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GENERALIZATIONS OF p-VALENT FUNCTIONS VIA THE HADAMARD PRODUCT

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ABSTRACT. The classes of univalent prestarlike functions R_α, $\alpha \geq -1$, of Ruscheweyh [1] and a certain generalization of R_α were studied recently by Al-Amiri [2]. The author studies, among other things, the classes of p-valent functions $R(\alpha + p - 1)$ where p is a positive integer and α is any integer with $\alpha + p > 0$. The author shows in particular that $R(\alpha + p) \subset R(\alpha + p - 1)$ and also obtains the radius of $R(\alpha + p)$ for the class $R(\alpha + p - 1)$.

KEY WORDS AND PHRASES. *p-valent starlike functions, p-valent close-to-convex functions, Hadamard product.*

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1. INTRODUCTION.

The classes of univalent prestarlike functions R_α, $\alpha \geq -1$, were studied by various authors [1,2]. The author extends these classes to the classes of p-valent starlike functions $R(\alpha + p - 1)$, where p is a positive integer and α is any integer greater that -p. The present studies give, along with other results, a method to determine the radius of $R(\alpha + p)$ for the class $R(\alpha + p - 1)$.

Let A_p denote the class of regular functions in the unit disc $D = \{z: |z| < 1\}$ having the power series

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad p \text{ a positive integer}, \quad z \in D. \tag{1.1}$$

We denote by $S^*(\beta)$, the subclass of A_1 whose members are starlike of order β, $0 \le \beta < 1$.

Ruscheweyh [1] introduced the following classes 'K_α' of univalent prestarlike functions:

$$K_\alpha = \{f(z) \mid f(z) \in A_1 \text{ and } \operatorname{Re} \frac{(z^\alpha f(z))^{(\alpha+1)}}{(z^{\alpha-1} f(z))^{(\alpha)}} > \frac{\alpha+1}{2}, \quad z \in D\},$$

$\alpha \in N_0 = \{0,1,2,\ldots\}$; where $F^{(n)}$ denotes the n-th derivative of the function F. As observed by Ruscheweyh, $f \in K_\alpha$ if and only if $\operatorname{Re} \frac{D^{\alpha+1} f(z)}{D^\alpha f(z)} > \frac{1}{2}$, $z \in D$ where $D^\alpha f(z) = f(z) * \frac{z}{(1-z)^{\alpha+1}}$. Here '*' denotes the Hadamard product of two regular functions, that is to say if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then $f(z)*g(z) = \sum_{n=0}^{\infty} a_n b_n z^n$.

Ruscheweyh proved that $K_{\alpha+1} \subset K_\alpha$ and $K_0 = S^*(\frac{1}{2})$. Hence for each $\alpha \in N_0$, K_α is a subclass of $S^*(\frac{1}{2})$. Recently, Al-Amiri [2] studied a certain generalization of K_α, in particular he obtained the radius of $K_{\alpha+1}$ in K_α, $\alpha \in N_0$. Further Singh and Singh [3] extended the classes K_α to the classes R_α, where

$$R_\alpha = \{f(z) \mid f(z) \in A_1 \text{ and } \operatorname{Re} \frac{D^{\alpha+1} f(z)}{D^\alpha f(z)} > \frac{\alpha}{\alpha+1}, \quad z \in D\}, \quad \alpha \in N_0.$$

They observed that R_α is a subclass of $S^*(0)$. In this note, we extend their ideas to the class of p-valent functions.

We call a function $f(z) \in A_p$ to be p-valent starlike if it satisfies $\operatorname{Re} \frac{zf'(z)}{f(z)} > 0$, $z \in D$. Further, we say that a function $f(z) \in A_p$ is p-valent close-to-convex if there exists a p-valent starlike function g(z) for which $\operatorname{Re}\left(\frac{zf'(z)}{g(z)} \right) > 0$, $z \in D$.

