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RESEARCH NOTES

A POINTWISE GROWTH ESTIMATE FOR ANALYTIC FUNCTIONS IN TUBES

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ABSTRACT. A class of analytic functions in tube domains $T^C = \mathbb{R}^n + iC$ in n-dimensional complex space, where C is an open connected cone in $\mathbb{R}^n$, which has been defined by V. S. Vladimirov is studied. We show that a previously obtained L^2 growth estimate concerning these functions can be replaced by a pointwise growth estimate, and we obtain further new properties of these functions. Our analysis shows that these functions of Vladimirov are exactly the Hardy H^2 class of functions corresponding to the tube T^C.

KEY WORDS AND PHRASES. Analytic Function in Tubes, Hardy H^2 Space, Cauchy Kernel and Integral, Fourier-Laplace Integral.

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1. INTRODUCTION.

All notation in this note is that of Vladimirov [1, p. 1]. Let C be an open connected cone in $\mathbb{R}^n$ and let C' be an arbitrary compact subcone of C [1, p. 218].

Let $f(z)$ be analytic in $T^C = \mathbb{R}^n + iC$ and for any $C' \subset C$ let $f(z)$ satisfy

$$||f(x+iy)||_{L^2} = (\int_{\mathbb{R}^n} |f(x+iy)|^2 \, dx)^{1/2} \leq M_{\epsilon,f}(C') \, e^{\epsilon|y|} \quad , \; y \in C' \subset C, \qquad (1)$$

for every $\epsilon > 0$ where the constant $M_{\epsilon,f}(C')$ depends on ϵ, f, and C' but not on $y \in C' \subset C$. Vladimirov has studied these analytic functions in [1, sections 25.3-25.4]. In this note we show that the L^2 growth estimate [1, p. 227, (74)] can be replaced by a pointwise growth estimate on the function with exactly the same growth on the right of the estimate and obtain further new information concerning these functions. Our analysis also shows that the analytic functions of Vladimirov defined above are in fact exactly the Hardy H^2 functions ([2, section 3] or [3, pp. 90-91];) thus the growth of Bochner [2, (13)] which defines the Hardy H^2 space for tubes T^C, namely

$$||f(x+iy)||_{L^2} \leq M_f \; , \; y \in C \; , \qquad (2)$$

is not a more restrictive condition than (1), contrary to the statement made in [1, p. 227, lines 4-5], in the sense that both (1) and (2) characterize the same space, the Hardy H^2 space corresponding to tubes T^C .

2. RESULTS.

To obtain our results we need three lemmas. The proof of Lemma 1 is like that of [1, p. 223, Lemma 2] and is omitted.

LEMMA 1. Let C be an open (not necessarily connected) cone. Let $y \in O(C)$, the convex envelope of C. There exists a $\delta = \delta_y > 0$ depending on y such that

$$yt \geq \delta \; |y| \; |t| \qquad (3)$$

for all $t \in C^* = \{t : yt \geq 0, \; y \in C\}$. Further, if C' is an arbitrary compact subcone of $O(C)$ there exists a $\delta = \delta(C') > 0$ depending only on C' such that (3) holds for all $y \in C'$ and all $t \in C^*$.

LEMMA 2. Let C be an open connected cone. We have

$$(I_{C^*}(t)\ e^{izt}) \in L^p \text{ for all } p\ ,\ 1 \leq p \leq \infty\ , \tag{4}$$

as a function of $t \in \mathbb{R}^n$ for arbitrary $z \in T^{O(C)} = \mathbb{R}^n + i\ O(C)$, where $I_{C^*}(t)$ denotes the characteristic function of C^* .

PROOF. Let $z = x+iy \in T^{O(C)}$. Using Lemma 1 we have

$$|I_{C^*}(t)\ e^{izt}| = I_{C^*}(t)\ e^{-yt} \leq I_{C^*}(t)\ \exp(-\delta|y||t|) \tag{5}$$

for some $\delta = \delta_y > 0$, and (5) holds for all $t \in \mathbb{R}^n$ since $I_{C^*}(t) = 0$ if $t \notin C^*$. (5) proves (4) for $p = \infty$. Now let $1 \leq p < \infty$. Using (5), [4, p. 39, Theorem 32], and integration by parts (n-1) times, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |I_{C^*}(t)\ e^{izt}|^p\ dt &\leq \int_{\mathbb{R}^n} \exp(-\delta p|y||t|)\ dt = \\ &= \Omega_n \int_0^\infty r^{n-1} \exp(-\delta p|y|r)\ dr = \Omega_n\ (n-1)!\ (\delta p|y|)^{-n}\end{aligned} \tag{6}$$

where Ω_n is the volume of the unit sphere in $\mathbb{R}^n$. (6) proves (4) for $1 \leq p < \infty$

