A THEORY OF NONLINEAR WAVE LOADING
ON OFFSHORE STRUCTURES

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ABSTRACT. A theoretical study is made of the nonlinear wave loading on offshore structures using the diffraction theory of hydrodynamics. A nonlinear modification of the classical Morison equation, $D \equiv F_\ell + F_D$ for estimating wave forces on offshore structures is suggested in this paper. The modified equation is found in the form $D \equiv F_\ell + F_{n\ell} + F_D$, where $F_{n\ell} \equiv F_d + F_w + F_q$ is the nonlinear contribution made up of the dynamic, waterline, and the quadratic forces associated with the irrotational-flow part of the wave loading on structures. The study has then been applied to calculate the linear and the nonlinear wave loadings on a large vertical cylinder partially immersed in an ocean of arbitrary uniform depth. All the linear and nonlinear forces exerting on the cylinder are determined explicitly.

A comparison is made between these two kinds of forces. Special attention is given to the nonlinear wave loadings on the cylinder. It is shown that all non-linear effects come from the interaction between the body's responses to the on-coming wave's fluctuating velocity and its fluctuating extension. It is found

that the nonlinear effects are dominated by the sum of the dynamic and waterline

forces. The nonlinear correction to Morison's equation increases with increasing

kb where b is the characteristic dimension of the body and k is the wave number.

This prediction is shown to be contrary to that of the linear diffraction theory

which predicted that the Morison coefficient decreases with increasing kb. Several

interesting results and limiting cases are discussed in some detail.

KEY WORDS AND PHRASES. Nonlinear Waves, Wave Forces, Morison equation, Wave
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1. INTRODUCTION.

One of the most common problems in hydrodynamics deals with waves as well as

wave loading estimation for offshore structures. This problem is of considerable

mathematical and engineering interest due to abundant applications to marine

hydrodynamics. In order to indicate our motivation and interest in the problem,

it is necessary to review some of the relevant works on the subject.

Based upon the classical works on linear diffraction theory for water waves by

Havelock [1] and for sound waves by Morse [2], MacCamy and Fuchs [3] developed a

linear diffraction theory for wave forces on a large cylindrical pile immersed in

oceans. This study was found to have limited applications because the nature of

water waves is inherently nonlinear.

In recent papers [4-5], Chakraborty has made an interesting extension of

MacCamy and Fuch's work, and obtained analytical solutions of the second

order wave forces on a vertical cylinder based upon the Stokes fifth-order wave

theory. It is unfortunate that his scattered potential fails to satisfy the non-

linear free surface boundary condition and the radiation condition at infinity.

In order to eliminate these drawbacks of Chakraborty's theory, Raman and

Venkatanarasaiah [6] have presented a nonlinear diffraction theory due to the

Stokes second-order waves for estimating the wave forces on a vertical cylinder.

Although their work has provided some interesting theoretical and experimental

results, the method of solution is extremely complicated.

Historically, the wave loading estimation for offshore structures has been based upon the work of Morison et al [7]. Morison's equation is generally used for calculating wave forces on solid structures in oceans. This equation expresses the total drag D as a sum of the inertial force, $\rho C_M V \dot{U}$ associated with the irrotational flow component, and the drag force, $\frac{1}{2}\rho C_D A U^2$ related to the vortex-flow component of the fluid flows. Mathematically, the Morison equation is

$$D = \rho C_M V \dot{U} + \frac{1}{2}\rho C_D A U^2 , \qquad (1.1)$$

where ρ is the fluid density, $C_M = (1 + \frac{M_a}{\rho V})$ is the Morison (or inertial) coefficient, M_a is the added mass, V is the volumetric displacement of the body, U is the fluctuating fluid velocity along the horizontal direction, A is the projected frontal area of the wake vortex, and C_D is the drag coefficient. It is assumed that the inertial and drag forces acting on a fixed solid object in an unsteady fluid flow are independent in the sense that there is no interaction between them.

There are two characteristic features of the Morison equation. One deals with the nature of the inertial force which is linear in velocity U. The other indicates that the drag force is nonlinaer in velocity U. It is generally believed that all nonlinear effects in experimental data are associated with drag forces. However, for real solid structures in real ocean waves, there is a significant nonlinear force associated with the irrotational component of the fluid flow because of the large amplitude of ocean waves. These waves are of special interest in wave loading estimation. It seems necessary to incorporate all significant nonlinear effects associated with the nonlinear free surface boundary condition in the irrotational flow part of the wave loading on structures.

Several studies have shown that the mathematical form of the Morison equation is satisfactory. However, certain difficulties in using it in the design and construction of offshore structures have been reported in the literature. These are concerned with the drag force which has relatively large scale effects. The shortage of reliable full-scale drag data in wave motions is also another problem.

There is another important question whether a linear theory of the irrotational-flow response is appropriate at all to water wave motions with a free surface. Despite these difficulties, the use of the Morison equation has extensively been documented in the literature through plentiful data for determining the coefficients C_M and C_D .

Recently, Hogben et al [8] and Lighthill [9] have published comprehensive review articles on the recent developments in the subject with a critical evaluation of both the methods and the results concerning the waves and hydrodynamic loading on offshore structures. Lighthill has also suggested that the Morison equation needs to be amended by the incorporation of significant nonlinear terms associated with free-surface effects on the irrotational-flow part of the wave loading. He then discussed a general theory and results with many interesting comments on the second-order wave forces on a large vertical cylinder immersed in deep oceans.

Motivated by the above discussion, especially on the nonlinear irrotational-flow response theory, we became interested in the study of nonlinear effects on wave loading estimation in order to examine the significance of these effects and to incorporate them in the Morison equation.

This paper is concerned with the study of the nonlinear wave loading on off-shore structures using the diffraction theory of hydrodynamics. A nonlinear modification of the Morison equation for estimating wave forces on offshore structures is presented in this paper. It is shown that this modification is made up of the dynamic, waterline, and the quadratic wave forces associated with the irrotational-flow part of the wave loading on structures. All linear and nonlinear wave loadings on a large vertical cylinder immersed partially in an ocean of arbitrary uniform depth are calculated explicitly. A comparison is made among the various nonlinear forces, and between the linear and nonlinear wave forces in order to determine their relative importance. It is shown that results of Lighthill for the case of deep oceans can be recovered as limiting cases of this analysis.