Let $R(\alpha + p - 1)$ denote the class of functions $f(z) \in A_p$ satisfying

$$\operatorname{Re} \left[\frac{(z^\alpha f(z))^{(\alpha+p)}}{(z^{\alpha-1} f(z))^{(\alpha+p-1)}} \right] > \alpha + p - 1, \; z \in D, \tag{1.2}$$

where α is any integer greater that -p. In Section 2 we shall show that

$$R(\alpha + p) \subset R(\alpha + p - 1). \tag{1.3}$$

Since R(0) is the class of functions which satisfy

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > p - 1 \ge 0 ,$$

it follows by our definition taken from [4] that such functions are p-valent starlike. Hence (1.3) implies that $R(\alpha + p - 1)$ contains p-valent starlike functions.

We denote by $H(\alpha + p - 1)$, the classes of functions $f(z) \in A_p$ that satisfy the condition

$$\mathrm{Re}\left[\frac{(z^{\alpha}f(z))^{(\alpha+p)} - \alpha(z^{\alpha-1}f(z))^{(\alpha+p-1)}}{(z^{\alpha-1}g(z))^{(\alpha+p-1)}}\right] > \frac{\alpha + p - 1}{\alpha + p}, \quad z \in D, \tag{1.4}$$

for some $g(z) \in R(\alpha + p - 1)$, α integer greater that $-p$.

In Section 4 we shall show that

$$H(\alpha + p) \subset H(\alpha + p - 1). \tag{1.5}$$

Again since $H(0)$ is the class of functions f that satisfy $\mathrm{Re}\,\frac{zf'(z)}{g(z)} > 0$, where g is starlike, (1.5) implies that $H(\alpha + p - 1)$ contains p-valent close-to-convex functions.

For $f \in A_p$, define

$$D^{\alpha+p-1}f(z) = f(z) * \frac{z^p}{(1 - z)^{\alpha+p}}, \tag{1.6}$$

where α is any integer greater than $-p$. Then

$$D^{\alpha+p-1}f(z) = \frac{z^p(z^{\alpha-1}f(z))^{(\alpha+p-1)}}{(\alpha + p - 1)!}. \tag{1.7}$$

It can be shown that (1.6) yields the following identity

$$z(D^{\alpha+p-1}f(z))' = (\alpha + p)D^{\alpha+p}f(z) - \alpha(D^{\alpha+p-1}f(z)). \tag{1.8}$$

From (1.2) and (1.7) it follows that a function f in A_p belongs to $R(\alpha + p - 1)$ if and only if

$$\mathrm{Re}\,\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)} > \frac{\alpha + p - 1}{\alpha + p}. \tag{1.9}$$

Note that for $p = 1$, the classes $R(\alpha + p - 1)$ reduce to the classes R_α of Singh and Singh [3]. Hence our results are generalizations of Singh and Singh.

From (1.4) and (1.7), it follows that a function f in A_p belongs to $H(\alpha + p - 1)$ if and only if

$$\mathrm{Re}\left[\frac{z(D^{\alpha+p-1}f(z))'}{D^{\alpha+p-1}g(z)}\right] > \frac{\alpha + p - 1}{\alpha + p}, \tag{1.10}$$

for some $g \in R(\alpha + p - 1)$.

In Sections 3 and 4 we shall describe some special elements of $R(\alpha + p - 1)$ and $H(\alpha + p - 1)$, respectively. These elements have integral representations. In Section 5, we introduce the classes $R_{\frac{1}{2}}(\alpha + p - 1)$ via the Hadamard product. Also the radii of $R(\alpha + p)$ in $R(\alpha + p - 1)$ and of $R_{\frac{1}{2}}(\alpha + p)$ in $R_{\frac{1}{2}}(\alpha + p - 1)$ are determined. In Section 6, the classes $R_{\frac{1}{2}}(\alpha + p - 1,\beta)$ which are extensions of the classes $R_{\frac{1}{2}}(\alpha + p - 1)$, are given. Many authors have considered a variation of these classes, notably Ruscheweyh [1], Suffridge [5], Goel and Sohi [6]. However, this note basically uses the techniques given by Al-Amiri [2].

