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ON GENERALIZED QUATERNION ALGEBRAS

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ABSTRACT. Let B be a commutative ring with 1, and G (={σ}) an automorphism group of B of order 2. The generalized quaternion ring extension B[j] over B is defined by S. Parimala and R. Sridharan such that (1) B[j] is a free B-module with a basis {1,j}, and (2) $j^2 = -1$ and $jb = \sigma(b)j$ for each b in B. The purpose of this paper is to study the separability of B[j]. The separable extension of B[j] over B is characterized in terms of the trace (= $1+\sigma$) of B over the subring of fixed elements under σ. Also, the characterization of a Galois extension of a commutative ring given by Parimala and Sridharan is improved.

KEY WORDS AND PHRASES. Quaternion Rings, Separable Algebras, and Galois Extensions.

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1. INTRODUCTION.

In [6], we studied the separable extension of group rings RG and quaternion rings R[i,j,k] over a ring R with 1. We have shown that R[i,j,k] is a separable extension of R if and only if 2 is a unit in R. Recently, S. Parimala and R. Sridharan ([5]) investigated another class of quaternion ring extensions B[j] over a commutative ring B with 1 and with an automorphism group G (= $\{\sigma\}$) of order 2, where B[j] is a free B-module with a basis $\{1,j\}$, $j^2 = -1$, and $jb = \sigma(b)j$ for each b in B. Their work is based on the following characterization of a Galois extension of a commutative ring ([5], Proposition 1.1): Let A be the set of elements in B fixed under σ. Assume 2 is a unit in A. Then, B is Galois over A if and only if $B\otimes_A B[j] \cong M_2(B)$, a matrix algebra over B of order 2, where the Galois extension is in the sense of Chase-Harrison-Rosenberg ([2]). The purpose of this paper is to study the separability of B[j]. Without the assumption that 2 is a unit in A, we shall characterize the separability of B[j] in terms of the trace (= $1+\sigma$) of B over A. This shows the existence of a separable generalized quaternion ring extension B[j] with 2 not a unit in A. When Char(A) = 2, we shall show that B[j] is a separable extension over B if and only if B is Galois over A. Thus we can improve the above theorem of Parimala and Sridharan. Then, the case in which 2 is a unit will be discussed, and several examples are constructed to illustrate our main results.

2. PRELIMINARIES.

Let us recall some basic definitions as given in [1],[2],[3],[4] and [6]. Let B be a commutative ring containing a subring A with the same identity 1. Then B is called a Galois extension over A ([2], or [3], Chapter 3) with a finite automorphism group G if (1) there exist elements $\{a_i, b_i$ in B / $i = 1,2,\ldots,n$ for some integer $n\}$ such that $\sum a_i b_i = 1$ and $\sum a_i \sigma(b_i) = 0$ whenever $\sigma \neq 1$ in G, and (2) $A = \{b$ in B / $\sigma(b) = b$ for all σ in $G\}$. The map $\sum\sigma$ is called the <u>trace</u> of B over A denoted by Tr. Let S be a ring (not necessarily commutative) containing a subring R with the same identity 1. Then S is called a <u>separable extension</u> of R if there exist elements, $\{c_i, d_i$ in S / $i = 1,2,\ldots,n$ for some integer $n\}$ such that (1) $a(\sum c_i \otimes d_i) = (\sum c_i \otimes d_i)a$ for all a in S where $\otimes$ is over R, and (2) $\sum c_i d_i = 1$. Such an element $\sum c_i \otimes d_i$ is called a <u>separable idempotent</u> for S. When R is contained in the center of S, S is called a <u>separable R-algebra</u>. The separable R-algebra S is called an <u>Azumaya R-algebra</u> if R is the center of S.

3. SEPARABLE QUATERNION ALGEBRAS.

Throughout, we assume that B is a commutative ring with 1, and G $(= \{\sigma\})$ an automorphism group of order 2 of B, and that $B[j]$ is the generalized quaternion algebra over A, where A is the subring of elements fixed under σ. Our main goal in the section is to study a separable extension $B[j]$ over B without the assumption that 2 is a unit in A. We begin with a description of the set of separable idempotents for $B[j]$ (if there are any) over B. Clearly, $\{1\otimes 1, 1\otimes j, j\otimes 1, j\otimes j\}$ is a basis for $B[j]\otimes_B B[j]$.

LEMMA 3.1. The element $x = a_{11}(1\otimes 1)+a_{12}(1\otimes j)+a_{21}(j\otimes 1)+a_{22}(j\otimes j)$ is a separable idempotent for $B[j]$ over B if and only if (1) $a_{22} = -\sigma(a_{11})$ such that $\mathrm{Tr}(a_{11}) = 1$, and (2) $a_{21} = \sigma(a_{12})$ such that $a_{12}((b-\sigma(b)) = 0$ for all b in B and $\mathrm{Tr}(a_{12}) = 0$.