2. BASIC EQUATIONS FOR WATER WAVES.

We consider a large rigid vertical cylinder of radius b which is acted upon by a train of two-dimensional, periodic, progressive waves propagating on the surface of a fluid of arbitrary uniform depth h. We assume that the fluid is inviscid, incompressible, and that the motion is irrotational in the region bounded by the free surface, rigid bottom boundary and the surface of the cylinder.

In this axisymmetric configuration with symmetry about the z-axis vertical positive upward, the velocity potential $\phi(r,\theta,z;t)$ and the free surface elevation $\eta(r,\theta;t)$ are governed by the following equations and boundary conditions:

$$\nabla^2\phi \equiv \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \; , \tag{2.1}$$

where $b \leq r < \infty$, $-h \leq z \leq \eta$ and $-\pi \leq \theta \leq \pi$.

The dynamic condition at the free surface is

$$\frac{\partial\theta}{\partial t} + g\eta + \frac{1}{2}(\nabla\phi)^2 = 0 \qquad \text{on } z = \eta, \; r \geq b, \tag{2.2}$$

where g is the acceleration due to gravity and the velocity components are

$$(u_r, u_\theta, u_z) \equiv \nabla\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{\partial\phi}{\partial z}\right) \, .$$

The kinematic condition at the free surface is

$$\frac{\partial\eta}{\partial t} - \frac{\partial\phi}{\partial z} + \left(\frac{\partial\phi}{\partial r}\right)\left(\frac{\partial\eta}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial\phi}{\partial\theta}\right)\left(\frac{\partial\eta}{\partial\theta}\right) = 0 \qquad \text{on } z = \eta, \; r \geq b. \tag{2.3}$$

The boundary condition at the rigid bottom is

$$\frac{\partial\phi}{\partial z} = 0 \qquad \text{on } z = -h, \tag{2.4}$$

The radiation condition at infinity is

$$\lim_{kr\to\infty} \left[(kr)^{\frac{1}{2}} \left(\frac{\partial}{\partial r} \pm ik\right) \phi_s\right] = 0, \tag{2.5}$$

where $k = \frac{2\pi}{\lambda}$ is the wave number of scattered wave, $\phi = \phi_\ell + \phi_s$ is the total velocity potential, ϕ_ℓ and ϕ_s represent the incident and the scattered wave potentials respectively.

3. IRROTATIONAL-FLOW PRESSURE DISTRIBUTIONS.

The irrotational-flow pressure distribution can be obtained from the
Bernoulli equation which is

$$\frac{p}{\rho} + gz + \frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla \phi)^2 = 0. \qquad (3.1)$$

This equation enables us to define three pressure distributions:

(i) The hydrostatic pressure distribution which has the value

$$p_o - \rho gz, \qquad (3.2)$$

at a height z above the level z = 0 where the atmospheric pressure is p_o.

(ii) The dynamic pressure distribution which has the well known Bernoulli value

$$\frac{1}{2}\rho(U^2 - q^2) , \qquad (3.3)$$

at a point where the fluid speed is q.

(iii) The transient pressure distribution which has the value

$$- \rho \frac{\partial \phi}{\partial t} \qquad (3.4)$$

There is a certain resultant force with which each of these pressure components
acts on a solid body in a stream of homogeneous liquid of variable speed U.

The total derivative of (3.1) with respect to time t has the form

$$\frac{D}{Dt}(\frac{p}{\rho}) + g\frac{Dz}{Dt} + \frac{D}{Dt}(\frac{\partial \phi}{\partial t}) + \frac{1}{2}\frac{D}{Dt}(\nabla \phi)^2 = 0, \qquad (3.5)$$

which can be written as

$$\frac{D}{Dt}(\frac{p}{\rho}) + (\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z}) + \frac{\partial}{\partial t}(\nabla \phi)^2 + \frac{1}{2}\underline{q}\cdot\nabla(\nabla \phi)^2 = 0, \qquad (3.6)$$

where $\underline{q}$ is the velocity vector.

Equation (3.6) has been evaluated at z = η to obtain

$$\frac{D}{Dt}(\frac{p_o}{\rho}) + (\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial t})_{z=\eta} + [\frac{\partial}{\partial t}(\nabla \phi)^2]_{z=\eta} + \frac{1}{2}[\underline{q}\cdot\nabla(\nabla \phi)^2]_{z=\eta} = 0 . \qquad (3.7)$$

If the atmospheric pressure is constant, the first term in (3.7) vanishes so that
(3.7) reduces to

$$\frac{\partial^2 \phi}{\partial t^2} + g(\frac{\partial \phi}{\partial z}) + \frac{\partial}{\partial t}(\nabla \phi)^2 + \frac{1}{2}[\underline{q}\cdot\nabla(\nabla \phi)^2] = 0 \text{ on } z = \eta. \qquad (3.8)$$

Retaining up to second-order term in (3.8), we obtain

$$\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} = - \frac{\partial}{\partial t}(\nabla \phi)^2 \ . \tag{3.9}$$

This is the equation derived by Lighthill [9]. It is noted here that a simple combination of equations (2.2) and (2.3) also gives the result (3.9).

4. PERTURBATION OF SOLUTION AND NONLINEAR WAVE FORCES.

We write the velocity potential, ϕ and the free-surface elevation function, η in the form

$$\phi = \phi_\ell + \phi_q + \dots \tag{4.1}$$

$$\eta = \eta_\ell + \eta_q + \dots \tag{4.2}$$

where ϕ_ℓ, η_ℓ are the ordinary solutions satisfying the linearized free-surface conditions and ϕ_q, η_q represent the quadratic correction of the order of the squares of the disturbances (and higher order corrections are neglected). Thus the following Taylor series expansion of ϕ about $z = 0$ holds good and has the form

$$\phi(r,\theta,\eta(r,\theta;t)) = \phi(r,\theta,0) + \eta \left(\frac{\partial \phi}{\partial z}\right)_{z=0} + \dots$$

$$= (\phi_\ell + \phi_q + \dots) + (\eta_\ell + \eta_q + \dots)\left[\frac{\partial \phi_\ell}{\partial z} + \frac{\partial \phi_q}{\partial z} + \dots\right]_{z=0} + \dots, \tag{4.3}$$

Therefore

$$\phi = \phi_\ell + \phi_q + \eta_\ell \left(\frac{\partial \phi_\ell}{\partial z}\right)_{z=0} + \text{(higher order terms)}, \tag{4.4}$$

where the first term is the first-order term and the second one represents the second-order term.