2. THE CLASSES $R(\alpha + p - 1)$.

We shall prove the following:

THEOREM 1. $R(\alpha + p) \subset R(\alpha + p - 1)$.

PROOF. Let $f \in R(\alpha + p)$. Define $w(z)$ by

$$\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)} = \frac{\alpha + p - 1}{\alpha + p} + \frac{1}{\alpha + p}\,\frac{1 - w(z)}{1 + w(z)}\,. \tag{2.1}$$

Here $w(z)$ is a regular function in D with $w(0) = 0$, $w(z) \neq -1$ for $z \in D$. It suffices to show that $|w(z)| < 1$, $z \in D$, since then (2.1) would imply that $f \in R(\alpha + p - 1)$.

Taking logarithmic derivative of both sides of (2.1) and using the identity (1.8) the following is obtained.

$$\frac{D^{\alpha+p+1}f(z)}{D^{\alpha+p}f(z)} = \frac{1}{(\alpha + p + 1)}\left[1 + \frac{(\alpha+p) + (\alpha+p-2)w(z)}{1 + w(z)} - \frac{2zw'(z)}{(1+w(z))(\alpha+p+(\alpha+p-2)w(z))}\right]. \tag{2.2}$$

The above equation must yield $|w(z)| < 1$ for all $z \in D$, for otherwise by using a lemma of Jack [7] one can obtain $z_0 \in D$ such that $z_0w'(z_0) = Kw(z_0)$, $|w(z_0)| = 1$ and $K \geq 1$. Consequently (2.2) would yield

$$\frac{D^{\alpha+p+1}f(z_0)}{D^{\alpha+p}f(z_0)} = \frac{1}{(\alpha+p+1)} + \frac{(\alpha+p) + (\alpha+p-2)w(z_0)}{(\alpha+p+1)(1+w(z_0))} - \frac{2Kw(z_0)}{(\alpha+p+1)(1+w(z_0))}\,\frac{(\alpha+p+(\alpha+p-2)\overline{w(z_0)})}{|\alpha+p+(\alpha+p-2)w(z_0)|^2}\,.$$

Since

$$\operatorname{Re} \frac{1}{1 + w(z_0)} = \frac{1}{2}, \quad \operatorname{Re} \frac{w(z_0)}{1 + w(z_0)} = \frac{1}{2},$$

the above equation implies

$$\operatorname{Re} \frac{D^{\alpha+p+1} f(z_0)}{D^{\alpha+p} f(z_0)} \le \frac{\alpha + p}{\alpha + p + 1}.$$

This is a contradiction to the assumption that $f \in R(\alpha + p)$. Hence $f \in R(\alpha + p - 1)$. This completes the proof of Theorem 1.

3. SPECIAL ELEMENTS OF $R(\alpha + p - 1)$.

In this section we form special elements of the classes $R(\alpha + p - 1)$ via the Hadamard product of elements of $R(\alpha + p - 1)$ and $h_\gamma(z)$, where

$$h_\gamma(z) = \sum_{j=p}^{\infty} \frac{\gamma + p}{\gamma + j} z^j, \quad \operatorname{Re} \gamma > -p.$$

THEOREM 2. Let $f \in A_p$ satisfy the condition

$$\operatorname{Re} \frac{D^{\alpha+p} f(z)}{D^{\alpha+p-1} f(z)} > \frac{2(\gamma+p-1)(\alpha+p-1) - 1}{2(\alpha+p)(\gamma+p-1)}, \quad z \in D, \tag{3.1}$$

p a positive integer, α any integer greater than $-p$ and $\gamma \ge -p + 2$.

Then

$$F(z) = f(z) * h_\gamma(z) = \frac{\gamma + p}{z^\gamma} \cdot \int_0^z t^{\gamma-1} f(t)\,dt \tag{3.2}$$

belongs to $R(\alpha + p - 1)$.

