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ON RANK 4 PROJECTIVE PLANES

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ABSTRACT. Let a finite projective plane be called rank m plane if it admits a collineation group G of rank m, let it be called strong rank m plane if moreover $G_P = G_l$ for some point-line pair (P,l). It is well known that every rank 2 plane is desarguesian (Theorem of Ostrom and Wagner). It is conjectured that the only rank 3 plane is the plane of order 2. By [1] and [7] the only strong rank 3 plane is the plane of order 2. In this paper it is proved that no strong rank 4 plane exists.

KEY WORDS AND PHRASES. *Projective planes, rank 4 groups.*

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1. INTRODUCTION.

In [6] Kallaher gives restrictions for the order n of a finite rank 3 projective plane and conjectures that no such plane exists if $n \neq 2$. Let a finite projective plane be called a strong rank m projective plane if it admits a rank m collineation group G such that $G_P = G_l$ for some point-line pair (P,l). By Bachmann [1] and Kantor [7] no strong rank 3 projective plane of order $n \neq 2$ exists. If the conjecture is true that for projective designs the representations on the points and on the blocks of an arbitrary transitive collineation group are similar (see Dembowski [2], p. 78), then every rank m projective plane is a strong rank m plane.

We shall prove in this article the following

THEOREM: No strong rank 4 projective plane exists.

To prove the Theorem we first divide the strong rank 4 planes into 3 classes (see Lemma 2 and 3). Then we associate with each such plane (0,1)-matrices A and C of trace 0 (see [3]). Finally we show that for each class the trace condition contradicts the integrality of the multiplicities of the eigenvalues of A or C.

We shall use the following notations, definitions and basic results (see Dembowski [2]):

A collineation group of a projective plane has equally many point orbits and line orbits. The rank of a transitive permutation group is the number of orbits of the stabilizer of one of the permuted elements. If G is a (point or line) transitive collineation group of a projective plane, then the point and line ranks are equal (Kantor [8]). A rank m projective plane is a projective plane which admits a transitive collineation group whose (point or line) rank is m ($m \geq 2$). The lines (points) are identified with the set of points (lines) on them. We write $P \in l^G$ if and only if $P \in l^\gamma$ for all $\gamma \in G$.

2. PROOF OF THE THEOREM.

Let $\mathbb{P} = (P, L, \in)$ be a projective plane of finite order n and let G be a rank 4 collineation group of $\mathbb{P}$ such that $G_{P_o} = G_{l_o}$ for some point-line pair (P_o, l_o). It is easily seen that $n \geq 3$. A bijective map $\sigma : P \longrightarrow L$ is defined by $P^\sigma = l$ if and only if $P = P_o^\gamma$ and $l = l_o^\gamma$ for some $\gamma \in G$. If $i \in \mathbb{N}$ we write l_i for P_i^σ. Clearly $P_o^\sigma = l_o$ and

$$P^{\sigma\gamma} = P^{\gamma\sigma} \ , \ l^{\sigma^{-1}\gamma} = l^{\gamma\sigma^{-1}} \quad \text{for all } P \in P,\ l \in L,\ \gamma \in G. \qquad (1)$$

For $P \in P$ G_P has exactly 4 orbits $\{P\}$, $\Delta(P)$, $\Gamma(P)$, $\Pi(P)$. We choose the notation in such a way that

$$(\Delta(P))^\gamma = \Delta(P^\gamma) \ , \ (\Gamma(P))^\gamma = \Gamma(P^\gamma) \ , \ (\Pi(P))^\gamma = \Pi(P^\gamma) \quad \text{for all } P \in P, \gamma \in G \qquad (2)$$

LEMMA 2.1: If $\Lambda_1, \Lambda_2, \Lambda_3 \in \{\Delta, \Gamma, \Pi\}$, then $|\Lambda_1(A) \cap \Lambda_2(B)| = |\Lambda_1(A') \cap \Lambda_2(B')|$ if $A \in \Lambda_3(B)$ and $A' \in \Lambda_3(B')$.

PROOF: If $A \in \Lambda_3(B)$, $A' \in \Lambda_3(B')$, then for some $\gamma \in G$, $\gamma_o \in G_B$

$$B' = B^\gamma = B^{\gamma_o\gamma} \ , \ A' = A^{\gamma_o\gamma} \text{, whence by (2)}$$

$$|\Lambda_1(A') \cap \Lambda_2(B')| = |\Lambda_1(A)^{\gamma_o\gamma} \cap \Lambda_2(B)^{\gamma_o\gamma}| = |(\Lambda_1(A) \cap \Lambda_2(B))^{\gamma_o\gamma}| = |\Lambda_1(A) \cap \Lambda_2(B)| .$$

LEMMA 2.2: Suppose that $P_o \in l_o$. Then $l_o - \{P_o\}$ and $P_o - \{l_o\}$ are G_{P_o}-orbits, say $\Delta(P_o) = l_o - \{P_o\}$ and $l_2^{G_{P_o}} = P_o - \{l_o\}$ with $P_2^{G_{P_o}} = \Gamma(P_o)$. $P_1 \in \Delta(P_o)$ and $P_3 \in \Pi(P_o)$ can be chosen such that $P_1 \in l_o$; $P_o, P_2, P_3 \in l_2$; $P_2 \in l_1$; $P_1 \notin l_3$ (Fig. 1).

