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SOLUTIONS OF SINGULAR INTEGRAL EQUATIONS

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ABSTRACT. Qualitative behavior of solutions of possibly singluar integral equations is studied. It includes properties such as positivity, boundedness and monotonicity of the solutions of the infinite interval.

KEY WORDS AND PHRASES. *Singular integral equations, Positivity of solutions, and Boundedness and monotonicity of solutions.*

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES.

1. INTRODUCTION.

Both qualitative and quantitative analyses of solutions to integral equations of Volterra type have been done in the past; see, for example, [1-11]. The purpose of this paper is to extend some of the results in [5] on properties of solutions to integral equations of the form

$$f(t) = 1 - \int_0^t K(t - \tau)f(\tau)d\tau \tag{1.1}$$

to cases where the kernel K(t) or its derivatives may be infinite at the origin. It includes properties such as positivity, boundedness and monotonicity of the solution on the infinite interval.

2. DECREASING KERNELS

In this section, properties of the solutions to (1.1) with the possibility of the kernel being monotonically decreasing are studied.

THEOREM 2.1. If (1) $K(0) = a \leq 0$ and (2) $4\overline{K}' \leq a^2$, where $\overline{K}'$ is a constant satisfying $K'(t) \leq \overline{K}'$ for all $t > 0$ ($K'(0)$ needs not be finite), then $f^{(n)}(t) > 0$ for $t > 0$, $n = 0,1,2$.

PROOF. The equation under consideration is $f = 1 - k * f$. By theorem 1.1.1 in [5], f also satisfies

$$f(t) = e^{-\gamma t} - L * f, \tag{2.1}$$

where γ is any constant and

$$L(t) = (a - \gamma)e^{-\gamma t} + K' * e^{-\gamma t}. \tag{2.2}$$

If $a \leq \gamma$, then $L(0) \leq 0$. Differentiation of (2.2) leads to

$$L'(t) = (\gamma^2 - a\gamma)e^{-\gamma t} + K' - \gamma e^{-\gamma t} * K'. \tag{2.3}$$

If $\gamma < 0$, then

$$\begin{aligned} L'(t) &\leq (\gamma^2 - a\gamma)e^{-\gamma t} + \overline{K}' - \gamma\overline{K}' \int_0^t e^{-\gamma\tau}\, d\tau \\ &= (\gamma^2 - a\gamma + \overline{K}')e^{-\gamma t} \end{aligned} \tag{2.4}$$

The polynomial in (2.3) is minimal for $\gamma = \frac{a}{2}$. Taking $\gamma = \frac{a}{2}$, (2.4) becomes

$$\begin{aligned} L'(t) &\leq (-\frac{a^2}{4} + \overline{K}')e^{-\gamma t} \\ &\leq 0, \end{aligned}$$

and so $L(t) \leq 0$ for all $t \geq 0$.

Differentiation of (2.1) leads to

$$f'(t) = -\gamma e^{-\gamma t} - L(t) - L * f'$$

and

$$f''(t) = \gamma^2 e^{-\gamma t} - L'(t) + aL(t) - L * f''.$$

From the Neumann Series [12], we conclude that $f^{(n)}(t) > 0$ for all $t > 0$, $n = 0,1,2$.

The following theorem on positive decreasing kernels has been obtained in [5].

THEOREM A. If (1) $K(t) > 0$ on $(0,\infty)$ ($K(0)$ needs not be finite) and (2) $K'(t) < 0$ on $(0,\infty)$, then $f(t) > 0$ for $t > 0$.

Under different conditions, monotonicity of the solution can be obtained, using Theorem 2.1.

THEOREM 2.2 If (1) $K(0) = a \geq 0$, (2) $K'(t) < 0$ for $0 \leq t \leq T$ ($K'(0)$ needs not be finite) and (3) $\overline{K'} \leq aK(T) - a^2$, where $\overline{K'}$ is a constant satisfying $K'(t) \leq \overline{K'}$ for $0 \leq t \leq T$, then (1) $f(t) > 0$ and (2) $f'(t)$ has at most one zero on $[0,T]$.