PROOF. Let $f \in A_p$ satisfy the condition (3.1). From (3.2) we obtain

$$z(D^{\alpha+p}F(z))' + \gamma(D^{\gamma+p}F(z)) = (p+\gamma)D^{\alpha+p}f(z), \tag{3.3}$$

and

$$z(D^{\alpha+p-1}F(z))' + \gamma(D^{\alpha+p-1}F(Z)) = (p+\gamma)D^{\alpha+p-1}f(z). \tag{3.4}$$

Define $w(z)$ by

$$\frac{D^{\alpha+p}F(z)}{D^{\alpha+p-1}F(z)} = \frac{\alpha + p - 1}{\alpha + p} + \frac{1}{\alpha + p} \cdot \frac{1 - w(z)}{1 + w(z)}. \tag{3.5}$$

Here $w(z)$ is a regular function in D with $w(0) = 0$, $w(z) \neq -1$ for $z \in D$. It suffices to show that $|w(z)| < 1$, $z \in D$.

Taking the logarithmic derivative of (3.5) and using (1.8) for $F(z)$ one can get

$$z(D^{\alpha+p}F(z))' = D^{\alpha+p}F(z) \cdot \left[(\alpha+p)\frac{D^{\alpha+p}F(z)}{D^{\alpha+p-1}F(z)} - \alpha - \frac{2zw'(z)}{(1+w(z))(\alpha+p+(\alpha+p-2)w(z))} \right]. \tag{3.6}$$

Now (3.3) and (3.6) yield

$$(p+\gamma)D^{\alpha+p}f(z) = D^{\alpha+p}F(z) \cdot \left[\gamma-\alpha + \frac{(\alpha+p)+(\alpha+p-2)w(z)}{1+w(z)} - \frac{2zw'(z)}{(1+w(z))(\alpha+p+(\alpha+p-2)w(z))} \right]. \tag{3.7}$$

Use (3.4) and (1.8) to eliminate the derivative and then apply (3.5) to get

$$(p+\gamma)D^{\alpha+p-1}f(z) = D^{\alpha+p-1}F(z) \cdot \left[\gamma - \alpha + \frac{(\alpha+p) + (\alpha+p-2)w(z)}{1+w(z)} \right]. \tag{3.8}$$

Therefore (3.7), (3.8) and (3.5) give

$$\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)} = \frac{\alpha+p-1}{\alpha+p} + \frac{1}{\alpha+p}\,\frac{1-w(z)}{(1+w(z))} - \frac{2zw'(z)}{(\alpha+p)(1+w(z))}\,\frac{(\gamma+p)+(\gamma+p-2)\overline{w(z)}}{|\gamma+p+(\gamma+p-2)w(z)|^2}. \tag{3.9}$$

Equation (3.9) should yield $|w(z)| < 1$ for all $z \in D$, for otherwise by Jack's lemma there exists $z_0 \in D$ with $z_0w'(z_0) = Kw(z_0)$, $K \geq 1$, and $|w(z_0)| = 1$. Applying this to (3.9) it follows that

$$\operatorname{Re}\left[\frac{D^{\alpha+p}f(z_0)}{D^{\alpha+p-1}f(z_0)}\right] \leq \frac{\alpha+p-1}{\alpha+p} - \frac{2}{(\alpha+p)}\,\frac{\gamma+p-1}{4(\gamma+p-1)^2} = \frac{2(\gamma+p-1)(\alpha+p-1)-1}{2(\alpha+p)(\gamma+p-1)}.$$

This contradicts the assumption on f given by (3.1). Hence $F \in R(\alpha+p-1)$. This completes the proof of Theorem 2.

REMARK 1. For $\gamma = 1$ and $p = 1$, Theorem 2 reduces to a result given in [3].

The following special cases of Theorem 2 represent some improvement on theorems due to Libera [8] in the sense that much weaker assumptions produce the same results.