With a similar expansion by the Taylor series, we obtain

$$\frac{\partial \phi}{\partial z} = \frac{\partial \phi_\ell}{\partial z} + \left\{ \frac{\partial \phi_q}{\partial z} + \eta_\ell \left[\frac{\partial}{\partial z}\left(\frac{\partial \phi_\ell}{\partial z}\right)\right]_{z=0} \right\} + \dots \tag{4.5}$$

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi_\ell}{\partial t^2} + \left\{ \frac{\partial^2 \phi_\ell}{\partial t^2} + \eta_\ell \left[\frac{\partial}{\partial z} \frac{\partial^2 \phi_\ell}{\partial t^2}\right]_{z=0} \right\} + \dots \tag{4.6}$$

$$\nabla \phi = \nabla \phi_\ell + \left\{ \nabla \phi_q + \eta_\ell \left[\frac{\partial}{\partial z}(\nabla \phi_\ell)\right]_{z=0} \right\} + \dots \tag{4.7}$$

Substituting these results into (3.9), we obtain equations for the linear potential ϕ_ℓ and the quadratic potential ϕ_q in the form

$$\frac{\partial^2 \phi_\ell}{\partial t^2} + g\frac{\partial \phi_\ell}{\partial z} = 0, \tag{4.8}$$

$$\frac{\partial^2 \phi_q}{\partial t^2} + g\frac{\partial \phi_q}{\partial z} = -\frac{\partial}{\partial t}(\nabla\phi_\ell)^2 - \eta_\ell \left[\frac{\partial}{\partial z}\left(\frac{\partial^2 \phi_\ell}{\partial t^2} + g\frac{\partial \phi_\ell}{\partial z}\right)\right]_{z=0}, \tag{4.9}$$

where $\eta_\ell = -\frac{1}{g}\left(\frac{\partial \phi_\ell}{\partial t}\right)$.

We would like to make some comments on the equation (4.9) satisfied by the quadratic potential ϕ_q, and the additional wave loading generated by ϕ_q. For making these comments, it is necessary to recall certain results of the theory of irrotational fluid flows which is extremely useful for the study of both steady and unsteady surface waves in oceans. In these studies when solid boundaries are absent, there is no principal source of generation of vorticity. With the linearized boundary condition at the free surface z = 0, the problem of determining the irrotational flow field becomes a purely linear problem because the velocity potential satisfies the well known Laplace equation. With such linear system of equations, the irrotational-flow response to oncoming waves of every frequency and direction in the ocean-wave spectrum can explicitly be determined. For example, the sinusoidal waves of amplitude a propagating freely in the x-direction in water of uniform depth h have the velocity of potential ϕ, and the free surface elevation η given by

$$\phi(x,z;t) = a\omega\,\frac{\cosh k(z+h)}{k\sinh kh}\cos(\omega t - kx), \tag{4.10}$$

$$\eta(x,t) = a\sin(\omega t - kx), \tag{4.11}$$

where ω is the frequency and k is the wave number.

The linearized boundary condition at the free surface z = 0 gives the well known dispersion relation

$$\omega^2 = gk\tanh kh. \tag{4.12}$$

In the case of deep water, these equations reduce to simple form

$$\phi = \frac{a\omega}{k} e^{kz} \cos(\omega t - kx), \quad \omega^2 = gk. \tag{4.13ab}$$

These simply mean that each water particle moves in a circular path at constant speed which decreases exponentially with depth of the liquid.

With the aid of the above results (4.10) - (4.12), we can rewrite the equation (4.9) in the form

$$\frac{\partial^2 \phi_q}{\partial t^2} + g \frac{\partial \phi_q}{\partial z} = -\frac{\partial}{\partial t} [(\nabla\phi_\ell)^2 + \frac{1}{2}(\tanh kh - 1) (k\phi_\ell)^2]. \tag{4.14}$$

In the case of infinitely deep water, $kh \to \infty$ so that $\tanh kh \to 1$, this equation takes the even simpler form

$$\frac{\partial^2 \phi_q}{\partial t^2} + g \frac{\partial \phi_q}{\partial z} = -\frac{\partial}{\partial t}(\nabla\phi_\ell)^2 . \tag{4.15}$$

In other words, for the deep water case, the second term on the right hand side of (4.9) or (4.14) disappears, and the quadratic potential ϕ_q satisfies (4.15) at the free surface with zero boundary conditions everywhere else.

Similarly, for a very shallow water, $kh \to 0$ so that $\tanh kh \to kh$, we can also write down the corresponding form of the equation (4.14).

It is also noted that the left hand side of equations (4.14) and (4.15) has no terms in ϕ_ℓ. This is because of the fact that the potential ϕ_ℓ satisfies the linearized free-surface condition not only on $z = 0$ but everywhere in the fluid including $z = \eta$. Physically, equation (4.14) implies that ϕ_q is the solution of the linearized wave motion generated in the presence of the body at rest by the fluctuating free-surface pressure distribution

$$\rho[(\nabla\phi_\ell)^2 + \frac{1}{2}(\tanh kh - 1)(k\phi_\ell)^2] \text{ or } \rho(\nabla\phi_\ell)^2$$

according as the fluid is of finite or infinite depth. The frequency of this fluctuating pressure is twice of that of the oncoming waves.

Making reference to Lighthill [9], the quadratic force F_q generated by the second-order potential ϕ_q is defined by the integral

$$F_q = - \int_{z=0} W\rho \left[(\nabla\phi_\ell)^2 + \frac{1}{2}(\tanh kh - 1) \, (k\phi_\ell)^2 \right] dS. \qquad (4.16)$$

where W is the vertical velocity of the fluid at the free surface due to the

horizontal translational oscillations of the body. The value W at z = 0 can be

determined by solving a simple linear problem.

Although the equation (4.16) is very convenient from the computational point

of view, however, this quadratic force F_q can also be determined by the traditional

method of fluid dynamics using Green's theorem combined with the application of

the radiation condition at infinity. We shall discuss this simple approach to

estimating F_q in a subsequent section.

In addition to the quadratic force F_q derived from the quadratic part ϕ_q of

the velocity potential, there is a significant contribution to the second-order

force generated by the linear potential, ϕ_ℓ. In fact, this additional second-

order force arises from the pressure distributions derived from the potential ϕ_ℓ.

