I + VK)^{-1}\ e_1.$$

This means that

(i) If $2k + a_0 - 1 = \delta = k_1 - k_2 \neq \pm n$ with $n = 1,2,\dots,$ then the operator $(I + VK)^{-1}$ always exists. Therefore, Equation (1.1) has two linearly independent solutions of the form

$$y_1(z) = z^{k_1}u(z) \quad \text{and} \quad y_2(z) = z^{k_2}u(z),$$

where k_1 and k_2 are the roots of Equation (1.2) and $u(z)$ belongs to $H_2(\Delta)$ and is given by the relation

$$u(z) = (u_z,u),\ u_z = \sum_{n=1}^{\infty} z^{n-1}e_n,\ u = c \cdot (I + VK)^{-1}\ e_1.$$

(ii) If $2k + a_0 - 1 = \delta = k_1 - k_2 = 0$, i.e. $k_1 = k_2$, then the operator $(I + VK)^{-1}$ always exists. Therefore, Equation (1.1) has only one solution of the form

$$y(z) = z^k u(z),$$

where k is the double root of Equation (2.1) and u(z) as in (i).

(iii) If $2k + a_0 - 1 = \delta = k_1 - k_2 = n$, $n = 1,2,\ldots$, then

$$2k_1 + a_0 - 1 = n, \quad n = 1,2,\ldots,$$

$$2k_2 + a_0 - 1 = -n, \quad n = 1,2,\ldots,$$

From the above and the Relations (2.6) and (2.7), we see that Equation (1.1) has always a solution of the form

$$y(z) = z^{k_1} u(z),$$

where k_1 is the greatest root of Equation (1.2) and u(z) as in (i). All the above complete the proof of the theorem.

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