By taking $\alpha = 0$, $p = 1$ in Theorem 2 we get

COROLLARY 1. Let $f \in A_1$ be such that $\operatorname{Re}\frac{zf'(z)}{f(z)} > \frac{-1}{2\gamma}$, $\gamma \geq 1$, $z \in D$. Then F is starlike in D, where

$$F(z) = \frac{\gamma + 1}{z^{\gamma}} \cdot \int_0^z t^{\gamma-1} f(t)\,dt. \tag{3.10}$$

For $\alpha = 1$, $p = 1$, Theorem 2 reduces to

COROLLARY 2. Let $f \in A_1$ be such that $\text{Re}[1 + \frac{zf''(z)}{f'(z)}] > -\frac{1}{2\gamma}$, $\gamma \geq 1$, $z \in D$. Then $F(z)$ as given in (3.10) above is convex in D.

Using the technique employed in the proof of Theorem 1 and Corollary 2 we obtain the following result.

COROLLARY 3. Let $f \in A_1$ be such that $\text{Re}\,\frac{f'(z)}{g'(z)} > 0$, $z \in D$ and g be such that $\text{Re}[1 + \frac{zg''(z)}{g'(z)}] > -\frac{1}{2\gamma}$, $\gamma \geq 1$, $z \in D$. Then $F(z)$ as given by (3.10), is close-to-convex, i.e., $\text{Re}\,\frac{F'(z)}{G'(z)} > 0$, $z \in D$ and where $G(z)$ is the convex function given by

$$G(z) = \frac{\gamma + 1}{z^{\gamma}} \cdot \int_0^z t^{\gamma-1} g(t)\,dt.$$

We state without proof the following theorem since its method of proof is similar to that of Theorem 2.

THEOREM 3. Let p be a positive integer and α be an integer greater than $-p$ and let $\text{Re}\,\gamma \geq -p + 1$. Then $F(z) = f(z)*h_{\gamma}(z)$, as given by (3.2), belongs to $R(\alpha + p - 1)$ for all $f \in R(\alpha + p - 1)$.

In case $\gamma = \alpha$, Theorem 3 can be improved as follows:

THEOREM 4. Let p be a positive integer, and α be any integer greater than $-p$. Then for $f(z) \in R(\alpha + p - 1)$,

$$F(z) = f(z)*h_{\alpha}(z) = \frac{p + \alpha}{z^{\alpha}} \cdot \int_0^z t^{\alpha-1} f(t)\,dt \in R(\alpha + p). \tag{3.11}$$

PROOF. Let $f(z) \in R(\alpha + p - 1)$. Differentiating (3.11) and then applying the operators $D^{\alpha+p}$, $D^{\alpha+p-1}$ we get, respectively, by using (1.8)

$$(\alpha + p) \cdot D^{\alpha+p} f(z) = (\alpha + p + 1) D^{\alpha+p+1} F(z) - D^{\alpha+p} F(z)$$

and

$$(\alpha + p) D^{\alpha+p-1} f(z) = (\alpha + p) D^{\alpha+p} F(z).$$

Therefore

$$\text{Re}\left[\frac{\alpha + p + 1}{\alpha + p} \frac{D^{\alpha+p+1}F(z)}{D^{\alpha+p}F(z)} - \frac{1}{\alpha + p}\right] = \text{Re}\,\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)} > \frac{\alpha + p - 1}{\alpha + p}.$$

This implies that

$$\text{Re}\ \frac{D^{\alpha+p+1}F(z)}{D^{\alpha+p}F(z)} > \frac{\alpha + p}{\alpha + p + 1}, \quad z \in D.$$

Hence $F(z) \in R(\alpha + p)$, and this completes the proof of Theorem 4.

REMARK 2. For $p = 1$, Theorem 4 reduces to a result of Singh and Singh [3].

4. THE CLASSES $H(\alpha + p - 1)$.

We state without proof Theorems 5 and 6 since their proofs use the same technique employed in Theorem 1. See Section 1 for the definition of the classes $H(\alpha + p - 1)$.

