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ERROR ESTIMATES FOR THE FINITE ELEMENT SOLUTIONS OF VARIATIONAL INEQUALITIES

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ABSTRACT. For piecewise linear approximation of variational inequalities associated with the mildly nonlinear elliptic boundary value problems having auxiliary constraint conditions, we prove that the error estimate for $u-u_h$ in the $W^{1,2}$- norm is of order h.

KEY WORDS AND PHRASES. Finite Element, Variational Inequalities, Approximation, Mildly nonlinear.

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1. INTRODUCTION.

In this paper, we derive the finite element error estimates for the approximate solution of mildly nonlinear boundary value problems having auxiliary constraint conditions. A much used approach with any elliptic problem is to reformulate it in a weak or variational form . It has been shown by Noor and Whiteman

[1] that in the presence of a constraint, such an approach leads to a variational inequality which is the weak formulation. An approximate formulation of the variational inequality is then defined, and the error estimates involving the difference between the solutions of the exact and approximate formulation in the $W^{1,2}$-norm is obtained, which is in fact of order h. This result is an extension of that obtained by Falk [2] and Mosco and Strang [3] for the constrained linear problem.

2. MAIN RESULTS

For simplicity, we consider the problem of the following type:

$$\left.\begin{aligned} -\Delta u(\underline{x}) &= F(\underline{x},u), & \underline{x} &\in \Omega \\ u(\underline{x}) &= 0, & \underline{x} &\in \partial\Omega \end{aligned}\right\} \tag{2.1}$$

where Ω is a convex polygon domain in R^n with boundary $\partial\Omega$ and $\bar{\Omega} = \Omega \cup \partial\Omega$, its closure. The given function $f(u) = F(\underline{x},u) \in C^o(\bar{\Omega} \times R)$ is a real-value function involving the unknown u. If f(u) is both antimonotone and Lipschitz continuous, then it is known that there does exist a unique solution of (2.1); see Noor [4, p. 57-62]. We study this problem in the usual Sobolev space $W_2^1(\Omega) = H^1$, the space of functions which together with their generalized derivatives of order one are in $L_2(\Omega)$. The subspace of functions from H^1, which in a generalized sense satisfy the homogeneous boundary conditions on $\partial\Omega$, is $\overset{o}{W}{}_1^2(\Omega) = H_o^1$.

It has been shown by Tonti [5] that, in its direct variational formulation, (2.1) is equivalent to finding $u \in H_o^1$ such that

$$I[u] \leq I[v] \qquad \text{for all } v \in H_o^1 ,$$

where

$$\begin{aligned} I[v] &= \int_\Omega \left(\frac{\partial v}{\partial x}\right)^2 d\Omega - 2\int_\Omega \int_o^v f\,(\eta)\, d\eta\, d\Omega \\ &= a(v,v) - 2F(v) \end{aligned} \tag{2.2}$$

is the energy functional associated with (2.1).

We now consider the case when the solution u of (2.1) is required to satisfy the condition $u \geq \psi$, where $\psi \in C^o(\bar{\Omega}) \cap H^1$, $\psi \leq 0$ on $\partial\Omega$; (see Glowinski [6, p. IV. 2]). In this situation, our problem is to find

$$u \in K \overset{\text{def}}{=} \{v;\ v \in H_o^1,\ v \geq \psi \text{ on } \Omega\},$$

a closed convex subset of H_o^1 (see Mosco [7]), such that u minimizes I [v] on K. It has been shown by Noor and Whiteman [1] that the minimum of I [v] on K can be characterized by a class of variational inequalities

$$a(u,v-u) \geq \langle F'(u), v-u \rangle \quad \text{for all } v \in K, \tag{2.3}$$

where F' (u) is the Frechet differential of F (u) and is in fact,

$$\langle F'(u), v \rangle = \int_\Omega f(u)\, v\, d\Omega. \tag{2.4}$$

The finite dimensional form of (2.3) is to find $u_h \in K_h$ such that

$$a(u_h, v_h-u_h) \geq \langle F'(u_h), v_h-u_h \rangle \text{ for all } v_h \in K_h. \tag{2.5}$$

Here K_h is a finite dimensional convex subset of H_o^1; for the construction of K_h, see Mosco [3] . Let Ω be the convex polygon. We partition it into triangles of side length less than h. We consider $S_h \subset H_o^1$, the subspace of continous piecewise linear functions on the triangulation of Ω , vanishing on the boundary $\partial\Omega$. Let ψ_h be the interpolant of ψ such that ψ_h agrees with ψ at all the vertices of the triangulation. For our purpose, it is enough to choose the finite dimensional convex subset $K_h = S_h \cap \{v_h \geq \psi_h \text{ on } \Omega\}$. For other choices of convex subsets K_h, see Nitsche [8] , where he has chosen $K_h = K \cap S_h$.