These pressure distributions have second-order terms which exert an additional

wave loading on a structural component. Actually, two forces constitute this

additional second-order contribution. One corresponds to the dynamic force, F_d

which is the resultant of the dynamic pressure, $-\frac{1}{2}\rho q^2$ $(q^2 = (\nabla\phi_\ell)^2)$ over the

surface of the body. Thus the force F_d has the following integral representation

$$F_d = \int_S \frac{1}{2}\rho (\nabla\phi_\ell)^2 \, dS . \qquad (4.17)$$

where S is the body surface and n_x is the direction cosine between the outward

normal and the direction of the force F_d. It should be pointed out that the

potential includes the responses of the body to both fluctuating motions and the

fluctuating extensional motions due to waves.

Another force provides a second-order contribution to the wave loading on

structures whenever the body penetrates the free surface of the liquid. It is

directly associated with the transient and hydrostatic pressures of the irrota-

tional flow. According to Lighthill, this additional second-order horizontal

x-component of the force, F_w acting at the waterline w (where the structural

component intersects the free surface of the fluid) is given by

$$F_w = \oint_w \frac{\rho}{2g} \left(\frac{\partial \phi_\ell}{\partial t}\right)^2 dy,$$

(4.18)

where the integrand is the horizontally resolved force per unit length acting at

the waterline and the integral gives its resultant in the x-direction.

We denote the linear force F_ℓ which is equal to the resultant of the transient

pressures $-(\rho \frac{\partial \phi_\ell}{\partial t})$. This force can therefore be written as

$$F_\ell = \int_S \left(-\rho \frac{\partial \phi_\ell}{\partial t}\right) dS.$$

(4.19)

This force is generally calculated using purely linear analysis.

It follows from the above discussion that the total second-order contribution

to the irrotational-flow loading on an offshore structure is equal to $F_d + F_w + F_q$,

whereas the linear wave loading on such structure is F_ℓ. The accuracy of the

wave loading can thus greatly be increased by combining F_ℓ with the nonlinear

contribution, $(F_d + F_w + F_q)$. The latter provides a nonlinear modification of the

Morison equation which has a new form

$$D \equiv F_\ell + F_{n\ell} + F_D ,$$

(4.20)

where $F_{n\ell} = F_d + F_w + F_q$ is the nonlinear contribution which is made up of the

dynamic, waterline, and the quadratic forces associated with the irrotational-

flow part of the wave loading on offshore structures. It will be shown in sub-

sequent sections of this paper that the nonlinear contribution to the wave loading

on structures is quite significant.

5. LINEAR FORCE AND MORISON'S COEFFICIENT BASED UPON THE LINEAR DIFFRACTION THEORY.

Havelock [1] developed the linear diffraction-theory potential for a long

vertical cylinder of radius b in an oncoming wave train in deep water described

by the potential given by (4.13a). This potential has the form

$$\phi_\ell = \text{Re}\left[\frac{a\omega}{k}\right] \exp(i\omega t + kz) \sum_{n=0}^{\infty} \alpha_n \phi_n(kr)\cos\theta,$$

(5.1)

where Re stands for real part, $\alpha_o = 1$, $\alpha_n = 2(-i)^n$ with n > 0, and

$$\phi_n(kr) = J_n(kr) - \frac{H_n^{(2)}(kr) \, J_n'(kb)}{H_n^{(2)'}(kb)} \quad . \tag{5.2}$$

Since our interest only in the $\cos \theta$ term in ϕ_ℓ, therefore

$$\phi_\ell = \text{Re}[\frac{a\omega}{k} \exp(i\omega t + kz)(-2i)\phi_\ell(kr)\cos \theta]. \tag{5.3}$$

Thus the linear force, $F_\ell(z,t)$ is given by

$$F_\ell(z,t) = \int_0^{2\pi} \left(-\rho\frac{\partial\phi_\ell}{\partial t}\right)_{r=b} (-\cos \theta)bd\theta, \tag{5.4}$$

$$= \text{Re}[\frac{2\rho\pi \, ab^2 \, \omega^2}{kb} \exp(i\omega t + kz)\phi_\ell(kb)], \tag{5.5}$$

$$= C_M \, (\rho ga\pi b^2) \, \cos(\omega t+\beta), \tag{5.6}$$

where Morison's coefficient C_M and the phase β are given by

$$C_M e^{i\beta} = \frac{2}{kb} \, \phi_1(kb) = -\frac{4i}{\pi(kb)^2 \, H_1^{(2)'}(kb)} \quad . \tag{5.7}$$

and the coefficient C_M specifies the cylinder's inertial reaction to a fluid acceleration advanced in phase by the phase lead β over its value on the cylinder axis.

In the case of a fluid of finite depth h, the corresponding results for C_M and β are given by

$$C_M \, e^{i\beta} = \frac{2}{kb} \, \phi_1(kb) \, (1-e^{-kh})\tanh kh, \tag{5.8}$$

$$= -\frac{4i(1-e^{-kh})\tanh kh}{\pi(kb)^2 \, H_1^{(2)'}(kb)} \quad . \tag{5.9}$$

where the value of $\phi_1(kb)$ is replaced by (5.2) along with the following results

$$J_1(kb) \, H_1^{(2)'}(kb) - H_1^{(2)}(kb) \, J_1'(kb) = \frac{2}{\pi ikb} \quad . \tag{5.10}$$

$$H_1^{(2)}(kb) = J_1(kb) - iY_1(kb) \quad . \tag{5.11}$$

Finally, the values of C_M and β are explicitly given by

$$C_M = \frac{4(1-e^{-kh})\tanh kh}{\pi(kb)^2 \left[J_1'^2(kb) + Y_1'^2(kb) \right]^{\frac{1}{2}}} \qquad (5.12)$$

$$\beta = -\tan^{-1}\left[\frac{J_1'(kb)}{Y_1'(kb)} \right] \qquad (5.13)$$

One of the most significant conclusions of this section is that the Morison coefficient C_M decreases or increases according as kb increases or decreases. Physically, when kb is large (the characteristic dimension b of the body is large compared with k^{-1}), the body becomes efficient as a generator of dipole wave radiation, and the force on it becomes more resistive in nature. This implies that at high frequencies the magnitude of the force increase more slowly than would a purely reactive force.

On the other hand, for small kb (the characteristic dimension b of the body is small compared with k^{-1}), the body radiates a very small amount of scattered-wave energy. This corresponds to a case of a rigid lid on the ocean inhibiting wave scattering.