THEOREM 5. $H(\alpha + p) \subset H(\alpha + p - 1)$.

THEOREM 6. If p is any positive integer, α is any integer greater than $-p$, and $\text{Re}\ \gamma \geq -p + 1$, then

$$F(z) = f(z)*h_\gamma(z) = \frac{p + \gamma}{z^\gamma} \cdot \int_0^z t^{\gamma-1} f(t)dt \in H(\alpha + p - 1)$$

whenever $f(z) \in H(\alpha + p - 1)$.

5. RADII OF THE CLASSES $R(\alpha + p)$ AND $R_{\frac{1}{2}}(\alpha + p)$.

Because discussing the problem concerning the radii of the classes $R(\alpha + p)$ and $R_{\frac{1}{2}}(\alpha + p)$ we define the classes $R_{\frac{1}{2}}(\alpha + p - 1)$. $R_{\frac{1}{2}}(\alpha + p - 1)$ contains functions $f(z) \in A_p$ that satisfy the condition

$$\text{Re}\left[\frac{(z^\alpha f(z))^{(\alpha+p)}}{(z^{\alpha-1} f(z))^{(\alpha+p-1)}}\right] > \frac{\alpha + p}{2}, \quad z \in D, \tag{5.1}$$

where α is any integer greater than $-p$. These classes have been studied by Goel and Sohi [6].

From (1.7) and (5.1), it follows that a function f in A_p belongs to $R_{\frac{1}{2}}(\alpha + p - 1)$ if and only if

$$\text{Re}\ \frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)} > \frac{1}{2}. \tag{5.2}$$

Our main interest is to determine the radius of the largest disc $D(r) = \{z: |z| < r\}$, $0 < r \leq 1$ so that if $f \in R(\alpha + p - 1)$ then $\text{Re}\ \frac{D^{\beta+p}f(z)}{D^{\beta+p-1}f(z)} > \frac{\beta + p - 1}{\beta + p}$, $\beta > \alpha$, z $D(r)$. A partial answer to this problem can be deduced by a simple application of a lemma due to (Ruscheweyh and Singh) [9]:

LEMMA 1. If $p(z)$ is an analytic function in the unit disc D with $p(0) = 1$, $\operatorname{Re} p(z) > 0$ and also

$$|z| < \frac{|\mu + 1|}{[A + (A^2 - |\mu^2 - 1|^2)^{\frac{1}{2}}]^{\frac{1}{2}}}, \tag{5.3}$$

$$A = 2(S + 1)^2 + |\mu|^2 - 1.$$

Then we have

$$\operatorname{Re}\left[p(z) + S\frac{zp'(z)}{p(z) + \mu}\right] > 0.$$

The bound given by (5.3) is best possible.

THEOREM 7. Let p be any positive integer, α any integer greater than $-p$. If $f(z) \in R(\alpha + p - 1)$ then

$$\operatorname{Re}\frac{D^{\alpha+p+1}f(z)}{D^{\alpha+p}f(z)} > \frac{\alpha + p}{\alpha + p + 1} \quad \text{for} \quad |z| < r_{\alpha,p},$$

where

$$r_{\alpha,p} = \frac{\alpha + p}{2 + \sqrt{3 + (\alpha+p-1)^2}}. \tag{5.4}$$

This result is sharp.

PROOF. Let $f(z) \in R(\alpha + p - 1)$. We define the regular function $q(z)$ on D by

$$\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)} = \frac{1}{(\alpha + p)}(q(z) + \alpha + p - 1), \quad z \in D. \tag{5.5}$$

Therefore $q(0) = 1$ and $\operatorname{Re} q(z) > 0$ in D.

