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A LEBESGUE DECOMPOSITION FOR ELEMENTS IN A TOPOLOGICAL GROUP

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ABSTRACT. Our aim is to establish the Lebesgue decomposition for strongly-bounded elements in a topological group. In 1963 Richard Darst established a result giving the Lebesgue decomposition of strongly-bounded elements in a normed Abelian group with respect to an algebra of projection operators. Consequently, one can establish the decomposition of strongly-bounded additive functions defined on an algebra of sets. Analagous results follow for lattices of sets. Generalizing some of the techniques yield decomposiitons for elements in a topological group.

KEY WORDS AND PHRASES. *Lebesgue decomposition, projection operator, strongly-bounded, topological group.*

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1. INTRODUCTION.

In 1963 R. B. Darst [2] established a result giving the Lebesgue decomposition of s-bounded elements in a normed Abelian group with respect to an algebra of projection operators. Consequently, one can establish the decomposition of s-bounded additive functions defined on an algebra of sets [4]. The set of corresponding restrictions of additive set functions defined on a lattice of sets corresponds to a lattice of projection operators [5]. The analagous result on lattices is established by using the same techniques [3]. More recently, Traynor has obtained decompositions of set functions with values in a topological group [6], [7]. The purpose here is to present a Lebesgue decomposition theorem for elements in a topological group by the use of projection operators. It is believed that this result would aid in obtaining decompositions of operators on non-locally convex lattices.

2. PRELIMINARIES.

Let G be an Abelian topological group under addition, and let T be an algebra of projection operators [1] on G. For $t_1, t_2 \in T$ define $t_1 \leq t_2$ to mean $t_1 t_2 = t_1$ and define $t_1 - t_2$ to mean $t_1 t_2'$. This relation induces a partial ordering on T, which in turn has a lattice structure if we set $t_1 \wedge t_2 = \sup\{t \in T: t \leq t_1, t \leq t_2\}$ and $t_1 \vee t_2 = \inf\{t \in T: t_1 \leq t, t_2 \leq t\}$ providing the sup and inf exist. But, we have $t_1 \vee t_2 = t_1 + t_2 - t_1 t_2 = (t_1' t_2')'$ and $t_1 \wedge t_2 = t_1 t_2$, so T is a Boolean algebra of operators. Let $\mathcal{U}$ be the set of all symmetric neighborhoods about $0 \in G$. For each $U \in \mathcal{U}$ and each positive integer n, define $nU = \{x + y: x \in (n-1)U \text{ and } y \in U\}$, where $0U = \{0\} \subset G$, whence $1U = U$. Then a subset $H \subset G$ is bounded if given $U \in \mathcal{U}$ there exists an integer n such that $H \subset nU$. It would make sense to even say $H \subset (m/n)U$ for this would mean $nH \subset mU$. We define an element $f \in G$ to be s-bounded (strongly bounded) if, for every sequence $\{t_i\} \subset T$ of pairwise disjoint elements, $t_i(f) \to 0$. For each positive real number x, T_x shall denote a non-empty subset of T with the properties

1) $t_x \varepsilon T_x$ and $t \varepsilon T$ implies $tt_x \varepsilon T_x$, and

2) $t_x \varepsilon T_x$ and $t_y \varepsilon T_y$ implies $t_x \vee t_y \varepsilon T_{x+y}$.

Several lemmas can now be stated, and their proofs follow as in [1] and [2].

LEMMA 1. Let $t_1, t_2 \varepsilon T$. Suppose $t_2(g) \varepsilon U$ implies $t_1(g) \varepsilon U$ for arbitrary $g \varepsilon G$ and $U \varepsilon \mathcal{U}$. Then $t_1 \leq t_2$.

LEMMA 2. If $\{t_i\}$ is a monotone sequence of elements of T, and if $f \varepsilon G$ is s-bounded, then $\{t_i(f)\}$ is Cauchy in G.

Given $U \varepsilon \mathcal{U}$ we write $U_0 = U$ and for each $n > 0$ we write U_n to represent some element of $\mathcal{U}$ where $U_n + U_n \subset U_{n-1}$, whence $2^n U_n \subset U$. This is possible since addition is continuous in G.

DEFINITION. T has Property A if given $g \varepsilon G$ and $U \varepsilon \mathcal{U}$ then there exists a $V \varepsilon \mathcal{U}$ such that if $a, b \varepsilon T$ and $(a'b)(g) \notin U$ then $a(g) \varepsilon V$ and $(a + a'b)(g) \notin V$.