6. NONLINEAR WAVE LOADINGS ON A VERTICAL CYLINDER.

We now calculate the linear wave loading, F_ℓ, and the nonlinear wave loading terms F_q, F_d and F_w on a long vertical cylinder of radius $b(<<k^{-1})$ which penetrates the free surface of water. With the axis of the cylinder as $x = y = 0$, and the presence of the oncoming wave train with potential (4.10), the fluctuating velocity is equal to the value of the horizontal velocity in the undisturbed wave on the cylinder axis. In other words, the fluctuating velocity $U(z,t)$ can be expressed as

$$U(z,t) = \left(\frac{\partial \phi}{\partial x}\right)_{x=0} = (a\omega k \sin \omega t)\frac{\cosh k(z+h)}{\sinh kh} , \qquad (6.1)$$

To a first approximation, the irrotational response to this fluctuating horizontal velocity produced by the cylinder of radius b is the simple dipole field. The associated velocity potential is given by

$$\phi_\ell = U(z,t)(r + \frac{b^2}{r}) \cos \theta ,\qquad (6.2)$$

in cylindrical polar coordinates (r,θ,z).

Thus the values of the potential and of the horizontal velocity on the cylinder surface $r = b$ are

$$(\phi_\ell)_{r=b} = 2bU \cos \theta, \quad \left(\frac{1}{r}\frac{\partial \phi_\ell}{\partial \theta}\right)_{r=b} = -2U \sin \theta,\qquad (6.3ab)$$

In addition to the fluctuating velocity produced near the cylinder by the oncoming waves, waves can also generate significant fluctuating extensional motions as indicated earlier. This extensional motion is the value of the horizontal velocity gradient $(\frac{\partial^2 \phi}{\partial x^2})$ in the undisturbed wave train (4.10) on the axis of the cylinder. In other words, the extensional motion is given by

$$E(z,t) = \left(\frac{\partial^2 \phi}{\partial x^2}\right)_{x=o} = \mathbf{k}(-a\omega \cos \omega t)\frac{\cosh k(z+h)}{\sinh kh}\qquad (6.4)$$

In fluid dynamics, this extension can be treated as a linear combination of a pure dilatation and a dilatationless straining. The local irrotational response to the pure dilatation is the simple monopole field with the velocity potential, $\frac{E}{4}(r^2 - 2b^2 \log r)$. The corresponding response to the dilatationaless straining is a quadrupole field with the potential, $\frac{E}{4}(r^2 + b^4 r^{-2})\cos 2\theta$. Thus the combined velocity potential has the form

$$\phi_\ell = \frac{E}{4}(r^2 - 2b^2 \log r) + \frac{E}{4}(r^2 + \frac{b^4}{r^2})\cos 2\theta,\qquad (6.5)$$

and the horizontal velocity on the cylinder surface is equal to

$$\left(\frac{1}{r}\frac{\partial \phi_\ell}{\partial \phi}\right)_{r=b} = -Eb \sin 2\theta,\qquad (6.6)$$

The equations (6.3ab) give a linear force equal to the resultant of the transient pressures $-\left(\rho\frac{\partial \phi_\ell}{\partial t}\right)$ so that the linear force per unit height of the cylinder, F_ℓ is

$$F_\ell(z,t) = \int_0^{2\pi} \left(-\rho\frac{\partial \phi_\ell}{\partial t}\right)(-\cos \theta) b \, d\theta = 2\rho\pi b^2 \dot{U} ,\qquad (6.7)$$

where the classical Morison coefficient in (6.7) is $C_M = 2$.

The total linear force, $\bar{F}_\ell$ exerting on the cylinder is given by

$$\bar{F}_\ell = \int_{-h}^{0} F_\ell(z,t)\,dz = 2\pi\rho \text{ ag } b^2 \text{ tanh kh } \cos\omega t . \qquad (6.8)$$

The total drag force, $\bar{F}_D$ exerting on the cylinder is equal to

$$\bar{F}_D = \int_{-h}^{0} \tfrac{1}{2}\rho \text{ A } C_D \text{ U}^2 \, dz, \qquad (6.9)$$

$$= \tfrac{1}{2}\rho g a^2 \, C_D \text{ tanh kh } \sin^2\omega t + \left[\frac{\sinh 2kh}{2 \sinh kh} - \frac{kh}{\sinh kh} \right], \qquad (6.10)$$

Thus the sum of (6.8) and (6.10) gives the total drag D involved in the Morison equation.

We next note that the function W involved in the formula (4.16) for the quadratic force represents the effect of the fluid flows at the free surface generated by the horizontal translational oscillations of the cylinder with unit amplitude and frequency 2ω. This function is also directly proportional to $\cos\theta$ which can also be confirmed by the exact diffraction theory. Therefore, the approximate value of W near the cylinder for small kb can be expressed by a simple dipole form

$$W \approx (4k) \left(\frac{b^2}{r} \cos\theta\right) , \qquad (6.11)$$

where the factor 4k is extracted from the linearized boundary condition $\phi_{tt} + g\phi_z = 0$ on $z = 0$, for oscillations of frequency 2ω in order to relate the vertical velocity W at the surface to a local velocity potential given by the second factor in (6.11) so that its radial derivative is $\cos\theta$ on the cylinder surface $r = b$.

In view of the proportionality of W to $\cos\theta$, the only $\cos\theta$ terms that may be present in $(\nabla\phi_\ell)^2$ and $(k\phi_\ell)^2$ involved in (4.16) are of interest in this analysis. The term

$$(\nabla\phi_\ell)^2 = \left(\frac{\partial\phi_\ell}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\phi_\ell}{\partial\theta}\right)^2 + \left(\frac{\partial\phi_\ell}{\partial z}\right)^2 , \qquad (6.12)$$

can be computed using the dipole potential (6.2), the combined potential (6.5)

and the basic potential (4.10). It follows from (6.2) and (6.5) that the square of the horizontal speed is

$$\left(\frac{\partial \phi_\ell}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial \phi_\ell}{\partial \theta}\right)^2 = \left[U(1-\frac{b^2}{r^2})\cos\theta + \frac{E}{2}(r - \frac{b^2}{r}) + \frac{E}{2}(r - \frac{b^4}{r^3})\cos 2\theta\right]^2$$

$$+ \left[U(1 + \frac{b^2}{r^2})\sin\theta + E(r + \frac{b^4}{r^3})\sin\theta\cos\theta\right]^2 , \qquad (6.13)$$