The case described by Lemma 2 will be called case I.

PROOF: If $l_o - \{P_o\}$ is not a G_{P_o} - orbit, then it is the union of 2 orbits, say $l_o - \{P_o\} = \Delta(P_o) \cup \Gamma(P_o)$. Then $P_o - \{l_o\}$ is a line orbit $l^{G_{P_o}}$ and $\Pi(P_o) = P^{G_{P_o}}$ with $l = P^\sigma$. This leads to the contradiction

$$n = |1^{G_{P_o}}| = |P^{\sigma\, G_{P_o}}| = |P^{G_{P_o}\sigma}| = |P^{G_{P_o}}| = |\mathcal{P} - 1_o| = n^2.$$

Hence we may assume that $\Delta(P_o) = 1_o - \{P_o\}$.

Dually: $P_o - \{1_o\}$ is a G_{P_o} - orbit, say $1_2^{G_{P_o}} = P_o - \{1_o\}$ where $P_2^{G_{P_o}} = \Gamma(P_o)$ (note that $P_o \notin 1_1$).

$|P_2^{G_{P_o}}| = |1_2^{G_{P_o}}| = |P_o - \{1_o\}| = n$ implies $\Gamma(P_o) \cap 1_2 = \{P_2\}$.

For any point $Q \neq P_o$, P_2 on 1_2 holds $Q^{G_{P_o}} = \Pi(P_o)$.

Let $P_1' \in \Delta(P_o)$. If $P_2 \in 1_1'$ put $P_1 = P_1'$. If $P_2 \notin 1_1'$ then

$$|P_1'^{G_{P_o,P_2}}| = |1_1'^{G_{P_o,P_2}}| \geq |(1_1' \cap 1_2)^{G_{P_o,P_2}}| = n-1 = |(1_1' \cap 1_2)^{\sigma\, G_{P_o,P_2}}|$$

$\geq |((1_1' \cap 1_2)^{\sigma} \cap 1_o)^{G_{P_o,P_2}}|$; this implies $P_1^{G_{P_o,P_2}} = P_1$ for some point $P_1 \in \Delta(P_o)$ and hence $P_2 \in 1_1$.

It remains to prove that $P_3 \in \Pi(P_o) \cap 1_2$ exists such that $P_1 \notin 1_3$. If no such P_3 exists then $P_1 \in Q^{\sigma}$ for all $Q \in 1_2 - \{P_o, P_2\}$ and hence $G_{P_o,P_2} \leq G_{P_o,P_1}$. Let $\gamma \in G$ be such that $P_2^{\gamma} = P_o$. Then $1_2^{\gamma} = 1_o$ and therefore $P_o^{\gamma} \in 1_o$, $P_o^{\gamma} \neq P_o$ and $P_o^{\gamma\gamma_o} = P_1$ for some $\gamma_o \in G_{P_o}$. It follows that $G_{P_o,P_1} = (\bar{\gamma}\gamma_o)^{-1} G_{P_o,P_2} \bar{\gamma}\gamma_o$.

Hence

$$G_{P_o,P_1} = G_{P_o,P_2}. \tag{3}$$

Further

$$P_2 \notin 1_1^{\gamma_o} \quad \text{for some } \gamma_o \in G_{P_o}, \tag{4}$$

for otherwise $P_2^{\gamma_o'} \in 1_1^{\gamma_o''}$ for all $\gamma_o', \gamma_o'' \in G_{P_o}$ which cannot occur.

$$P_2^{\gamma_o} \in 1_1 \text{ for some } \gamma_o \in G_{P_o} \quad \text{if and only if } \gamma_o \in G_{P_2}. \tag{5}$$

To prove (5) note that by (4) through any point of $1_2 - \{P_o\}$ goes at least one and hence exactly one line of $1_1^{G_{P_o}}$; (3) and $P_2 \in 1_1$ then imply (5).

Let's apply (5) to G_{P_1} in place of G_{P_o}:

$$\Delta(P_1) = 1_1 - \{P_1\}; \quad \Gamma(P_1) = \Gamma(P_o^{\gamma}) = (\Gamma(P_o))^{\gamma} = P_2^{G_{P_o}\gamma} = P_o^{\gamma^{-1}G_{P_o}\gamma} = P_o^{G_{P_1}}$$

where $\gamma \in G$ such that $P_o^{\gamma} = P_1$, $P_2^{\gamma} = P_o$; $\Pi(P_1) = S^{G_{P_1}}$ for some $S \in l_o - \{P_o, P_1\}$; hence $P_o^{\gamma_1} \in l_2$ for some $\gamma_1 \in G_{P_1}$ if and only if $\gamma_1 \in G_{P_o}$.

It follows that $R \notin P_o^{G_{P_1}}$ for any $R \in l_2 - \{P_o, P_2\}$. Let $r = R^{\sigma}$ for some such R.