We also want to know the regularity of the function $u \in K$ satisfying (2.3). In this case Brézis and Stampacchia [9] have shown that, if ψ lies in both H_o^1 and H^2, then the solution $u \in K$ satisfying (2.3) also lies in H^2. Its norm can be estimated from the data:

$$||u||_2 \leq e||\psi||_2.$$

Moreover, if $\tilde{u}$ is the interpolant of u, which agrees with u at every vertex of Ω, then $\tilde{u}$ lies in K_h. It is well known from approximation theory (see Strang and Fix [10] that

$$||u-\tilde{u}|| \leq eh||u||_2. \tag{2.6}$$

We also note that in certain cases, the equality holds instead of the inequality in (2.3). This happens when v, together with 2u-v, also lies in K. In this case, we get

$$a(u,v-u) = \langle F'(u), v-u \rangle. \tag{2.7}$$

Finally let C and C_h be the cones composed of non-negative functions on H_o^1 and its subspace S_h. Thus, it is clear that

$$U = u - \psi \text{ is in } C$$

$$U_h = u_h - \psi_h \text{ is in } C_h.$$

From these relations, it follows that

$$u-u_h = U-U_h + \psi - \psi_h. \tag{2.8}$$

DEFINITION. An operator T on H_o^1 is said to be quasi-monotone, if for all $z,u,v,w, \in H_o^1$,

$$\langle Tu-Tv, w-z \rangle \geq 0. \tag{2.9}$$

We also need the following result of Mosco and Strang [3].

THEOREM 1: Suppose that U $\geq$ 0 in the plane polygon Ω and that U lies in both H_o^1 and H^2. Then, there exists a V_h in S_h such that

$$0 \leq V_h \leq U \text{ in } \Omega$$

and

$$||U-V_h|| \leq eh||U||_2. \tag{2.10}$$

Now we state and prove the main result.

THEOREM 2. Let $a(u,v)$ be a continuous coercive bilinear form and $F'(u)$ be a quasi-monotone operator on H_o^1. If $V_h \varepsilon C_h$ and $2U-V_h \varepsilon C$, then

$$||u-u_h|| = O(h),$$

where u and u_h are the solutions of (2.3) and (2.5) respectively.

PROOF. Since both $v = \psi + V_h$ and $2u-v = \psi + (2U-V_h)$ are in K, we have from (2.3) and (2.7) that

$$a(u,V_h-U) = \langle F'(u), V_h-U\rangle. \tag{2.11}$$

Letting $v_h = \psi_h + V_h$ and $u_h = \psi_h + U_h$ in (2.5), we have

$$a(u_h,V_h-U_h) \geq \langle F'(u_h) V_h- U_h\rangle. \tag{2.12}$$

Using $v = \psi + U_h$ in (2.3), we get

$$a(u, U_h-U) \geq \langle F'(u), U_h-U\rangle. \tag{2.13}$$

From (2.11) and 13), we obtain

$$a(u,U_h-V_h) \geq \langle F'(u), U_h-V_h\rangle, \tag{2.14}$$

and from (2.12) and (2.14), we get

$$a(u-u_h, U_h-V_h) \geq \langle F'(u) - F'(u_h), U_h-V_h\rangle .$$

Thus, using the quasi-monotonicity of $F'(u)$, we have

$$a(u-u_h, U_h-V_h) \geq 0,$$

which can be written as

$$a(u-u_h, U-U_h) \leq a(u-u_h, U-V_h). \tag{2.15}$$

Now by coercivity of $a(u,v)$, it follows that there exists a constant $\alpha > 0$ such that

$$\begin{aligned} \alpha ||u-u_h||^2 &\leq a(u-u_h, u-u_h) \\ &= a(u-u_h, \psi-\psi_h) + a(u-u_h, U-U_h), \text{ from (2.8)} \\ &\leq a(u-u_h,\psi-\psi_h) + a(u-u_h, U-V_h) \\ &\leq \beta||u-u_h|| \{ ||\psi-\psi_h|| + ||U-V_h|| \}, \end{aligned}$$

where $\beta > 0$ is a continuity constant of the bilinear form $a(u,v)$.

Hence, it follows that

$$||u-u_h|| \leq \frac{\beta}{\alpha} ||\{ \quad ||\psi-\psi_h|| \quad + || U-V_h||\}||$$
$$\leq \frac{\beta}{\alpha} Ch\{ \quad || \psi ||_2 + \quad ||U ||_2\}, \text{ by } (2.6)$$

and (2.10), from which the required estimate follows.

Remark: The problem of deriving the L^∞ -norm estimates for the mildly nonlinear problems having constraint conditions is still open.

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