The free surface value of this quantity gives a term in $\cos\theta$ in the form

$$-\frac{1}{2}g(ak)^2 \sin 2\omega t \coth kh \left[2r - \frac{2b^2}{r} + \frac{b^4}{r^3} + \frac{b^6}{r^5}\right]\cos\theta . \qquad (6.14)$$

Similarly, using (6.2) and (4.11), the $\cos\theta$ term in the veritcal speed squared is given by

$$\left[\left(\frac{\partial \phi_\ell}{\partial z}\right)^2\right]_{z=0} = \left\{a\omega\cos\omega t + a\omega k\sin\omega t\cos\theta(r + \frac{b^2}{r})\right\}^2 . \qquad (6.15)$$

The $\cos\theta$ term in this expression is involved only with the cross-term between the two terms in the third bracket of (6.15), and is equal to

$$-\frac{1}{2}g(ak)^2 \sin 2\omega t \tanh kh \ (2r + \frac{2b^2}{r})\cos\theta. \qquad (6.16)$$

A simple addition of (6.14) and (6.15) gives the $\cos\theta$ term in the total fluid speed squared at the free surface in the form

$$\frac{1}{2}g(ak)^2 \sin 2\omega t \coth kh \left[(\frac{4b^2}{r} - \frac{b^4}{r^3} - \frac{b^6}{r^5}) + (2r + \frac{2b^2}{r})(\tanh^2 kh-1)\right], \qquad (6.17)$$

It turns out from (4.10) and (6.2) that the free surface value of $(k\phi_\ell)^2$ is

$$\left[a\omega\cos\omega t \coth kh + Uk(r + \frac{b^2}{r})\cos\theta\right]^2 . \qquad (6.18)$$

The $\cos\theta$ term involved in this expression is also obtained only from the cross-term between the two terms within the third bracket of (6.18), and has the value

$$g(ak)^2(r + \frac{b^2}{r}) \coth kh \ \sin 2\omega t \ \cos\theta. \qquad (6.19)$$

Invoking the results (6.11), (6.17) and (6.19), the quadratic force F_q is

given by

$$F_q = -\rho \int_b^B rdr \int_0^{2\pi} (-4k)(\frac{b^2}{r} \cos \theta) \left[(\nabla\phi_\ell)^2 + \frac{1}{2}(\tanh kh - 1)(k\phi_\ell)^2 \right] d\theta , \qquad (6.20)$$

when $b < r < B$ and the upper limit B will be determined later on.

Evaluation of (6.20) gives

$$F_q = 2\pi\rho g \ k^3 (ab)^2 \sin 2\omega t \left[(4 \ b^2 \log \frac{B}{b} - \frac{3b^2}{4} + \frac{b^4}{2B^2} + \frac{b^6}{4B^4}) \right.$$

$$\left. + (B^2 - b^2 + 2b^2 \log \frac{B}{b})(\tanh^2 kh + \frac{1}{2} \tanh kh - \frac{3}{2}) \right] . \qquad (6.21)$$

The overturning moment about the bottom of the cylinder is given by $M_q = h \, F_q$.

It may be pointed out that the result (6.17) has an interesting implication for the magnitude of the quadratic force F_q. In fact, the positive coefficient of the first term in (6.17) clearly indicates that the nonlinear correction to Morison's equation increases with kb. This shows a striking contrast with the result (5.13) based on the linear diffraction theory which predicted that the Morison coefficient C_M decreases with kb. In other words, the nonlinear effects become significant in the diffraction regime.

We next estimate the dynamic force, F_d which is the resultant of the dynamic pressure, $-\frac{1}{2}\rho(\nabla\phi_\ell)^2$ where $(\nabla\phi_\ell)^2$ is to be evaluated on the cylinder surface $r = b$. It follows from (6.13) that the square of the horizontal speed on $r = b$ is

$$\left[\left(\frac{\partial\phi_\ell}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\phi_\ell}{\partial\theta}\right)^2 \right]_{r=b} = (2U \sin \theta + E b \sin 2\theta)^2 . \qquad (6.22)$$

Similarly, the square of the vertical speed on $r = b$ is evaluated from (6.14) and (6.2) to obtain

$$\left[\left(\frac{\partial\phi_\ell}{\partial z}\right)^2 \right]_{r=b} = (2b \ ka\omega \sin \omega t \cos \theta + a\omega \cos \omega t) \cdot \frac{2 \sinh^2 k(z+h)}{\sinh^2 kh} , \qquad (6.23)$$

so that a simple addition of (6.22) and (6.23) gives the required result for $(\nabla\phi_\ell)^2$ on $r = b$ in the form

$$(\nabla\phi_\ell)^2 = [2U \sin \theta + E b \sin 2\theta]^2 +$$

$$+ [2bk \ a\omega \sin \omega t \cos \theta + a\omega \cos \omega t]^2 \ \frac{\sinh^2 k(z+h)}{\sinh^2 kh} . \qquad (6.24)$$

With this result, the dynamic force per unit lenght of the cylinder is given by

$$F_d(z,t) = \int_0^{2\pi} [-\tfrac{1}{2}\rho(\nabla\phi_\ell)^2](-\cos\theta)\, b\, d\theta, \qquad (6.25)$$

$$= 2\pi\rho\, b^2\, UE[\tfrac{1}{2} - \tanh^2 k(z+h)], \qquad (6.26)$$

where this result is made up of the contributions only from the cross-terms between the terms in the square bracket of the expression (6.24).