Of the 3 orbits $(P_o, l_1)^G$, $(P_o, l_2)^G$, $(P_o, r)^G$ induced by G on $\mathcal{P} \times \mathcal{L} - (P_o, l_o)^G$ only one consists of flags. Thus (P_1, l_o) and (P_1, r) and then also (P_o, l_1) and (R, l_1) belong to the same G-orbit. This contradicts $R \notin P_o^{G_{P_1}}$. Hence there exists $P_3 \in \Pi(P_o) \cap l_2$ such that $P_1 \notin l_3$.

LEMMA 2.3: Suppose that $P_o \notin l_o$. Then l_o and dually P_o are G_{P_o}-orbits, say $\Delta(P_o) = l_o$. $P_1 \in \Delta(P_o)$, $P_2 \in \Gamma(P_o)$, $P_3 \in \Pi(P_o)$ can be chosen such that either $P_o, P_2, P_3 \in l_1$; $P_1 \in l_2, l_3$; $P_2 \notin l_3$; $P_3 \notin l_2$
or $P_o, P_1, P_3 \in l_2$; $P_1 \in l_o$; $\Gamma(P_o) \cap l_2 = \{P_2^{\gamma_o}\}$ for some $\gamma_o \in G_{P_o}$; $P_2^{\gamma_o} \in l_1$; $P_1, P_2, P_3 \notin l_1, l_3$. In both cases $n \geq 4$.

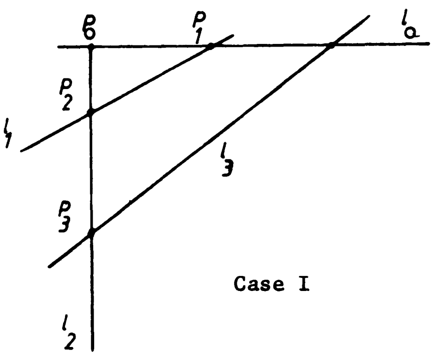

Figure 1

The 2 cases described by Lemma 3 will be called case II1 resp. case II2 (Fig. 2).

PROOF: It is easily seen that l_o and P_o are G_{P_o} - orbits; say $\Delta(P_o) = l_o$. Let $P_1 \in \Delta(P_o)$. We have to distinguish 2 cases:

Case II1: $P_o \in l_1$

Case II2: $P_o \notin l_1$.

CASE II1: Clearly $P_o = l_1^{G_{P_o}}$ and $\Gamma(P_o) = P_2^{G_{P_o}}$, $\Pi(P_o) = P_3^{G_{P_o}}$ for some $P_2, P_3 \in l_1 - \{P_o, l_o \cap l_1\}$. If $P_2 \in l_3$ then $(P_2, l_3) \in (P_o, l_1)^G$, hence $(P_3, l_2) \in (P_1, l_o)^G$, so $P_3 \in l_2$.

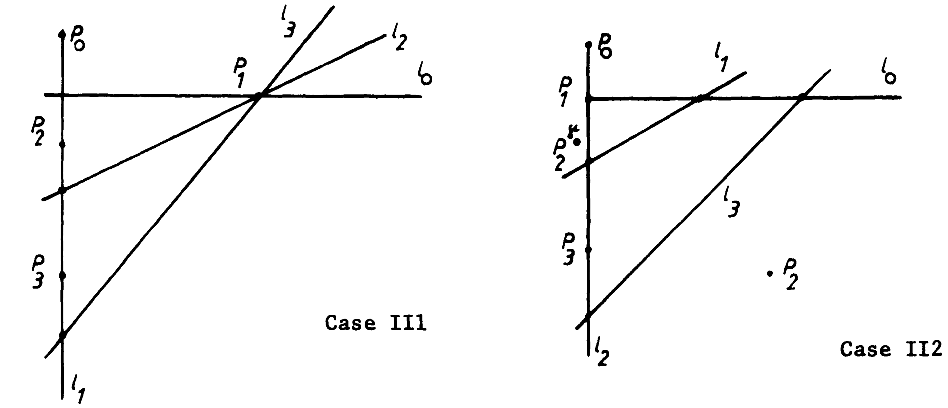

Figure 2

Analogously $P_2 \in l_3$ if $P_3 \in l_2$. Thus

$$P_2 \in l_3 \quad \text{if and only if} \quad P_3 \in l_2 . \tag{6}$$

Similarly one proves

$$P_1 \in l_2,\ l_3 . \tag{7}$$

If $n > 3$ then, by (6), we can choose P_2, P_3 such that $P_2 \notin l_3$, $P_3 \notin l_2$. Let's show that $n > 3$ (Fig. 3). Suppose that $n = 3$. Put $P_4 = l_o \cap l_1$.