PROOF. Consider the fundamental equation corresponding to (2.1), namely,

$$h(t) = 1 - L * h. \tag{2.5}$$

If $a \leq \gamma$, then $L(0) \leq 0$. If γ is positive, then from (2.3),

$$\begin{aligned} L'(t) &\leq (\gamma^2 - a\gamma) + \overline{K'} - \gamma \int_0^t K'(\tau)d\tau \\ &= (\gamma^2 - a\gamma) + \overline{K'} - \gamma K(t) + a\gamma \\ &\leq \gamma^2 - K(T)\gamma + \overline{K'}. \end{aligned} \tag{2.6}$$

Let the bound on $L'(t)$, in (2.6), be denoted by $\overline{L'}$. To apply Theorem 2.1, γ must be chosen so that $a \leq \gamma$ and $4\overline{L'} \leq L^2(0)$. The last inequality leads to

$$4(\gamma^2 - K(T)\gamma + \overline{K'}) \leq (a - \gamma)^2$$

or

$$3\gamma^2 + 2[a - 2K(T)]\gamma + (4\overline{K'} - a^2) \leq 0. \tag{2.7}$$

Let the polynomial in (2.7) be denoted by $p(\gamma)$. The roots of $p(\gamma) = 0$ are

$$\gamma = \frac{[2K(T) - a] \pm 2\sqrt{a^2 - aK(T) + K^2(T) - 3\overline{K'}}}{3}. \tag{2.8}$$

Let the large and small roots in (2.8) be denoted by γ_+ and γ_- respectively. Clearly, $\gamma_- < a$, and in requiring that $a \leq \gamma_+$, we obtain

$$3a \le 2K(T) - a + 2\sqrt{a^2 - aK(T) + K^2(T) - 3\overline{K'}}$$
$$[2a - K(T)]^2 \le a^2 - aK(T) + K^2(T) - 3\overline{K'},$$
$$a^2 - aK(T) + \overline{K'} \le 0.$$

Therefore at $\gamma = a$, $L(0) \le 0$ and $p(\gamma) \le 0$. By Theorem 2.1, $h^{(n)}(t) > 0$ for $0 < t \le T$, $n = 0,1,2$.

By a convolution theorem [13], the solutions f and h are related by

$$f(t) = e^{-\gamma t} + e^{-\gamma t} * h', \tag{2.9}$$

from which it follows that $f(t) > 0$ for $0 \le t \le T$. Differentiation of (2.9) leads to

$$f'(t) = -\gamma e^{-\gamma t} + h'(t) - \gamma e^{-\gamma t} * h', \tag{2.10}$$

and from (2.9) and (2.10), we obtain

$$\gamma f(t) + f'(t) = h'(t),$$

so

$$\gamma f'(t) + f''(t) = h''(t). \tag{2.11}$$

Since $f'(0) = -a \le 0$ and $h''(t) > 0$, it follows from (2.11) that f' has at most one zero.

3. INCREASING KERNELS

Equations of the form (1.1) with monotonically increasing kernels are now studied. A result from [5] will be stated first.

THEOREM B. If (1) $K(t) > 0$, $K'(t) > 0$ and $K''(t) \le 0$ for $0 \le t < \infty$ and (2) $4b \le a^2$, where $a = K(0)$ and $b = K'(0)$, then $|f(t)| \le 1$ for $0 \le t < \infty$.

Boundedness of the solutions for the next class of increasing kernels can be obtained, using Theorem B.

THEOREM 3.1. If (1) $K(t) > 0$, $K'(t) > 0$, $K''(t) > 0$ and $K'''(t) \le 0$ for $0 \le t < \infty$, (2) $a^2 < 4b$, (3) $3b \le a^2$ and (4) $2a^3 - 9ab + 27c \le 0$, where $a = k(0)$, $b = K'(0)$ and $c = K''(0)$, then $|f(t)| \le 2$ for $0 \le t < \infty$.