Note that Property A is a condition yielding information about the growth of elements from G; a condition on the manner in which projections affect the relative location of elements in symmetric neighborhoods. We also look at a smaller class of neighborhoods by selecting an arbitrary bounded set $\hat{U}$ from and forming the sets $n\hat{U}$ with $n = 1,2,\cdots$. Choosing $\hat{U}_1, \hat{U}_2, \cdots$ we then form $S = \{\cdots, \hat{U}_2, \hat{U}_1, \hat{U}, 2\hat{U}, \cdots\}$ and set $\hat{\mathcal{U}}$ equal to the set

$$\left\{ \sum_{i=1}^{n} S_i :\ S_i \varepsilon S,\ S_i \neq S_j \text{ if } i \neq j \right\}.$$

It follows that $\hat{\mathcal{U}}$ possesses the following property inherited from $\mathcal{U}$: if $U \varepsilon \hat{\mathcal{U}}$ then there exists $U_1 \varepsilon \hat{\mathcal{U}}$ such that $U_1 + U_1 \subset U$. This yields the result,

LEMMA 3. If T has Property A with respect to $\hat{\mathcal{U}}$, and if $t_1, t_2 \varepsilon T$ with $t_1 \leq t_2$, then $t_2(g) \varepsilon U$ implies $t_1(g) \varepsilon U$ for arbitrary $g \varepsilon G$ and $U \varepsilon \hat{\mathcal{U}}$.

From now on we shall assume T has Property A with respect to $\hat{\mathcal{U}}$.

LEMMA 4. Let $f \varepsilon G$ be s-bounded, $\{t_k\} \subset T$ and $U \varepsilon \hat{\mathcal{U}}$. Then there exists a positive integer n such that if $j \geq i > n$ then

$$\left(\bigvee_{i \le k \le j} t_k - \bigvee_{k \le n} t_k\right)(f) \in U.$$

For $g \in G$ and $n \in N$ let $S(n,g) = \{U \in \hat{\mathcal{U}} : t(g) \in U \text{ for all } t \in T_{1/n}\}$. Lemma 3 guarantees that no $S(n,g)$ is empty.

LEMMA 5. If $t(g) \neq 0$ for some $t \in T_{1/n}$, then there exists a $W \in S(n,g)$ such that $W_1 + W_2 + \cdots + W_n \notin S(n,g)$ for all choices of $W_i \in \hat{\mathcal{U}}$.

PROOF. Let $U \in S(n,g)$ and construct a sequence $\{A_k\} \subset \hat{\mathcal{U}}$ as follows. Set $A_1 = U_1 + U_2 + \cdots + U_n$ for arbitrary U_i, and set $A_{k+1} = (A_k)_1 + (A_k)_2 + \cdots + (A_k)_n$ for arbitrary $(A_k)_i$. Now $A_1 \subset \left[(2^n-1)/2^n\right]U$, and then $A_2 \subset \left[(2^n-1)/2^n\right]^2 U$. In general $A_k \subset \left[(2^n-1)/2^n\right]^k U$. But the coefficient $\left[(2^n-1)/2^n\right]^k$ can be made as small as we like (consequently given $i > 0$ there exists $k > 0$ such that $A_k \subset U_i$), so if the lemma is not true then given $U \in S(n,g)$ there would exist $U_1, U_2, \cdots, U_n$ such that $U_1 + \cdots + U_n \in S(n,g)$. Setting $A_1 = U_1 + \cdots + U_n$ we apply the hypothesis again and get $(A_1)_1 + \cdots + (A_1)_n \in S(n,g)$. Continuing this procedure yields sets A_k which contain $\{t(g): t \in T_{1/n}\}$, and which continue to get smaller. This is impossible since $t(g) \neq 0$ for some t.

Let us denote this set W by $W(n,g)$. This lemma implies that out of all the neighborhoods containing $\{t(g): t \in T_{1/n}\}$, $W(n,g)$ is one of the "smallest." Since $\{t(g): t \in T_{1/(n+1)}\}$ is contained in $\{t(g): t \in T_{1/n}\}$ we can choose our $W(n,g)$ to be nested, $W(n,g) \supset W(n+1,g)$. Assuming this sequence of neighborhoods converges, we are led to defining the following function.