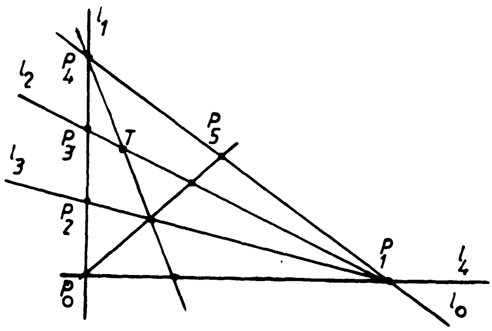

Figure 3

Then, since $P_o \in l_1$ and $P_1 \in l_o$, $l_4 = P_o P_1$. Let $P_5 \in l_o - \{P_1, P_4\}$. Then $P_o \in l_5$ and then $l_5 \cap l_2 = (P_o P_5 \cap l_3)P_4 \cap l_2$. Denote this point by T. Clearly $P_2P_5 \cap l_2 = T$. Since $(P_2P_5)^{\sigma^{-1}} \in l_2 \cap l_5$ we obtain the contradiction $(P_2P_5)^{\sigma^{-1}} \in P_2P_5$.

CASE II2: We may assume that $P_o, P_1, P_3 \in l_2$ where $P_2 \in \Gamma(P_o)$. Then $G_{P_o,P_1} = G_{P_o,P_2}$. We first assume that $n>3$. $|P_2^{G_{P_o}}| = |l_2^{G_{P_o}}| = n+1$, hence $|l_2 \cap \Gamma(P_o)| = 1$; let $l_2 \cap \Gamma(P_o) = \{P_2^{\gamma_o}\}$ with some $\gamma_o \in G_{P_o}$. Then $P_3^{G_{P_o,P_1}} = l_2 - \{P_o, P_1, P_2^{\gamma_o}\}$ and hence, since $P_2^{\gamma_o}$ is invariant under G_{P_o,P_1} and since $n>3$, $l_1 \cap l_2 = P_2^{\gamma_o}$.

The only G-orbit of $\mathcal{P} \times \mathcal{L}$ consisting of flags is $(P_o, l_2)^G$. Hence (P_1, l_2), (P_3, l_2), $(P_1, l_o) \in (P_o, l_2)^G$. $P_o \notin l_1$ then implies that (P_2, l_1), (P_2, l_3), (P_o, l_1), $(P_2, l_o) \notin (P_o, l_2)^G$, in particular $P_2 \notin l_o, l_1, l_3$.

If $P_1 \in l_3$ then $(P_1, l_3) \in (P_o, l_2)^G$ and hence $(P_3, l_1) \in (P_2, l_o)^G$. Since also $(P_o, l_1) \in (P_2, l_o)^G$ we have $P_o^{\gamma_1} = P_3$ for some $\gamma_1 \in G_{l_1} = G_{P_1}$. This implies that G_{P_1} is transitive on $l_2 - \{P_1, P_2^{\gamma_o}\}$ which is impossible. Hence $P_1 \notin l_3$.

If $n=3$ then $l_2 = \{P_o, P_1, P_2^{\gamma_o}, P_3\}$. γ_o is of order 4, for if $\gamma_o^2 = 1$ then $(P_2, l_2^{\gamma_o}) \in (P_2^{\gamma_o}, l_2)^{G_{P_o}}$ which is impossible. Moreover $P_2^{\gamma_o^2} \neq P_2$ since otherwisw $\gamma_o^2 \in G_{P_o, P_2} = 1$. It follows that $|(P_2P_2^{\gamma_o^2})^{G_{P_o}}| = 4$ which contradicts $(P_2P_2^{\gamma_o^2})^{G_{P_o}} = \{P_2P_2^{\gamma_o^2}, P_2^{\gamma_o}P_2^{\gamma_o^3}\}$. This completes the proof of the Lemma.

Let us now associate with $(G, \mathbb{P})$ 3 (0,1)-matrices.

If $\rho(P)$ is a G-orbit then let $\rho'(P)$ denote the paired orbit (see Wielandt [9]). If $Q \in \rho(P)$ then $Q = P^\gamma$ for some $\gamma \in G$ and $Q^\gamma \in (\rho(P))^\gamma = \rho(P^\gamma) = \rho(Q)$. Hence $Q^{\gamma^{-1}} = P \in \rho'(Q)$, i.e.

$$Q \in \rho(P) \text{ implies that } P \in \rho'(Q). \tag{8}$$

This implies that in

Case I:	Case II1:	Case II2:
$\Delta'(P) = \Gamma(P)$	$\Delta'(P) = \Delta(P)$	$\Delta'(P) = \Gamma(P)$
$\Gamma'(P) = \Delta(P)$	$\Gamma'(P) = \Pi(P)$ resp. $\Gamma(P)$	$\Gamma'(P) = \Delta(P)$
$\Pi'(P) = \Pi(P)$	$\Pi'(P) = \Gamma(P)$ resp. $\Pi(P)$	$\Pi'(P) = \Pi(P)$.

Now let $P = \{P_1, P_2, \ldots, P_v\}$, $L = \{l_1, l_2, \ldots, l_v\}$, $l_k = P_k^\sigma$ $(k = 1, 2, \ldots, v)$. Let A be the (0,1)-matrix with rows enumerated by the points P_k and columns by $\Delta(P_k)$ and such that $(P_k, \Delta(P_i)) = 1$ if and only if $P_k \in \Delta(P_i)$. Let B, C be the analogous matrices with $\Gamma(P_k)$ resp. $\Pi(P_k)$ in place of $\Delta(P_k)$.

