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A LOWER BOUND ON THE NUMBER OF FINITE SIMPLE GROUPS

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ABSTRACT. Let $S(n) = |\{m < n\text{: there is a (non-cyclic) simple group of order } m\}|$. Investigation of known families of simple groups provides the lower bound $S(n) \gg n^{1/4}/\log n$.

KEY WORDS AND PHRASES. Simple group, asymptotic formula.

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The non-specialist reader should refer first to Hurley and Rudvalis (4).

Write $S(n) = |\{m < n\text{: there is a simple group of order } m\}|$ and $S'(n) = |\{G\text{: } G \text{ is a simple group and } |G| < n\}|$. Dornhoff (1), Dornhoff and Spitznagel (2), and Erdös (3) got successively better upper bounds for S(n) by refining an argument which uses the Sylow theorems to generate a necessary criterion for a simple group of order m to exist. From the observation that $S(n) \le |\{m < n\text{: for any prime } p|m \text{ there is a } d|m \text{ such that } d > 1 \text{ and } d \equiv 1 \pmod p\}|$ Dornhoff found that

$S(n) = o(n)$ and Erdös derived a complicated bound better than that of Dornhoff but not as good as $o(n^{1-\epsilon})$. It should be noted that in general $S(n) < S'(n)$ because it occasionally (in fact infinitely often) happens that non-isomorphic simple groups of the same order exist.

We offer the following lower bound for $S(n)$, hence for $S'(n)$

THEOREM. $S(n) \gg n^{1/4}/\log n$.

PROOF. We estimate the number of integers $m < n$ which can be the order of a simple group in one of the known families and note that in all but finitely many cases the orders of the groups in that family are distinct.

From a list of known families of simple groups (4, p. 708) we see that one family dominates in the sense that for $F_i(n) = |\{m < n\text{: } m \text{ is the order of a simple group in family } i\}|$, $F_i(n) = O(F_1(n))$ for any i. $F_1(n)$ is the number of simple projective special linear groups of order less than n.

Thus to estimate $S(n)$ from below, we count tripletons (k,p,a) such that

1) k is an integer greater than 1,

2) a is an integer ≥ 1, and if $p = 2$ or $p = 3$ and $k = 2$ then $a > 1$, and

3) p is a prime, and writing $q = p^a$ we have

$$f(k,p,a) = q^{k(k-1)/2} \prod_{i=2}^{k} (q^i - 1)/(k,\ p-1) = |\ PSL_k(q)| < n.$$

Artin (5) showed that in exactly two cases distinct tripletons give rise to isomorphic groups, and in one case there are non-isomorphic groups of the same order in that family. Since $f(k,p,a) < q^{k(k-1)/2}\ q^{(k(k+1)/2)-1} < q^{k^2}$, $S(n) \gg |\{m < n\text{: there exists } (k,p,a) \text{ satisfying 1), 2), and 3) such that } m = p^{ak^2}\}|$. Such tripletons may be counted by a triple sum, and we have $S(n) \gg \sum_{a=1}^{\infty} \sum_{k=2}^{\infty} \sum_{p < n^{1/ak^2}} 1$. Constraining a and k so that $n^{1/ak^2} \geq 2$,

$$S(n) \gg \sum_{a=1}^{\log n/4\log 2} \ \sum_{k=2}^{(\log n/a\log 2)^{1/2}} \pi(n^{1/ak^2}),$$

and the Prime Number Theorem using $a = 1$ and $k = 2$ yields $S(n) \gg n^{1/4}/\log n$.

This theorem is of interest because it has been conjectured (3) that $S'(n) = o(n^{1-\varepsilon})$, or even $S'(n) = o(n^{1/3})$. We have that $1/4 - \varepsilon$ is a lower bound on the exponent of n, and if when all simple groups are classified no new family denser than the projective special linear groups appears, analyzing a perhaps more complicated triple sum carefully should yield the best exponent b in the estimate $S'(n) = o(n^b)$.

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