The total dynamic force, $\bar{F}_d$ exerting on the cylinder is given by

$$\bar{F}_d = \int_{-h}^0 F_d(z,t)\,dz, \qquad (6.27)$$

$$= \tfrac{1}{8}\pi\rho(ab)^2 gk\, \tanh kh \left[\frac{\sinh 2kh - 6kh}{\sinh^2 kh}\right]\sin 2\omega t. \qquad (6.28)$$

The overturning moment M_d for this case is obtained in the following form

$$M_d = \int_{-h}^0 (z+h) F_d(z,t)\,dz$$

$$= \frac{\pi\rho(ab\omega)^2 \sin 2\omega t}{16k\, \sinh kh}\left[(2kh)\sinh(2kh) - \cosh(2kh)+1-6k^2 h^2\right]. \qquad (6.29)$$

Finally, the magnitude of the waterline force F_w is given by

$$F_w = \int_0^{2\pi} \frac{\rho}{2g}\left(\frac{\partial\phi_\ell}{\partial t}\right)(-\cos\theta)\, b\, d\theta, \qquad (6.30)$$

where

$$\left[\left(\frac{\partial\phi_\ell}{\partial t}\right)^2\right]_{z=o} = (2ab\, gk\, \cos\omega t\, \cos\theta - ag\, \sin\omega t)^2. \qquad (6.31)$$

Evaluation of the integral (6.30) gives the following result

$$F_w = \pi\rho(ak)ag\, b^2\, \sin 2\omega t. \qquad (6.32)$$

The overturning moment of F_w about the bottom of the cylinder is simply

$$M_w = hF_w = \pi\rho g(ak)ah\, b^2\, \sin 2\omega t. \qquad (6.33)$$

It is important to point out that in the above calculation, neither the response to the fluctuating velocity by itself nor the response to the fluctuating extension by itself has produced any nonlinear contributions to various wave forces. In fact, all nonlinear contributions come from the interaction of the two kinds of response. In other words, these contributions were found to result only from the cross-terms involving products of the local responses to the fluctuating velocity (6.1) and to the fluctuating extension (6.5).

For the deep water case (kh → ∞), the forces $\bar{F}_\ell$ and $\bar{F}_d$ are calculated from (6.8) and (6.28) in the limit kh → ∞ to obtain simpler results due to Lighthill [9].

$$\bar{F}_\ell = 2\pi\rho g\ ab^2 \cos \omega t, \tag{6.34}$$

$$\bar{F}_d = \frac{1}{4}(ak)\pi\ \rho g\ ab^2 \sin 2\omega t. \tag{6.35}$$

A simple comparison of these forces reveals that the linear force is independent of the wave number k, whereas the dynamic force varies linearly with k. This means that the relative importance of the dynamic force increases with the wave steepness ak.

Finally, the total nonlinear contribution to the Morison equation force is equal to the sum of (6.21), (6.28) and (6.32) for the case of fluid of finite depth h.

For the case of deep ocean (kh → ∞), the quadratic force assumes the following limiting form

$$F_q = 2\pi g\rho\ (ak)(bk)^2\left[4b^2\log \frac{B}{b} - \frac{3b^2}{4} + \frac{b^4}{2B^2} + \frac{b^4}{4B^4}\right]\sin 2\omega t, \tag{6.36}$$

so that the total nonlinear contribution to the total drag is the sum of (6.32), (6.35) and (6.36).

It is interesting to compare these nonlinear contributions in order to determine their relative importance. In the Morison equation limit of small kb, the contribution from F_q is relatively small compared with those from $\bar{F}_d$ and F_w which give

$$(\bar{F}_d + F_w) = (\frac{1}{2}+2)(ak)g\rho\pi\, ab^2 \sin\omega t \cos\omega t. \tag{6.37}$$

It follows from this result that the waterline force F_w is four times larger than the dynamic force $\bar{F}_d$. It is also evident that in the limit $kb \to 0$, the non-linear effects are dominated by the sum of the dynamic and waterline forces. On the other hand, the contribution of the quadratic force to the Morison equation increases with increasing value to kb. This means that in the diffraction regime, the nonlinear effects become more important.

For comparison, the Morison equation (1.1) gives a value of the total drag as

$$D = 2g\rho a\pi b^2 \cos\omega t + \frac{1}{2}\rho ga^2 b\, C_D \sin^2\omega t. \tag{6.38}$$

The significance fo the nonlinear contribution (6.37) is that it gives an increased positive peak of the total drag force when both $\cos\omega t$ and $\sin\omega t$ are positive.

7. SIMPLE APPROACH TO CALCULATING THE QUADRATIC FORCE.

We assume the following form of the linear potential ϕ_ℓ and the quadratic potential ϕ_q:

$$\phi_\ell = \text{Re}[\Phi_\ell\, e^{2i\omega t}], \tag{7.1}$$

$$\phi_q = \text{Re}[\Phi_q\, e^{2i\omega t}]. \tag{7.2}$$

Inserting these values in equation (4.9), we obtain

$$\left[\frac{\partial \Phi_q}{\partial z} - K\, \Phi_q\right]_{z=0} = -\frac{2i\omega}{g}(\nabla\Phi_\ell)^2 + \frac{i\omega}{g^2}\, \Phi_\ell\left[-\omega^2\frac{\partial\Phi_\ell}{\partial z} + g\frac{\partial^2\Phi_\ell}{\partial z^2}\right]_{z=0}, \tag{7.3}$$

where

$$K = \frac{4\omega^2}{g}.$$

With the potential due to a unit translation oscillation of the body in the form $\text{Re}[\Psi\, e^{2i\omega t}]$, it follow that on the surface S of the body,

$$\left[\frac{\partial\Psi}{\partial n}\right]_S = n_x \qquad \text{and} \qquad \left[\frac{\partial\Phi_q}{\partial n}\right]_S = 0, \tag{7.4ab}$$

where n_x is the x-component of a unit inward normal to the surface S.

Furthermore

$$[\frac{\partial \Psi}{\partial n} - K\Psi] = 0 \text{ on } z = 0, \tag{7.5}$$

where Ψ, like Ψ_q satisfies the radiation condition at infinity.

This leads us to apply Green's theorem

$$\iiint_V [\Phi_q \nabla^2 \Psi - \Psi \nabla^2 \nabla_q] dV = \int_{S \cup z=o} [\Phi_q \frac{\partial \Psi}{\partial n} - \Psi \frac{\partial \Phi_q}{\partial n}] dS, \tag{7.6}$$

where the surface integral is taken over the boundaries of the fluid (S and z =0) and n is a normal outward from the fluid.