We have in

case I:	case II1:		case II2:
$A^t = B,\ C^t = C$	$A^t = A,\ B^t = C$	if $\Gamma'(P)=\Pi(P)$	$A^t = B,\ C^t = C$
	$A^t = A,\ B^t = B,\ C^t = C$	if $\Gamma'(P)=\Gamma(P)$	

Let $k = |\Delta(P)|$, $l = |\Gamma(P)|$, $m = |\Pi(P)|$,

$$|\Delta(P) \cap \Delta(Q)| = \begin{Bmatrix} \lambda \\ \mu \\ \nu \end{Bmatrix} \quad \text{if} \quad Q \in \begin{cases} \Delta(P) \\ \Gamma(P) \\ \Pi(P) \end{cases}$$

$$|\Pi(P) \cap \Pi(Q)| = \begin{Bmatrix} \lambda' \\ \mu' \\ \nu' \end{Bmatrix} \quad \text{if} \quad Q \in \begin{cases} \Pi(P) \\ \Delta(P) \\ \Gamma(P) \end{cases} .$$

A straightforward calculation shows that

$I + A + B + C = J$, the $v \times v$-matrix with 1's in every entry

$$A^t A = k I + \lambda A + \mu B + \nu C$$

$$C^t C = m I + \mu' A + \nu' B + \lambda' C$$

$A\,J = k\,J$

$B\,J = l\,J$

$C\,J = m\,J.$

Now we determine the eigenvalues of A in case II1 and of C in the cases I and II2.

CASE II1: $k = n + 1$

$l = n_2(n + 1)$ where $n_2 = |P_2^{G_{P_o,P_1}}|$

$m = n_3(n + 1)$ where $n_3 = |P_3^{G_{P_o,P_1}}|$

$k + l + m + 1 = v = n^2 + n + 1,\ n_2 + n_3 = n - 1,\ \lambda = \mu = \nu = 1.$

It follows that $A^2 = A^tA = (n + 1)\,I + A + B + C = n\,I + J$; hence $(A - (n + 1)\,I)(A^2 - n\,I) = 0$. This gives the eigenvalues of A:

$\lambda_1 = n + 1,\ \ \lambda_{2,3} = \pm\sqrt{n}$.

CASE I: $k = l = n,\ m = n\,(n - 1),\ k + l + m + 1 = v = n^2 + n + 1.$

We have
$$\lambda' = |\Pi(P_o) \cap \Pi(P_3)|$$
$$\mu' = |\Pi(P_o) \cap \Pi(P_1)|$$
$$\nu' = |\Pi(P_o) \cap \Pi(P_2)|.$$

Let's calculate λ':

$$n(n - 1) = |\Pi(P_3)| = |\Pi(P_3) \cap \Delta(P_o)| + |\Pi(P_3) \cap \Gamma(P_o)| + |\Pi(P_3) \cap \Pi(P_o)| + 1 \quad (9)$$

(note that $\Gamma(P_3) = P_2^{G_{P_3}}$ and hence $P_o \in \Pi(P_3)$).

$$n = |\Delta(P_o)| = |\Delta(P_o) \cap \Delta(P_3)| + |\Delta(P_o) \cap \Gamma(P_3)| + |\Delta(P_o) \cap \Pi(P_3)| . \quad (10)$$

Clearly

$$|\Delta(P_o) \cap \Delta(P_3)| = 1 \quad (11)$$

$$|\Delta(P_o) \cap \Gamma(P_3)| = |\Delta(P_3) \cap \Gamma(P_o)| = 2. \quad (12)$$

PROOF of (12): $P_o \in \Pi(P_3)$ and $P_3 \in \Pi(P_o)$, hence, by Lemma 1, $|\Delta(P_o) \cap \Gamma(P_3)| = |\Delta(P_3) \cap \Gamma(P_o)|$. $P_2 \notin 1_1^{\gamma_o}$ for some $\gamma_o \in G_P$. Thus $|P_3^{G_{P_o,P_2}}| = n-1$ implies that $|1_1^{\gamma_o G_{P_o,P_2}}| \geq n-1$. Hence $|1_1^{\gamma_o G_{P_o,P_2}}| = n-1$. Since $P_1 \notin P_1^{\gamma_o G_{P_o,P_2}}$ we then have $P_1^{G_{P_o,P_2}} = P_1$, i.e. $G_{P_o,P_2} \leq G_{P_o,P_1}$. Since both groups are conjugate (see the proof of Lemma 2) this gives $G_{P_o,P_1} = G_{P_o,P_2}$.

Moreover $P_2 \in 1_1^{\gamma_o^*}$ if and only if $\gamma_o^* \in G_{P_1}$. Thus $P_2 \in 1_3^{\gamma_o'}$, $P_2^{\gamma_o''} \in 1_3^{\gamma_o'}$, $P_2^{\gamma_o''} \neq P_2$ for some $\gamma_o', \gamma_o'' \in G_{P_o}$. Since , by the above, G_{P_o,P_2} is transitive on $1_o - \{P_o, P_1\}$, $|P_2^{\gamma_o'' G_{P_o,P_2}}| = n-1$; hence $|1_3^{\gamma_o''} \cap P_2^{G_{P_o}}| = |1_3 \cap P_2^{G_{P_o}}| = 2$. This proves (12).