PROOF. The equation is $f(t) = 1 - K * f$. As before, let h be the solution to the fundamental equation (2.5) corresponding to the equivalent equation (2.1). Differentiation of (2.2) leads to

$$L'(t) = (\gamma^2 - a\gamma + b)e^{-\gamma t} + K'' * e^{-\gamma t} \tag{3.1}$$

and

$$L''(t) = (-\gamma^3 + a\gamma^2 - b\gamma + c)e^{-\gamma t} + K''' * e^{-\gamma t}. \tag{3.2}$$

Since $a^2 < 4b$, $L'(t) > 0$ for any γ. The requirement that $4L'(0) \le L^2(0)$ leads to

$$4(\gamma^2 - a\gamma + b) \le (a - \gamma)^2$$

or

$$3\gamma^2 - 2a\gamma + (4b - a^2) \le 0. \tag{3.3}$$

Let the polynomial in (3.3) be $p(\gamma)$. The discriminant of $p(\gamma) = 0$ is $16(a^2 - 3b) \ge 0$ and the vertex of $p(\gamma)$ is at $\gamma = \frac{a}{3}$, so $p(\frac{a}{3}) \le 0$. Taking $\gamma = \frac{a}{3}$, $L(t) > 0$ for all t, and

$$\begin{aligned} -\gamma^3 + a\gamma^2 - b\gamma + c &= \frac{1}{27}(2a^3 - 9ab + 27c) \\ &\le 0, \end{aligned}$$

so from (3.2), $L''(t) \le 0$. By Theorem B, $|h(t)| \le 1$ for $0 \le t < \infty$.

By the convolution theorem in [13], f(t) and h(t) are related by

$$f(t) = h(t) - e^{-\gamma t} * h,$$

so

$$\begin{aligned} |f(t)| &\le 1 + \gamma \int_0^t e^{-\gamma\tau}\, d\tau \\ &= 1 - (e^{-\gamma t} - 1) \\ &\le 2. \end{aligned}$$

The case of $4b \le a^2$ is examined in the following theorem.

THEOREM 3.2. If (1) $K(t) > 0$, $K'(t) > 0$, $K''(t) > 0$ and $K'''(t) \le 0$ for

$0 \leq t < \infty$, (2) $\frac{1}{3} a < \frac{a - \sqrt{a^2 - 4b}}{2}$ and (3) $2a^3 - 9ab + 27c \leq 0$, where $a = K(0)$, $b = K'(0)$, $c = K''(0)$, then $|f(t)| \leq 2$ for $0 \leq t < \infty$.

PROOF. As in Theorem 3.1, taking $\gamma = \frac{a}{3}$ leads to $4L'(0) \leq L^2(0)$, $L(t) > 0$ and $L''(t) \leq 0$. But since $4b \leq a^2$ now, condition (2) in this theorem guarantees that $L'(t) > 0$. The proof can be complete as before.

The next class of increasing kernels can be studied accordingly.

THEOREM 3.3. If (1) $K(t) > 0$, $K'(t) > 0$, $K''(t) > 0$, $K'''(t) > 0$ and $K^{(4)}(t) \leq 0$ for $0 \leq t < \infty$, (2) $a^2 < 3b$, (3) $\frac{8}{3} b \leq a^2$, (4) $a^3 - 4ab + 8c \leq 0$ and (5) $-3a^4 + 16a^2b - 64ac + 256d \leq 0$, where $a = K(0)$, $b = K'(0)$, $c = K''(0)$ and $d = K'''(0)$, then $|f(t)| \leq 4$ for $0 \leq t < \infty$.

PROOF. Differentiation of (3.2) leads to

$$L'''(t) = (\gamma^4 - a\gamma^3 + b\gamma^2 - c\gamma + d)e^{-\gamma t} + K^{(4)} * e^{-\gamma t}. \tag{3.4}$$

The requirement that $3L'(0) \leq L^2(0)$ in Theorem 3.1 leads to

$$3(\gamma^2 - a\gamma + b) \leq (a - \gamma)^2$$

or

$$2\gamma^2 - a\gamma + (3b - a^2) \leq 0. \tag{3.5}$$

Let the polynomial in (3.5) be $p(\gamma)$. Since the discriminant of $p(\gamma) = 0$ is $3(3a^2 - 8b) \geq 0$, we have $p(\frac{a}{4}) \leq 0$. In order for the condition (4) in Theorem 3.1 to be satiisfied, we must have

$$2(a - \gamma)^3 - 9(a - \gamma)(\gamma^2 - a\gamma + b) + 27(-\gamma^3 + a\gamma^2 - b\gamma + c) \leq 0,$$

$$2(a^3 - 3a^2\gamma + 3a\gamma^2 - \gamma^3) - 9a(\gamma^2 - a\gamma + b) + 18(-\gamma^3 + a\gamma^2 - b\gamma) + 27c \leq 0$$

or

$$-20\gamma^3 + 15a\gamma^2 + 3(a^2 - 6b)\gamma + 2a^3 - 9ab + 27c \leq 0. \tag{3.6}$$