Since both Φ_q and Ψ satisfy the Laplace equation, the left hand side of (7.6) vainshes so that (7.6) gives

$$\int_{S \cup z=o} [\Phi_q \frac{\partial \Psi}{\partial n} - \Psi \frac{\partial \Phi_q}{\partial n}] dS = 0. \tag{7.7}$$

After performing the indicated integration in (7.7) and applying the conditions (7.4ab) and (7.5), it follows from (7.7) that

$$\int_{z=o} \Psi[\frac{\partial \Phi_q}{\partial z} - K \Phi_q] dx \, dy = \int_S \Phi_q n_x \, dS. \tag{7.8}$$

Thus the quadratic force F_q can be written as

$$F_q = \int_S [- \rho \frac{\partial \Phi_q}{\partial t}] n_x \, dS, \tag{7.9}$$

$$= Re[- \rho \, 2i\omega \, e^{2i\omega t} \int_S \Phi_q n_x \, dS] \tag{7.10}$$

Invoking (7.8) and (7.3), it follows that

$$F_q = Re\left[(-2i\omega\rho) e^{2i\omega t} \frac{1}{K} \int \left(\frac{\partial \Psi}{\partial z}\right)_{z=o} [\frac{\partial \Phi_q}{\partial z} - K \Phi_q] dx \, dy\right], \tag{7.11}$$

$$= Re\left[- \frac{4\rho\omega^2}{gK} e^{2i\omega t} \int \left(\frac{\partial \Psi}{\partial z}\right)_{z=o} \{(\nabla\Phi_\ell)^2 - \frac{1}{2g} \Phi_\ell (-\omega^2 \frac{\partial \Phi_\ell}{\partial z} + g \frac{\partial^2 \Phi_\ell}{\partial z^2})\} dx \, dy\right], \tag{7.12}$$

$$= \mathrm{Re}\left[-\int\limits_{z=o} W\left\{(\nabla\phi_\ell)^2 + \frac{1}{2}(\tanh kh - 1)(k\phi_\ell)^2\right\}dx\ dy\right], \tag{7.13}$$

where $W = \left(\dfrac{\partial\Psi}{\partial z}\right)_{z=o}$ is the vertical velocity on the free surface associated with the unit translational oscillation of the body.

In order to determine the vertical velocity W, it is necessary to find the solution for the potential Ψ. As stated earlier, the function Ψ satisfies the Laplace equation, the condition (7.4ab) on the cylinder $r = b$, the free surface condition on $z = o$ and the radiation condition at infinity. The solution Ψ can be expressed in the following form

$$\Psi(r,\theta,z) = \left[\frac{i\kappa}{\pi}(P)\int_{-\infty}^{\infty}\frac{K_1(|\alpha|r)}{|\alpha|K_1'(|\alpha|b)}\frac{e^{i\alpha z}d\alpha}{\alpha(\kappa-i\alpha)} + \frac{2e^{Kz}H_1^{(2)}(Kr)}{K\ H_1^{(2)'}(Kb)}\right]\cos\theta, \tag{7.14}$$

where (P) denotes the Cauchy principle value of the integral and $K_1(x)$ is the Bessel function of imaginary argument.

It is noted that the integral in (7.14) represents the standing wave part of the motion and the non-integral part is the radiation component of the motion. The r-derivative of the integral on $r = b$ is a simple Fourier integral which can easily be evaluated as $1 - e^{2Kz}$, and the r-derivative of the radiation component on $r = b$ is $2e^{Kz}$ so that the potential Ψ satisfies the boundary condition,

$$\left(\frac{\partial\Psi}{\partial r}\right)_{r=b} = \cos\theta.$$

Some simple algebraic calculation gives the real integral form of W

$$W = \left[\frac{2}{\pi}\int_o^\infty\left\{-\frac{K_1(Kr\alpha)}{\alpha K_1'(Kb\alpha)}\right\}\frac{d\alpha}{1+\alpha^2} + \frac{2H_1^{(2)}(Kr)}{H_1^{(2)'}(Kb)}\right]\cos\theta. \tag{7.15}$$

From the asymptotic nature of the Bessel function for small Kr and Kb, it turns out that

$$\lim_{\substack{Kr\to o \\ Kb\to o}}\left[-\frac{K_1(Kra)}{\alpha\ K_1'(Kba)}\right] = \frac{Kb^2}{r}\ . \tag{7.16}$$

Thus the integral in (7.15) is asymptotically equal to $\frac{Kb^2}{r}$, and the non-integral

term is approximately $-\frac{2Kb^2}{r}$ as both Kr and Kb tend to zero. This leads to the

final asymptotic value of W as

$$W \sim -\frac{Kb^2}{r}\cos\theta , \qquad (7.17)$$

which is identical with the form (6.10).

In order to estimate the effective range b < r < B within which (7.17) holds,

we can define B by the equation

$$Re\left[\int_b^\infty W\,dr\right] = -Kb^2\log(\frac{B}{b})\cos\theta. \qquad (7.18)$$

On the other hand, it follows from (7.15) that

$$\int_b^\infty W\,dr = \left[\frac{2}{\pi}\int_o^\infty\left\{-\frac{K_o(Kb\alpha)}{\alpha^2 K\,K_1'(Kb\alpha)}\right\}\frac{d\alpha}{1+\alpha^2} + \frac{2\,H_o^{(2)}(Kb)}{K\,H_1^{(2)'}(Kb)}\right]\cos\theta. \qquad (7.19)$$

We next use the asymptotic result for small Kb as

$$\left[-\frac{K_o(Kb\alpha)}{K\alpha^2 K_1'(Kb\alpha)}\right] \sim -Kb^2[\log(\frac{1}{2}Kb\alpha) + \gamma] \qquad (7.20)$$

to obtain the asymptotic representation for (7.19) in the form

$$\int_b^\infty W\,dr \sim [Kb^2\{\log(\frac{Kb}{2}) + \gamma\} + 2Kb^2\{\log(\frac{Kb}{2}) + \gamma + \frac{i\pi}{2}\}]\cos\theta, \qquad (7.21)$$

where γ is Euler's constant.

This result agrees with (7.18) provided that

$$B = \frac{1}{K}(2e^{-\gamma}) = (\frac{1}{2}e^{-\gamma})(k\tanh kh)^{-1} \equiv C(k\tanh kh)^{-1}, \qquad (7.22)$$

where $C \equiv \frac{1}{2}e^{-\gamma}$ is a numerical constant.

It is noted here that beyond the limiting radius r = B, W is oscillatory and

hence makes a small contribution to the quadratic force F_q discussed in the

previous section.

8. CONCLUDING REMARKS.

This work has been devoted to a nonlinear modification of the classical Morison equation. It has been shown that the nonlinear contribution to the Morison equation is $(F_d + F_w + F_q)$ which is the sum of the dynamic, waterline, and the quadratic wave forces associated with the irrotational component of the fluid flows. The salient feature of the nonlinear theory has been illustrated by means of a large vertical cylinder immersed in an ocean.

With regard to the geometrical shape of offshore structures, it can be added here that the present theory is general enough to incorporate any other axisymmetric solid structures of practical interest.

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