Equations (10), (11), (12) imply

$$|\Pi(P_3) \cap \Delta(P_o)| = n-3. \tag{13}$$

To determine $|\Pi(P_3) \cap \Gamma(P_o)|$ we use

$$n = |\Gamma(P_o)| = |\Gamma(P_o) \cap \Delta(P_3)| + |\Gamma(P_o) \cap \Gamma(P_3)| + |\Gamma(P_o) \cap \Pi(P_3)|. \tag{14}$$

By (12) $|\Gamma(P_o) \cap \Delta(P_3)| = 2$. Since $\Gamma(P_3) = P_2^{G_{P_3}}$, $|\Gamma(P_o) \cap \Gamma(P_3)| = |P_2^{G_{P_o}} \cap P_2^{G_{P_3}}| = |1_2^{G_{P_o}} \cap 1_2^{G_{P_3}}| = 1$. It follows that

$$|\Pi(P_3) \cap \Gamma(P_o)| = n-3. \tag{15}$$

Equations (9), (13) and (15) imply that

$$\lambda' = n^2 - 3n + 5. \tag{16}$$

Analogously we calculate μ' and ν':

$n(n-1) = |\Pi(P_1)| = |\Pi(P_1) \cap \Delta(P_o)| + |\Pi(P_1) \cap \Gamma(P_o)| + |\Pi(P_1) \cap \Pi(P_o)|$

with $|\Pi(P_1) \cap \Delta(P_o)| = n-1$.

In $n = |\Gamma(P_o)| = |\Gamma(P_o) \cap \Delta(P_1)| + |\Gamma(P_o) \cap \Gamma(P_1)| + |\Gamma(P_o) \cap \Pi(P_1)|$

$|\Gamma(P_o) \cap \Delta(P_1)| = 1$ by the proof of (12) and $|\Gamma(P_o) \cap \Gamma(P_1)| = |P_2^{G_{P_o}} \cap P_o^{G_{P_1}}| = |1_2^{G_{P_o}} \cap 1_o^{G_{P_1}}| = 0$. Hence $|\Pi(P_1) \cap \Gamma(P_o)| = n - 1$ and thus

$$\mu' = (n - 1)(n - 2). \tag{17}$$

$n(n - 1) = |\Pi(P_2)| = |\Pi(P_2) \cap \Delta(P_o)| + |\Pi(P_2) \cap \Gamma(P_o)| + |\Pi(P_2) \cap \Pi(P_o)|$.

In $n = |\Delta(P_o)| = |\Delta(P_o) \cap \Delta(P_2)| + |\Delta(P_o) \cap \Gamma(P_2)| + |\Delta(P_o) \cap \Pi(P_2)|$ $|\Delta(P_o) \cap \Delta(P_2)| = 0$ and $|\Delta(P_o) \cap \Gamma(P_2)| = |P_1^{G_{P_o}} \cap P_1^{G_{P_2}}| = |1_1^{G_{P_o}} \cap 1_1^{G_{P_2}}| = 1$ (note that $|1_1^{G_{P_2}}| = |P_1^{G_{P_2}}| = n$ and hence $\Gamma(P_2) = P_1^{G_{P_2}}$). Hence $|\Pi(P_2) \cap \Delta(P_o)|$ $= n - 1$.

Further $|\Pi(P_2) \cap \Gamma(P_o)| = |\Gamma(P_o)| - |\Gamma(P_o) \cap \Delta(P_2)| - |\Gamma(P_o) \cap \Gamma(P_2)| - 1$ where $|\Gamma(P_o)| = n$, $|\Gamma(P_o) \cap \Delta(P_2)| = 0$ and $|\Gamma(P_o) \cap \Gamma(P_2)| = |P_2^{G_{P_o}} \cap P_1^{G_{P_2}}| = |1_2^{G_{P_o}} \cap 1_1^{G_{P_2}}| = 0$. Hence $|\Pi(P_2) \cap \Gamma(P_o)| = n - 1$. It follows that

$$\nu' = (n - 1)(n - 2). \tag{18}$$

Equations (16), (17) and (18) imply that $C^2 = C^tC = n(n - 1) I + (n - 1)(n - 2)(A + B) + (n^2 - 3n + 5) C = n(n - 1) I + (n - 1)(n - 2)(J - I) + 3C$ and then $(C - n(n - 1) I)(C^2 - 3C - 2(n - 1) I) = 0$.

The eigenvalues of C are $\lambda_1 = n(n - 1)$; $\lambda_{2,3} = (3 \pm \sqrt{8n + 1})/2$.