DEFINITION. Let $Y: G \to \hat{\mathcal{U}}$ by $Y(g) = \lim W(n,g)$.

This function is the counterpart to the function y in [2]. Our last lemma is the following.

LEMMA 6. Let G be complete and $f \in G$ be s-bounded. Let $W(n,f)$ be an associated sequence of neighborhoods as above that contain $\{t(f): t \in T_{1/n}\}$. Then given $M > 0$ there exists a decreasing sequence $\{a_i\}\downarrow$ in T such that

1) if $x > 0$ then there exists an integer i such that $a_i \in T_x$, and

2) $\lim a_i(f) \notin W_1 + W_2 + \cdots + W_M$ where $W = \lim W(n,f)$, and for all W_i.

PROOF. To just sketch the essentials of the lemma, we let $t_i \in T_{1/2^{i+1}}$ such that $t_i(f) \notin W_1 + W_2 + \cdots + W_{M+2}$ for all W_i, $i = 1,2,\cdots,M+2$. This is possible by the choice of $W(n,f)$. By Lemma 4 there exists a positive integer n_1 such that $j \geq i > n_1$ implies $(\bigvee_{i \leq k \leq j} t_k - \bigvee_{k \leq n_1} t_k)(f) \in W_{M+3}$. Applying Lemma 4 again to the sequence $t_{n_1+1}, t_{n_1+2}, \cdots, t_{n_2}, \cdots$ produces a positive integer n_2 such that $(\bigvee_{i \leq k \leq j} t_k - \bigvee_{n_1 < k \leq n_2} t_k)(f) \in W_{M+4}$ for $j \geq i > n_2$. Continuing this process we get an increasing sequence $\{n_j\}\uparrow$ of positive integers such that $(\bigvee_{q \leq k \leq p} t_k - \bigvee_{n_{j-1} < k \leq n_j} t_k)(f) \in W_{M+j+2}$ whenever $p \geq q > n_j$. If $u_j = \bigvee_{n_j < i \leq n_{j+1}} t_i$ then $u_j \in T_{1/2^{n_j+1}}$ and $k > j$ implies $(\bigvee_{j < p \leq k} u_p - u_j)(f) \in W_{M+j+3}$. Setting $a_k = \bigwedge_{j \leq k} u_j$ produces the desired decreasing sequence.

We now can state and prove our main decomposition result.

THEOREM. Let G be an Abelian topological group, and let T be an algebra of projection operators on G. Assume T_x, $\mathcal{U}$ and $\hat{\mathcal{U}}$ are as before with G being complete, and with T possessing Property A with respect to $\hat{\mathcal{U}}$. If $f \in G$ is s-bounded then there exists unique elements $h, s \in G$ such that

1) $f = h + s$,

2) given $U \in \hat{\mathcal{U}}$ there exists a positive real number x such that if $t \in T_x$ then $t(h) \in U$,

3) given $U \in \hat{\mathcal{U}}$ and $\varepsilon > 0$ there exists $t \in T_\varepsilon$ such that $t'(s) \in U$.