For any open connected cone C the Cauchy kernel corresponding to $T^{O(C)}$ ([5, p. 201] or [1, p. 223, (61)]) is

$$K(z-t) = \int_{C^*} \exp(i(z-t)\eta)\ d\eta\ ,\ z \in T^{O(C)}\ ,\ t \in \mathbb{R}^n\ .$$

LEMMA 3. Let C be an open connected cone. $K(z-t)$ is an analytic function of $z \in T^{O(C)}$ for fixed $t \in \mathbb{R}^n$. We have

$$|K(z-t)| \leq \Omega_n\ (n-1)!\ \delta^{-n}\ |y|^{-n}\ ,\ z = x+iy \in T^{O(C)}\ ,\ t \in \mathbb{R}^n\ , \tag{7}$$

where Ω_n is the volume of the unit sphere in $\mathbb{R}^n$ and $\delta = \delta_y > 0$ is the number of (3); and for $1 < p \leq 2$, $(1/p) + (1/q) = 1$, we have

$$||K(z-t)||_{L^q} \leq (\Omega_n(n-1)!)^{1/p}\ (p\delta)^{-n/p}\ |y|^{-n/p}\ ,\ z = x+iy \in T^{O(C)}\ , \tag{8}$$

where the L^q norm is with respect to the variable $t \in \mathbb{R}^n$. Further, if C' is an arbitrary compact subcone of $O(C)$ then (7) and (8) hold for $z = x+iy \in T^{C'}$ and

$t \in \mathbb{R}^n$ with δ depending only on $C' \subset O(C)$ and not on $y \in C' \subset O(C)$.

PROOF. The fact that $K(z-t)$ is analytic in $z \in T^{O(C)}$ for fixed $t \in \mathbb{R}^n$ follows by [1, p. 223]. (7) follows by the analysis of (6) for $p = 1$. For $1 < p \leq 2$ and $(1/p) + (1/q) = 1$, $K(z-t) = \mathfrak{F}^{-1}[I_{C^*}(\eta)\, e^{iz\eta}\ ; t]$ in the L^q sense as noted in [5, p. 202, proof of Theorem 1]; hence by the Parseval inequality

$$||K(z-t)||_{L^q} \leq ||I_{C^*}(\eta)\, e^{iz\eta}||_{L^p}\ ,\ z \in T^{O(C)}\ . \tag{9}$$

Inequality (8) now follows from (9) and a computation as in (6) for $1 < p \leq 2$. If C' is an arbitrary compact subcone of $O(C)$ we use the second part of Lemma 1 to obtain (5) and (6) for $z \in T^{C'}$, $C' \subset O(C)$, where δ now depends only on $C' \subset O(C)$ and not on $y \in C' \subset O(C)$. This fact and the above analysis yields (7) and (8) holding for $z = x+iy \in T^{C'}$ with δ depending only on $C' \subset O(C)$.

The result [1, p. 223, (62)] is a special case of Lemma 3 for $p = 2$.

We now obtain our result which adds information to [1, p. 227, Corollary] and hence to the analytic functions considered in [1, sections 25.3-25.4]. First note a misprint in the statement of [1, p. 227, Corollary]; "equality (63)" in [1, p. 227, line 2 of the Corollary] should read "inequality (64)".

THEOREM. Let C be an open connected cone. Let $f(z)$ be analytic in T^C and satisfy (1). Then $f(z)$ has an analytic extension $F(z) \in H^2(T^{O(C)})$ to $T^{O(C)}$ such that for any compact subcone $C' \subset O(C)$

$$|F(z)| \leq M(C')\ ||h||_{L^2}\ |y|^{-n/2}\ ,\ z = x+iy \in T^{C'} \tag{10}$$

where $M(C')$ is a constant which depends at most on $C' \subset O(C)$ and $h \in L^2$ is the L^2 boundary value of $F(x+iy)$ as $y \to 0$, $y \in O(C)$. Further,

$$\sup_{y \in C} ||f(x+iy)||_{L^2} = \sup_{y \in O(C)} ||F(x+iy)||_{L^2}\ , \tag{11}$$

and if $\overline{O(C)}$ contains an entire straight line then $F(z) \equiv 0$.