Taking logarithmic derivative of (5.5) and using (1.8) we get

$$\frac{D^{\alpha+p+1}f(z)}{D^{\alpha+p}f(z)} - \frac{\alpha + p}{\alpha + p + 1} = \frac{1}{(\alpha+p+1)}\left[q(z) + \frac{zq'(z)}{q(z) + \alpha + p - 1}\right]. \tag{5.6}$$

Using Lemma (1) with $S = 1$, $\mu = \alpha + p - 1$, (5.6) and (5.3) show that

$$\operatorname{Re}\left[\frac{D^{\alpha+p+1}f(z)}{D^{\alpha+p}f(z)}\right] > \frac{\alpha + p}{\alpha + p + 1} \quad \text{for}$$

$$|z| < \frac{\alpha + p}{[A + (A^2 - ((\alpha+p-1)^2 - 1)^2)^{\frac{1}{2}}]^{\frac{1}{2}}}, \tag{5.7}$$

where

$$A = (\alpha + p)^2 - 2(\alpha + p) + 8.$$

Minor computations yield the following:

$$A + (A^2 - ((\alpha+p-1)^2 - 1)^2)^{\frac{1}{2}} = (2 + \sqrt{3 + (\alpha+p-1)^2}\,)^2. \tag{5.8}$$

Thus (5.7) yields the radius $r_{\alpha,p}$ as given by (5.4).

The method of Al-Amiri [2] is used to determine the extremal functions. The extremal functions thus obtained for this theorem are rotations of f(z) where f(z) is given by

$$\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)} = \frac{1}{(\alpha + p)}\left[\frac{1 + z}{1 - z} + \alpha + p - 1\right], \quad z \in D.$$

This completes the proof of Theorem 7.

REMARK 3. For $\alpha = 0$, $p = 1$, Theorem 7 gives the well-known radius of convexity for the class of starlike functions: $r_{0,1} = 2 - \sqrt{3}$.

Now an easy modification of the method used by Al-Amiri [2, Theorem 4] gives the following result.

THEOREM 8. Let p be any positive integer, α any integer greater than $-p$. If $f(z) \in R_{\frac{1}{2}}(\alpha + p - 1)$, then f(z) satisfies (5.2) with α replaced by $\alpha + 1$ for $|z| < r_{\alpha,p}$ where

$$r_{\alpha,p} = \left[\frac{(\alpha + p - 1) + 2(\alpha + p + 2)^{\frac{1}{2}}}{(\alpha + p + 3) + 2(\alpha + p + 2)^{\frac{1}{2}}}\right]^{\frac{1}{2}}$$

This result is sharp.

REMARK 4. For $p = 1$, Theorem 8 becomes a special case of a result due to Al-Amiri [2, Theorem 4].

6. THE CLASSES $R_{\frac{1}{2}}(\alpha + p - 1,\beta)$.

By $R_{\frac{1}{2}}(\alpha + p - 1,\beta)$, we denote the classes of functions $f(z) \in A_p$ that satisfy

$$\text{Re}\left[(1 - \beta)\,\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)} + \beta\,\frac{D^{\alpha+p+1}f(z)}{D^{\alpha+p}f(z)}\right] > \frac{1}{2}, \quad z \in D, \tag{6.1}$$

for some $\beta \geq 0$, p any positive integer and α any integer greater than $-p$. Again using the technique employed in [2], the following theorem is obtained.

THEOREM 9. Let p be any positive integer, α any integer greater than $-p$. If $f(z) \in R_{\frac{1}{2}}(\alpha + p - 1)$, then f(z) satisfies (6.1) for $|z| < r_{\alpha,p,\beta}$ where

$$r_{\alpha,p,\beta} = \left[\frac{(\alpha + p + 1 - 2\beta) + 2(\beta(\alpha + p + 1 + \beta))^{\frac{1}{2}}}{(\alpha + p + 1 + 2\beta) + 2(\beta(\alpha + p + 1 + \beta))^{\frac{1}{2}}} \right]^{\frac{1}{2}} .$$

This result is sharp.

REMARK 5. For $\beta = 1$, Theorem 9 reduces to Theorem 8. Also for $p = 1$, Theorem 9 represents a special case of a theorem due to Al-Amiri [2, Theorem 8].

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