Let the polynomial in (3.6) be $q(\gamma)$. Then

$$q(\tfrac{a}{4}) = -\frac{5}{16}a^3 + \frac{15}{16}a^3 + 3(a^2 - 6b)\frac{a}{4} + 2a^3 - 9ab + 27c$$

$$= \frac{27}{8}a^3 - \frac{27}{2}ab + 27c$$

$$= \frac{27}{8}(a^3 - 4ab + 8c)$$

$$\leq 0.$$

Note that $a^3 - 4ab + 8c \leq 0$ implies that $a^2 - 4b \leq 0$. The remaining inequality required in Theorem (3.1) leads to

$$(a - \gamma)^2 < 4(\gamma^2 - a\gamma + b)$$

or

$$3\gamma^2 - 2a\gamma + (4b - a^2) > 0. \tag{3.7}$$

Let the polynomial in (3.7) be $r(\gamma)$. The discriminant of $r(\gamma) = 0$ is $16(a^2 - 3b) < 0$, so $r(\gamma) > 0$ for any γ.

Taking $\gamma = \frac{a}{4}$, we have from (2.2) that $L(t) > 0$. Since $a^2 \leq 4b$, (3.1) leads to $L'(t) \geq 0$. If $\gamma = \frac{a}{4}$, then

$$-\gamma^3 + a\gamma^2 - b\gamma + c = -\frac{a^3}{64} + \frac{a^3}{16} - \frac{ab}{4} + c$$

$$= \frac{1}{64}(3a^3 - 16ab + 64c)$$

$$> 0$$

by condition (5) in this theorem, and

$$\gamma^4 - a\gamma^3 + b\gamma^2 - c\gamma + d = \frac{a^4}{256} - \frac{a^4}{64} + \frac{a^2b}{16} - \frac{ac}{4} + d$$

$$= -\frac{3a^4}{256} + \frac{a^2b}{16} - \frac{ac}{4} + d$$

$$= \frac{1}{256}(-3a^4 + 16a^2b - 64ac + 256d)$$

$$\leq 0.$$

It follows from (3.2) and (3.4) that $L''(t) > 0$ and $L'''(t) \leq 0$. By theorem (3.1), $|h(t)| \leq 2$ for $0 \leq t < \infty$.

As before, f(t) and h(t) are related by

$$f(t) = h(t) - \gamma e^{-\gamma t} * h$$

and so

$$|f(t)| \leq 2 + 2\gamma \int_0^t e^{-\gamma\tau}\, d\tau$$

$$= 2 - 2(e^{-\gamma t} - 1)$$

$$\leq 4.$$

The case of $3b \leq a^2$ is examined in the following theorem.

THEOREM 3.4. If (1) $K(t) > 0$, $K'(t) \geq 0$, $K''(t) > 0$, $K'''(t) > 0$ and $K^{(4)}(t) \leq 0$ for $0 \leq t < \infty$, (2) $\frac{a}{4} < \frac{a - 2\sqrt{a^2 - 3b}}{3}$, (3) $a^3 - 4ab + 8c \leq 0$ and (4) $-3a^4 + 16a^2b - 64ac + 256d \leq 0$, where $a = K(0)$, $b = K'(0)$, $c = K''(0)$ and $d = K'''(0)$, then $|f(t)| \leq 4$ for $0 \leq t < \infty$.

PROOF. As in the proof of Theorem 3.3, the choice of $\gamma = \frac{a}{4}$ satisfies conditions (3) and (4) in Theorem 3.1. The zeroes of the polynomial $r(\gamma)$ in (3.7) are at

$$\gamma = \frac{a - 2\sqrt{a^2 - 3b}}{3},$$

so $r(\frac{a}{4}) > 0$ and the remaining inequality (2) of Theorem 3.1 is true.

The rest of the proof is the same as that of Theorem 3.3.

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