CASE II2: $k = l = n + 1$, $m = (n - 2)(n + 1)$, $k + l + m + 1 = v = n^2 + n + 1$. By the proof of Lemma 3 $n > 4$. Let's determine λ', μ', ν':

$(n + 1)(n - 2) = |\Pi(P_3)| = |\Pi(P_3) \cap \Delta(P_o)| + |\Pi(P_3) \cap \Gamma(P_o)| + |\Pi(P_3) \cap \Pi(P_o)| - 1$. In $n + 1 = |\Delta(P_o)| = |\Delta(P_o) \cap \Delta(P_3)| + |\Delta(P_o) \cap \Gamma(P_3)| + |\Delta(P_o) \cap \Pi(P_3)|$ clearly $|\Delta(P_o) \cap \Delta(P_3)| = 1$. Let's show that

$$|\Delta(P_o) \cap \Gamma(P_3)| = |\Delta(P_3) \cap \Gamma(P_o)| = 2 \tag{19}$$

$$|\Delta(P_1) \cap \Gamma(P_o)| = 2. \tag{20}$$

PROOF of (19) and (20): By Lemma 1 $|\Delta(P_o)\cap\Gamma(P_3)| = |\Delta(P_3)\cap\Gamma(P_o)|$.

For $\gamma'_o\in G_{P_o}$

$$P_2^{\gamma_o}\in l_1^{\gamma'_o},\quad l_1^{\gamma'_o}\neq P_2^{\gamma_o}P_2 \text{ if and only if } \gamma'_o\in G_{P_o,P_1}, \tag{21}$$

for otherwise $P_2^{\gamma_o}\in l_1^{\gamma'_o G_{P_o,P_1}}$, $l_1\notin l_1^{\gamma'_o G_{P_o,P_1}}$, $|l_1^{\gamma'_o G_{P_o,P_1}}| = n-2$, $P_2\in l_1^{\gamma'_o G_{P_o,P_1}\gamma_o^{-1}}$, which leads to the contradiction $n+1 = |l_1^{G_{P_o}}| \geq (|P_2^{\gamma_o}|-2) + (|P_2|-2) = 2(n-1)$.

Further

$$P_2^{\bar\gamma_o}\notin l_1\cup P_2^{\gamma_o}P_2 \quad\text{for some } \bar\gamma_o\in G_{P_o}: \tag{22}$$

otherwise, since $|P_2^{G_{P_o}}|\geq 5$, every line of $l_1^{G_{P_o}}$ would contain at least 3 points of $P_2^{G_{P_o}}$ and this would imply that a point of $P_2^{G_{P_o}} - (l_1\cup P_2^{\gamma_o}P_2)$ exists. By (21) and (22) $P_2^{\gamma_o}, P_2^{\bar\gamma_o}\in l_3^{\gamma^*_o}$, $P_2^{\gamma_o}\neq P_2^{\bar\gamma_o}$ for some $\gamma^*_o\in G_{P_o}$. Since $|l_3^{\gamma^*_o G_{P_o,P_1}}| = n-2$, $|P_2^{G_{P_o}}\cap l_3| = |\Gamma(P_o)\cap\Delta(P_3)| = 2$. This proves (19).

Each of the $n-2$ lines of $l_3^{G_{P_o}} - \{P_2^{\gamma_o}P_2\}$ through $P_2^{\gamma_o}$ contains exactly one point of $P_2^{G_{P_o}} - \{P_2^{\gamma_o}\}$. Together with $P_2^{\gamma_o}$, P_2 this gives n points of $P_2^{G_{P_o}}$. It follows that exactly one point of $P_2^{G_{P_o}} - \{P_2^{\gamma_o}\}$ lies on l_1. This proves (20).

By means of (19) we obtain $|\Pi(P_3)\cap\Delta(P_o)| = n-2$.

In $n+1 = |\Gamma(P_o)| = |\Gamma(P_o)\cap\Delta(P_3)| + |\Gamma(P_o)\cap\Gamma(P_3)| + |\Gamma(P_o)\cap\Pi(P_3)|$ $|\Gamma(P_o)\cap\Delta(P_3)| = 2$ by (19) and $|\Gamma(P_o)\cap\Gamma(P_3)| = |P_2^{G_{P_o}}\cap P_2^{G_{P_3}}| = |l_2^{G_{P_o}}\cap l_2^{G_{P_3}}| = 1$. Hence $|\Pi(P_3)\cap\Gamma(P_o)| = n-2$. It follows that $\lambda' = n^2-3n+1$.

$\mu' = |\Pi(P_o)\cap\Pi(P_1)| = |\Pi(P_1)| - |\Pi(P_1)\cap\Delta(P_o)| - |\Pi(P_1)\cap\Gamma(P_o)|$ where $|\Pi(P_1)| = (n+1)(n-2)$, $|\Pi(P_1)\cap\Delta(P_o)| = n-2$ and $|\Pi(P_1)\cap\Gamma(P_o)| = |\Gamma(P_o)| - |\Gamma(P_o)\cap\Delta(P_1)| - |\Gamma(P_o)\cap\Gamma(P_1)|$. $|\Gamma(P_o)\cap\Delta(P_1)| = 2$ by (20) and $|\Gamma(P_o)\cap\Gamma(P_1)| = |P_2^{G_{P_o}}\cap P_2^{G_{P_1}}| = |l_2^{G_{P_o}}\cap l_2^{G_{P_1}}| = 1$. Hence $|\Pi(P_1)\cap\Gamma(P_o)| = n-2$ and $\mu' =$

$(n - 2)(n - 1)$.

