

RATIONAL MODULES AND HIGHER ORDER CAUCHY TRANSFORMS

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ABSTRACT. We apply the higher order Cauchy transforms to describe the closures of rational modules with respect to the L^p norms, the uniform norm and different Lipschitz norms on a compact set in the plane.

KEY WORDS AND PHRASES. Rational Module, Cauchy Transform

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1. INTRODUCTION.

Let X be a compact subset of the complex plane. Let the module $R(X)\overline{P}_m$ be the space $R + R\overline{z} + \dots + R\overline{z}^m$

$$= \{r_0(z) + r_1(z)\overline{z} + \dots + r_m(z)\overline{z}^m\},$$

where each r_i is a rational function with poles off X .

In recent years the problem of approximation by rational functions in the areal mean has received great attention (see e.g. [1], [3], [8]) as well as the problem of uniform approximation (see [5], [12]), but little work has been devoted to the corresponding problem of approximation by functions in a rational module due to the lack of analyticity, perhaps. The concept of rational modules arises in a natural fashion when one attempts to study rational approximation in Lipschitz norms. In [10], O'Farrell studied the relation of the problems of approximation by rational modules in different Lipschitz norms, and in the uniform norms, etc., to one another. Not long ago the author proved in [13] that $R(X)\overline{P}_1$

is dense in $L^p(X)$ for all $1 \leq p < \infty$ and $R(X)\overline{P}_2$ is dense in $C(X)$ if X has no interior. For an introductory survey of rational modules and all the standard notations, we refer the readers to the paper of O'Farrell [10].

However, the presence of an interior really complicates the situation. Let $L_a^p(X)$ be the closed subspace of $L^p(X)$ which consists of functions analytic in the interior of X . In this note we employ a method of proof which goes back to Brennan [4] to describe the closures of $R(X)\overline{P}_m$ in $L^p(X)$ and $C(X)$ respectively in terms of the Cauchy transforms of functions in $L_a^p(X)$ for an arbitrary compact subset X .

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2. PRELIMINARIES.

The following notations will be used. If X is a compact set, $\overset{\circ}{X}$ will be its interior, ∂X its topological boundary. If V is any space of functions on X , $[V]_p$ will be the closure of V with respect to the $L^p(X)$ norm and $[V]_u$ the uniform closure of V in $C(X)$.

Let μ be a (finite Borel) measure on X . The Cauchy transform $\hat{\mu}$ is defined by

$$\hat{\mu}(z) = \int \frac{d\mu(\zeta)}{\zeta - z}.$$

Some basic properties for $\hat{\mu}$ can be found in [6]. If g is a function on X , we will write \hat{g} for \widehat{gdm} , where dm denotes the two-dimensional Lebesgue measure.

We write $\mu \perp V$ if $\int f d\mu = 0$ for all f in V and $g \perp V$ if $gdm \perp V$.

The following lemmas play important roles in this theory. Lemma 1 is a special case of the key lemma in [10], and lemma 2 is used by the author in [13].

Throughout this paper, m will be a non-negative integer.

LEMMA 1. Let μ be a measure on X . Then $\mu \perp R(X)\overline{P}_{m+1}$ if and only if $\hat{\mu} \perp R(X)\overline{P}_m$.

LEMMA 2. If $g \in L^p(X)$, then g is continuous when $p > 2$ and \hat{g} is continuous when $1 < p \leq 2$.

If V is a space of functions on X , we put $\hat{V} = \{\hat{f} : f \in V\}$. Notice that f is regarded to be zero off X . By induction, we can define the n th Cauchy transform:

$\hat{V}_1 = \hat{V}$, $\hat{V}_m = (\hat{V}_{m-1})^\wedge$, $m = 2, 3, 4, \dots$. In general, we don't have $V \subset \hat{V}$, nor $V \supset \hat{V}$, on X .

REMARK: As a consequence of Lemma 1, it is not hard to see that

$$[R(X)\overline{P}_m]_p = [R(X) + R(X)^\wedge + \dots + R(X)_m^\wedge]_p \text{ and } [R(X)\overline{P}_m]_u = [R(X) + R(X)^\wedge + \dots + R(X)_m^\wedge]_u \text{ for all } m \geq 1, 1 \leq p < \infty.$$

3. MAIN RESULTS.

It is well known that $[R(X)]_p = L_a^p(X)$ for all $1 \leq p < 2$. This is due to Sinanjan [11] in the case of no interior, to Bers [2] for $p = 1$ (and open sets), and to Havin [7] for $1 < p < 2$. For $p \geq 2$, however, $R(X)$ is not always dense in $L_a^p(X)$. Some necessary and sufficient conditions are given by Brennan [3], Hedberg [8], etc., in terms of various capacities.

