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A TOPOLOGICAL PROPERTY OF β (N)

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ABSTRACT. In this paper we prove that the Stone-Cech-compactification of the natural numbers does not admit a countable infinite decomposition into subsets homeomorphic to each other and to the said compactification.

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1. INTRODUCTION.

Let N be the discrete topological space of all positive integers and let $\beta(N)$ be its Stone-Cech-compactification. Then to each decomposition of N into a finite union of infinite sets, viz. homeomorphic to N, corresponds a finite decomposition of $\beta(N)$ into subsets homeomorphic to each other and to $\beta(N)$. This property fails to hold for countable decomposition of N. Indeed, if $N = \bigcup_{i=1}^{+\infty} Z_i$, let f map Z_i into $1/i$, $i = 1,2,\ldots$, and let V be an infinite subset of N such that $V \cap Z_i$ is finite for all i. Then f has a continuous extension over $\beta(N)$ which we shall also denote by f and the following hold:

a. $\bar{V} \neq V$ since $\bar{V}$ is compact but V is not.

b. f maps $\bar{V}-V$ onto 0.

c. $\bar{V}-V$ is not a subset of $\bigcup_{i=1}^{+\infty}\bar{Z}_i$ because f maps $\bar{Z}_i$ onto $1/i$, $i=1,2,\ldots$

Nevertheless, it is impossible that $\beta(N)$ might have some other countable decomposition into subsets homeomorphic to each other and to $\beta(N)$. Indeed, an assertion that this is the case, wrong as we will prove below, appears as an exercise in Dugundji's Topology (Exercise 9, page 256, Chapter XI, Eleventh Printing).

2. RESULTS.

In this section we will prove that the assumption that $\beta(N)$ admits a countable decomposition into subspaces homeomorphic to $\beta(N)$ is self-contradictory. We suppose therefore that $\beta(N) = \bigcup_{i=1}^{+\infty} \bar{U}_i$, that this union is disjoint and that all U_i's are discrete subspaces of $\beta(N)$. Under this assumption we will prove that N intersects infinitely many U_i's and that it is contained in their union. Then, we will pick from each U_i the least element of $U_i \cap N$ and we will prove that the set thus formed has a closure that is not a subset of $\bigcup_{i=1}^{+\infty} \bar{U}_i$. That, of course, will be a contradiction.

We have then:

LEMMA 1. For every subset T of N and every subset A of $\beta(N)$,

$$T \cap \bar{A} = T \cap A \tag{1}$$

PROOF. T is open in N (N is discrete) and N is locally compact.

LEMMA 2. There are infinitely many i's such that $U_i \cap N$ is not void.

PROOF. By the previous remarks $U_i \cap N = \bar{U}_i \cap N$, and therefore N is the union of the $(U_i \cap N)$'s. If $U_i \cap N = \emptyset$ for $i = k+1, k+2, \ldots$, then

$$\beta(N) = \bar{N} = \overline{\bigcup_{i=1}^{k} (N \cap U_i)} = \bigcup_{i=1}^{k} (\overline{N \cap U_i}) \subseteq \bigcup_{i=1}^{k} \bar{U}_i \tag{2}$$

and this is impossible. Hence the lemma.

On the basis of this lemma, we can consider the family $\{U_i \mid i = 1,2,3,\ldots\}$ as the union of two families:

The first is infinite countable and contains all U_i's which intersect N. We shall refer to it in the future as $\{V_j\}$.

The second may be void, finite or infinite countable and contains all U_i's which do not intersect N. We shall refer to it as $\{W_j\}$.

Let
$$Z_i = V_i \cap N, \quad i=1,2,3,\ldots \tag{3}$$
and let A be the set formed by picking the least element of each Z_i. Then we can construct a family $\{M(t) \mid t \text{ is a real number}\}$ whose elements are infinite subsets of A and satisfy: if $s \neq t$, then $M(s) \cap M(t)$ is void or finite.