PROOF. From [1, pp. 225-226] we have the existence of a function $g \in L^2$

with support in C^* almost everywhere such that

$$f(z) = \int_{\mathbb{R}^n} g(t)\, e^{izt}\, dt \ , \ z \in T^C \ , \tag{12}$$

and the Fourier-Laplace integral on the right of (12) is well defined because of the properties of $g(t)$ and (4). Now put

$$F(z) = \int_{\mathbb{R}^n} g(t)\, e^{izt}\, dt \ , \ z \in T^{O(C)} \ . \tag{13}$$

The fact that $F(z)$ is analytic in $T^{O(C)}$ follows as a special case of [6, Theorem 2.1]. Because of (4) and the properties on $g(t)$, we have that $(g(t)\exp(-yt)) \in L^1 \cap L^2$ as a function of $t \in \mathbb{R}^n$ for $y \in O(C)$; hence the integral on the right of (13) can be interpreted to be the L^2 Fourier transform of $(g(t)\exp(-yt))$, $y \in O(C)$. The Parseval equality and the fact that the support of g is in C^* almost everywhere now yield

$$||F(x+iy)||_{L^2} = ||g(t)\, e^{-yt}||_{L^2} \leq ||g||_{L^2} \ , \tag{14}$$

and we conclude that $F(z) \in H^2(T^{O(C)})$. (This fact also follows by [3, p. 101, Theorem 3.1].) We now apply the proof of [1, p. 227, Bochner's formula] or [3, p. 103, Theorem 3.6] to obtain the existence of a function $h \in L^2$ which is the L^2 Fourier transform of g and is the L^2 boundary value of $F(x+iy)$ as $y \to 0$, $y \in O(C)$, such that

$$F(z) = (2\pi)^{-n} \int_{\mathbb{R}^n} h(t)\, K(z-t)\, dt, \ z \in T^{O(C)} \ . \tag{15}$$

Using (15), the Hölder inequality, and the estimate (8) for $p = 2$ valid for $z \in T^{C'}$, C' being an arbitrary compact subcone of $O(C)$, we have

$$|F(z)| \leq (2\pi)^{-n}\, ||h||_{L^2}\, ||K(z-t)||_{L^2}$$

$$\leq (2\pi)^{-n}\, ||h||_{L^2}\, (\Omega_n\, (n-1)!)^{1/2}\, (2\delta)^{-n/2}\, |y|^{-n/2}$$

for $z = x+iy \in T^{C'}$ which proves (10) with $M(C') = ((2\pi)^{-n}\, (\Omega_n\, (n-1)!)^{1/2}\, (2\delta)^{-n/2})$

depending only on $C' \subset O(C)$ since δ does. The proof of [3, p. 93, Corollary 2.4] now yields (11). If $\overline{O(C)}$ contains an entire straight line then $F(z) \equiv 0$ because of [1, p. 222, Lemma 1] and the fact that the support of $g(t)$ is in C^* almost everywhere. The proof is complete.

Since any $H^2(T^C)$ function satisfies (12) with $g \in L^2$ having support in C^* almost everywhere ([6, Corollary 4.1] or [3, p. 101, Theorem 3.1]) and hence satisfies (15) for $z \in T^C$ and some $h \in L^2$ by the proof of [1, p. 227, (72)], the proof of our Theorem shows that any $H^2(T^C)$ function satisfies (10). As another consequence of the proof of our Theorem, any function $f(z)$ which is analytic in T^C and satisfies (1), i.e. [1, p. 224, (64)], has the representation (12) and hence is an $H^2(T^C)$ function because of analysis as in (14). Thus the analytic functions considered by Vladimirov in [1, sections 25.3-25.4] are exactly the $H^2(T^C)$ functions. The statement made in [1, p. 227, lines 4-5] that (2) is a more restrictive condition than (1) is thus not correct in the sense that both (1) and (2) characterize exactly the same space, the Hardy H^2 space corresponding to tubes T^C .

However, the growth (1) of Vladimirov has suggested to us a way to define analytic functions in tubes which do generalize the Hardy spaces. The definitions of these new spaces and our representations and analysis concerning them will appear in [6].

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