PROOF. First, as counterparts to the classical Lebesgue decomposition theorem, the element h is to represent the continuous portion of f, while s represents the singular portion. Again, to just sketch some of the essentials of the proof, we bypass the uniqueness and, turning our attention to existence note that if $h = f$ satisfies condition (2) then there is nothing to prove. Denoting $Y(f)$ by $W(f)$, we assume $W(f)$ contains points other than $0 \varepsilon G$. Then, from Lemma 6, there exists a sequence $\{a_{1i}\}\downarrow$ in T such that $\lim a_{1i}(f) \not\varepsilon W_1(f) + W_2(f)$ for all $W_i(f)$. Let $s_1 = \lim a_{1i}(f) \varepsilon G$ and $f_1 = f - s_1 = \lim a_{1i}'(f)$. If $Y(f_1) = \{0\}$, then f_1 satisfies (2) and the proof is completed because $a_{1i}(f) \to s_1$ implies $a_{1i}'(s_1) \to 0$, and thus f_1 is also s-bounded. So given $U \varepsilon \hat{\mathcal{U}}$ and $\varepsilon > 0$ there exists $t \varepsilon T_\varepsilon$ such that $t'(s) \varepsilon U$, namely $t = a_{1i}$ for large i. If f_1 does not satisfy (2), applying Lemma 6 to f_1 produces another sequence $\{a_{2i}\}\downarrow$ in T such that $\lim a_{2i}(f_1) \not\varepsilon W_1(f_1) + W_2(f_1)$. Let $s_2 = \lim a_{2i}(f_1)$ and $f_2 = f_1 - s_2 = \lim a_{2i}'(f_1)$. Then f_2 is s-bounded and $f = f_2 + (s_1 + s_2)$. To show $s_1 + s_2$ satisfies condition (3) we let $U \varepsilon \hat{\mathcal{U}}$ and $\varepsilon > 0$. We have $a_{1i}'(s_1) \to 0$ and $a_{2i}'(s_2) \to 0$. So there exists a positive integer N such that $a_{1i}'(s_1) \varepsilon U_1$ and $a_{2i}'(s_2) \varepsilon U_1$ for all i greater than N. Then $(a_{1i} \vee a_{2i})'(s_1 + s_2) = (a_{1i}' \wedge a_{2i}')(s_1) + (a_{1i}' \wedge a_{2i}')(s_2) \varepsilon U$. Condition (3) is satisfied by letting $t = a_{1i} \vee a_{2i}$ for large i. So if $Y(f_2) = \{0\}$ then let $h = f_2$ and $s = s_1 + s_2$ and the proof is completed. If not, continue the process. If for some positive integer k, $Y(f_k) = \{0\}$, we are through. Otherwise we obtain a sequence $\{(s_k, f_k)\}$ of pairs of elements of G and a sequence $\{\{a_{ki}\}_{i=1}^{\infty}\}$ of non-increasing sequences of elements of T such that for each positive integer k we have

1) there exists a sequence $\{x_{ki}\}_{i=1}^{\infty}$ of positive reals where $x_{ki} \to 0$ and $a_{ki} \varepsilon T_{x_{ki}}$,

2) $s_k = \lim_i a_{ki}(f_{k-1})$ with $f_0 = f$,

3) $f_k = f_{k-1} - s_k = \lim_i a_{ki}'(f_{k-1})$,

4) $s_k \not\varepsilon W_1(f_{k-1}) + W_2(f_{k-1})$ for all $W_i(f_{k-1})$,

5) $f = f_k + \sum_{i=1}^{k} s_i$.

In the end we will have our decomposition $f = h + s$ with $s = \sum_{i=1}^{\infty} s_i$ and $h = f - s$. Toward this goal, although the steps shall be omitted, the next step is to show $\lim s_k = 0$ by showing that s_k eventually belongs to an arbitrarily selected $U \in \hat{\mathcal{U}}$. And then it must be established that $\lim_n \sum_{i=1}^{n} s_i$ exists. Assuming this, we then let $s = \lim_n \sum_{i=1}^{n} s_i$ and $h = f - s$. We shall show that s satisfies condition (3) of the theorem. We have $s_k = \lim_i a_{ki}(f_{k-1})$. Let $U \in \hat{\mathcal{U}}$. Then $a_{ki}'(s_k) \to 0$, so $a_{ki}'(s_k) \in U_{k+1}$ for all i greater than some positive integer M_k. Since $s = \lim \sum s_i$ then there exists a positive integer N such that $\sum_{i=N+1}^{\infty} s_i \in U_1$, and then

$$\Big[\bigvee_{k \le N} a_{ki}\Big]'(s) = \Big[\bigvee_{k \le N} a_{ki}\Big]' \sum_{j=1}^{N} s_j + \Big[\bigvee_{k \le N} a_{ki}\Big]' \sum_{j > N} s_j$$

$$= \sum_{j=1}^{N} \bigwedge_{k = N} a_{ki}'(s_j) + \Big[\bigvee_{k \le N} a_{ki}\Big]' \sum_{j > N} s_j$$

$$\in U_2 + \cdots + U_{N+1} + U_1 \quad \text{for large } i = \max\{M_1, \cdots, M_N\}$$

$$\in U.$$

So, let $t = \bigvee_{k \le N} a_{ki}$ where $i = \max\{M_1, \cdots, M_N\}$ and condition (3) is satisfied. Now $h = f - s = \lim f_n$ and $Y(f_n) \to \{0\}$. Then $Y(h) = \{0\}$ and the decomposition is finished.

These results are part of the author's dissertation from Colorado State University.

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