One way to construct such a family is to enumerate the set of all rationals Q, $Q = \{r_n \mid n \in N\}$, to choose for each real t a sequence $\{r_{n(k,t)}\}_{k=1}^{+\infty}$ which converges to t and has infinitely many terms different from t, and to put,

$$M(t) = \{z \mid \text{for some } k, z \text{ is the least element in } Z_{n(k,t)}\}. \tag{4}$$

Then,

LEMMA 3. For every t, $\overline{M(t)} \neq M(t)$. (5)

PROOF. $\overline{M(t)}$ is compact and $M(t)$ is not.

LEMMA 4. For every t, $\overline{M(t)}$ is an open subset of $\beta(N)$.

PROOF. $\bar{N} = \overline{M(t)} \cup \overline{N - M(t)}$ and the union is disjoint. [Extend continuously over N a function that is 0 on $M(t)$ and 1 on $N-M(t)$] .

LEMMA 5. If $t \neq t'$, then $\overline{M(t)} \cap \overline{M(t')} = M(t) \cap M(t')$. (6)

PROOF. $M(t) \cap M(t')$ is finite and $M(t')$ is open in $\beta(N)$ which is Hausdorff. Therefore,

$$\overline{M(t)} \cap M(t') = M(t) \cap M(t'). \tag{7}$$

By the same token, ($\overline{M(t)}$ is open and $\overline{M(t)} \cap M(t')$ is finite)

$$\overline{M(t)} \cap \overline{M(t')} = \overline{M(t)} \cap M(t') = M(t) \cap M(t'). \tag{8}$$

Let

$$P = \beta(N) - \bigcup_{i=1}^{+\infty} \overline{Z}_i \tag{9}$$

and

$$U(t) = P \cap \overline{M(t)}. \tag{10}$$

LEMMA 6. For every t, $U(t) \neq \emptyset$.

PROOF. Let f be a function which maps Z_i onto $1/i$. If x is in $\overline{M(t)} - M(t)$, then $f(x) = 0$. On the other hand, $f(\overline{Z}_i) = 1/i$. Hence the lemma.

LEMMA 7. For every t, $U(t)$ is an open-closed set in P and for every t and t', $t \neq t'$,

$$U(t) \cap U(t') = \emptyset. \tag{11}$$

PROOF. That $U(t)$ is open-closed follows from the construction of $M(t)$ and Lemma 4. That $U(t) \cap U(t') = \emptyset$ follows from Lemma 5, and the fact that $M(t)$ and $M(t')$ are subsets of N.

Since the union of all V_j's and W_j's is a countable set and $\{U(t) \mid t \in R\}$ is an uncountable set of pairwise disjoint sets, there is a t' such that $U(t')$ does not meet any V_j's or W_j's. Let x be a point in $U(t')$. Then,

LEMMA 8. x does not belong to $\overline{V}_j$, $j=1,2,3,\ldots$

PROOF. x does not belong to $\overline{Z}_j$ by the very construction of $U(t')$. On the other hand, $V_j - Z_j$ is a subset of $\beta(N) - \bigcup_{i=1}^{+\infty} \overline{Z}_i$. Indeed, we remark that:

(a) The V_j's have the discrete topology and

$$(V_j - Z_j) \cap \bar{Z}_j = (V_j - Z_j) \cap Z_j \neq \emptyset \tag{12}$$

and

(b) If $k \neq j$, then

$$(V_j - Z_j) \cap \bar{Z}_k \subseteq \bar{V}_j \cap \bar{V}_k = \emptyset. \tag{13}$$

If every neighborhood of x in $\beta(N)$ met $V_j - Z_j$, then U(t') would meet it. Since this is not the case, then x is not in $\overline{V_j - Z_j}$. Hence the lemma.

LEMMA 9. x does not belong to $\bar{W}_j$, j=1,2,3,...

PROOF. $\bar{W}_j$ is a subset of P. If x were in $\bar{W}_j$, every neighborhood of x, and in particular U(t') would meet W_j. But this is not the case.

As a result of Lemmas 8 and 9, we see that U(t'), by construction a non-empty subset of $\beta(N)$, does not have any element in common with $\beta(N)$. Therefore, our supposition was false.

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