$\nu' = |\Pi(P_o) \cap \Pi(P_2)| = |\Pi(P_2)| - |\Pi(P_2) \cap \Delta(P_o)| - |\Pi(P_2) \cap \Gamma(P_o)|$ where $|\Pi(P_2)| = (n + 1)(n - 2)$,

$$|\Pi(P_2) \cap \Delta(P_o)| = |\Pi(P_o) \cap \Delta(P_1)| \quad \text{by Lemma 1}$$
$$= |\Delta(P_1)| - |\Delta(P_1) \cap \Delta(P_o)| - |\Delta(P_1) \cap \Gamma(P_o)|$$
$$= (n + 1) - 1 - 2 = n - 2 \quad \text{by (20)},$$

$$|\Pi(P_2) \cap \Gamma(P_o)| = |\Pi(P_o) \cap \Gamma(P_1)| \quad \text{by Lemma 1}$$
$$= |P_3^{G_{P_o}} \cap P_o^{G_{P_1}}| = |\{l_3^{\gamma_o} : \gamma_o \in G_{P_o},\ P_1 \in l_3^{\gamma_o}\}| = n - 2$$

since through any point on l_o goes exactly one line of $l_1^{G_{P_o}}$ and one of $l_2^{G_{P_o}}$. Hence $\nu' = (n - 2)(n - 1)$.

It follows that $C^2 = C^tC = (n + 1)(n - 2)\,I + (n - 1)(n - 2)(A + B) + (n^2 - 3n + 1)\,C = (n + 1)(n - 2)\,I + (n^2 - 3n + 2)(A + B + C) - C$ and $C^2 + C - 2(n - 2)\,I = (n - 1)(n - 2)\,J$ whence $(C - (n + 1)(n - 2)\,I)(C^2 + C - 2(n - 2)\,I) = 0$. The eigenvalues of C are $\lambda_1 = (n + 1)(n - 2)$, $\lambda_{2,3} = (-1 \pm \sqrt{8n - 15})/2$.

REMARK: Let $\Phi: G \to GL_v(\mathbb{C})$ be the matrix representation of G obtained by associating with each $\gamma \in G$ the corresponding permutation matrix $\Phi(\gamma)$ (the ordering of P is the same as used in constructing the matrices A, B, C). By (2) $\Phi(\gamma)$ commutes with A, B, C for all $\gamma \in G$. Hence, by [9] Theorem 28.4, $\{I, A, B, C\}$ is the basis of the commuting algebra $\mathbb{V}(G)$ of Φ. By [9] Theorem 29.5 $\mathbb{V}(G)$ is commutative and hence, by [9] Theorem 29.4, the representation Φ has 4 irreducible constituents $D_1 = 1, D_2, D_3, D_4$, each with multiplicity 1. If f_i is the degree of D_i then $f_1 = 1$ and $\sum_{i=1}^{4} f_i = v$.

Let us finally show how the fact that A and C have trace 0 contradicts the integrality of the multiplicities of $\lambda_1, \lambda_2, \lambda_3$.

In the 3 cases λ_1 appears with multiplicity 1. Let f denote the multiplicity of λ_2; then v - f - 1 is the multiplicity of λ_3. This leads to

$$0 = n(n-1) + f(3+\sqrt{8n+1})/2 + (n(n+1)-f)(3-\sqrt{8n+1})/2 \quad \text{in case I,}$$

$$0 = (n+1) + f\sqrt{n} + (-\sqrt{n})(n(n+1)-f) \quad \text{in case II1,}$$

$$0 = (n+1)(n-2) + f(-1+\sqrt{8n-15})/2 + (n(n+1)-f)(-1-\sqrt{8n-15})/2$$

in case II2.

In any case this contradicts the fact that $n \geq 2$ and $f \geq 1$ are integers:

In case II1 this is clear.

In case I suppose that a prime p divides $\sqrt{8n+1}$. Then $p \nmid n$, hence $p \mid 5n+1$ and then $p \mid 3n$, i.e. p = 3. This implies that $8n+1 = 3^{2i}$ for some $i \geq 2$ and that $n(5n+1)/\sqrt{8n+1} = (3^{2i}-1)(5\cdot 3^{2i-1}+1)/8^2\cdot 3^{i-1} \notin \mathbb{N}$.

In case II2 suppose that a prime p divides $\sqrt{8n-15}$. Then $p \in \{17, 23\}$ and $p^2 \nmid n+1$, $p^2 \nmid n-4$. Hence $8n-15 \in \{17^2, 23^2, 17^2\cdot 23^2\}$, i.e. $n \in \{38, 68, 19112\}$. Suppose that n = 38. Since G_{P_o,P_2} is transitive on $1_2 - \{P_o, P_1, P_2^{\gamma_o}\}$ $|G|$ is even. This contradicts the fact that if $n \equiv 2 \mod 4$, then the full collineation group is of odd order (Hughes [5]).

Suppose that $n \in \{68, 19112\}$. Then n is not a square and $n^2 + n + 1$ not a prime. Hence, since G is flag-transitive, n is a prime power (Higman and Mc Laughlin [4]) which is absurd.

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