In this section, we will show that the higher order Cauchy transforms of $R(X)$ and $L_a^p(X)$ always have the same closure in $L^p(X)$ (modulo $R(X)$), for all $1 \leq p < \infty$.

THEOREM 3. Let X be a compact set. Then $[R(X)\overline{P}_m]_p = [R(X) + L_a^p(X)^\wedge + \dots + L_a^p(X)_m^\wedge]_p$ for all $m \geq 1$ and $1 \leq p < \infty$.

PROOF. It is enough to show that $[R(X)\overline{P}_1]_p = [R(X) + L_a^p(X)^\wedge]_p$ for all $1 \leq p < \infty$ since the arguments for $m > 1$ are similar.

It is clear that $[R(X)\overline{P}_1]_p \subset [R(X) + L_a^p(X)^\wedge]_p$ by the remark in section 2. To prove the other inclusion, we let g be any function in $L^q(X)$, $1 < q \leq \infty$, $p^{-1} + q^{-1} = 1$ such that $g \perp R(X)\overline{P}_1$. We shall show that $g \perp L_a^p(X)^\wedge$.

Lemma 1 implies $\hat{g} \perp R(X)$ and therefore $\hat{g} = 0$ off X . Also \hat{g} is continuous by Lemma 2. It follows that $\hat{g} = 0$ everywhere on ∂X . An argument similar to J. Brennan [4] using the theory of singular integrals and Schwartz lemma concludes that both \hat{g} and g vanishes almost everywhere on ∂X . Hence it suffices to prove that $\int_{\hat{X}} \hat{f}g \, dm = 0$ for all f in $L_a^p(X)$. Again, a similar construction in [4] gives a sequence of functions ρ_n , $n = 1, 2, \dots$ such that each ρ_n has support in $\overset{\circ}{X}$ and, writing $\bar{\partial}$ for $\partial/\partial x + i\partial/\partial y$, $\bar{\partial}\rho_n \rightarrow \hat{g}$ in the $L^q(X)$ norm. Thus for any f in $L_a^p(X)$,

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \int_{\hat{X}} f(\bar{\partial}\rho_n) \, dm = \int_{\hat{X}} f\hat{g} \, dm \\ &= \int_{\hat{X}} \hat{f}g \, dm . \end{aligned}$$

The following theorem shows that, among other things, every n th Cauchy transform of a function in $L_a^p(X)$ ($n = 2, \dots, m$) belongs to the uniform closure of $R(X)\overline{P}_m$ for all $m \geq 2$, $2 < p < \infty$.

THEOREM 4. Let X be a compact set. Then $[R(X)\overline{P}_m]_u = [R(X) + R(X)^\wedge + L_a^p(X)_2^\wedge + \dots + L_a^p(X)_m^\wedge]_u$ for all $m \geq 2$ and $2 < p < \infty$.

PROOF. This follows from Theorem 3 and the fact that $\hat{\mu} \in L_{loc}^q$ for all $1 \leq q < 2$ and any measure μ on X (see [6], p.37)

We may consider the Cauchy transform as a linear map from C_c^∞ , the space of infinitely differentiable functions on the complex plane with compact support, into C^∞ . In [10], O'Farrell obtains various pairs of norms so that the Cauchy transform is bounded, and thus we have the following corollaries.

COROLLARY 5. Let X be a compact set. Then $[R(X)\overline{P}_m]_{lip \alpha} = [R(X) + R(X)^\wedge + L_a^p(X)_2^\wedge + \dots + L_a^p(X)_m^\wedge]_{lip \alpha}$ for all $m \geq 2$, $2 < p < \infty$ and $0 < \alpha < 1$.

COROLLARY 6. Let X be a compact set. Then $[R(X)\overline{P}_m]_{lip 1} = [R(X) + \hat{V}]_{lip 1}$ where V is any space of functions on X such that $[V]_{lip \alpha} = [R(X)\overline{P}_{m-1}]_{lip \alpha}$, for each fixed m and $0 < \alpha < 1$.

REMARK. L. I. Hedberg has pointed out to the author that the problem of approximation in L^p by functions in $R(X)\overline{P}_1$ is closely related to the problem of approximation by harmonic functions (see [9]).

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