PROOF. Let x be a separable idempotent for $B[j]$ over B. Then $xu = ux$ for each u in $B[j]$. Hence $xj = jx$; that is,

$$\sigma(a_{11})(j\otimes 1)+\sigma(a_{12})(j\otimes j)-\sigma(a_{21})(1\otimes 1)-\sigma(a_{22})(1\otimes j) =$$

$a_{11}(1\otimes j)-a_{12}(1\otimes 1)+a_{21}(j\otimes j)-a_{22}(j\otimes 1)$. Equating corresponding coefficients, we have $\sigma(a_{11}) = -a_{22}$, $a_{12} = \sigma(a_{21})$; that is, $a_{22} = -\sigma(a_{11})$ and $a_{21} = \sigma(a_{12})$ for $\sigma^2 = 1$. Also, bx = xb for all b in B, so $b_{12}(b-\sigma(b)) = 0$. Thus $x = a_{11}(1\otimes 1)+a_{12}(1\otimes j)+\sigma(a_{12})(j\otimes 1)-\sigma(a_{11})(j\otimes j)$ with $a_{12}(b-\sigma(b)) = 0$. Moreover, by the second condition of a separable idempotent, $a_{11}+(a_{12}+\sigma(a_{12}))j+\sigma(a_{11}) = 1$, so $Tr(a_{11}) = 1$ and $Tr(a_{12}) = 0$. Conversely, it is straightforward to verify that any x satisfying all equations as given is a separable idempotent.

THEOREM 3.2. B[j] is a separable extension over B if and only if there is an element c in B such that Tr(c) = 1.

PROOF. The necessity is a consequence of Lemma 3.1. For the sufficiency, if Tr(c) = 1, we take $a_{11} = c$, $a_{12} = a_{21} = 0$. Then $a_{11}(1\otimes 1)-\sigma(a_{11})(j\otimes j)$ is a separable idempotent for B[j] by Lemma 3.1. Thus B[j] is a separable extension over B.

Using Theorem 3.2, we can obtain a characterization of a separable extension B[j] over B when Char(A) = 2.

THEOREM 3.3. Assume Char(A) = 2. Then, B[j] is a separable extension over B if and only if B is a Galois extension over A.

PROOF. Let B be a Galois extension over A. Corollary 1.3 on P. 85 in [3] implies that Tr(c) = 1 for some c in B. Thus B[j] is a separable extension over B by Theorem 3.2. Conversely, by Theorem 3.2 again, there exists an c in B such that Tr(c) = 1, so $(c+\sigma(c)) = 1$. By hypothesis, Char(A) = 2, $\sigma(c) = \sigma(-c) = -\sigma(c)$, so $c-\sigma(c) = 1$. Hence the ideal generated by $\{(b-\sigma(b)) \,/\, b \text{ in } B\} = B$. This implies that B is Galois over A by the statement 5 in Proposition 1.2 on P. 81 in [3].

Let us recall that the theorem of Parimala and Sridharan (Proposition 1.1 in [5]): Assume 2 is a unit in A. Then, B is Galois over A if and only if $B\otimes_A B[j] \cong M_2(B)$, a matrix algebra over B of order 2. We are going to improve it without the assumption that 2 is a unit in A.

THEOREM 3.4. If B is Galois over A, then $B\otimes_A B[j] \cong M_2(B)$.

PROOF. If B is Galois over A, there exists an c in B such that $Tr(c) = 1$ ([3], Corollary 1.3, P. 85). Hence B[j] is a separable extension over A by Theorem 3.2. But B is also a separable extension over A by Proposition 1.2 in [3], so the transitive property of separable extensions ([4], Proposition 2.5) implies that B[j] is a separable A-algebra. Moreover, we claim that (1) B[j] is an Azumaya algebra over A, and (2) B is a maximal commutative subalgebra of B[j]. The proof of these facts was given in [7]. For completeness, we give an outline here. For part (1), it suffices to show that A is the center of B[j]. Clearly, A is contained in the center. Now, let $b+b'j$ be in the center. Then $j(b+b'j) = (b+b'j)j$ and $c(b+b'j) = (b+b'j)c$ for each c in B. Equating coefficients of the basis $\{1,j\}$ in the above equations, we have that b is in A and $b' = 0$ by Statement 5 in Proposition 1.2 on P. 81 in [3]. For part (2), to show that B is a maximal commutative subalgebra of B[j] is to show that the commutant of B in B[j] is B. The computation is similar to part (1).

Moreover, noting that B is separable over A, we then conclude that $B\otimes_A (B[j])^o \cong \mathrm{Hom}_B(B[j],B[j])$ by Theorem 5.5 on P. 65 in [3], and this implies that $B\otimes_A B[j] \cong M_2(B)$, where $(B[j])^o$ is the opposite ring.

In [7], the sufficiency of the Parimala and Sridharan theorem was shown by a different method from [5]. Now we slightly improve the statement without the assumption that 2 is a unit in A.

THEOREM 3.5. Let B[j] be a separable extension over B. If $B\otimes_A B[j] \cong M_2(B)$, then B is Galois over A.

PROOF. Since $B[j]$ is a separable extension over B, there exists an element c in B such that $Tr(c) = 1$ by Theorem 3.2. Hence the sequence $B\to A\to 0$ is exact under the trace map. But A is projective over A, so the sequence splits, and then A is an A-direct summand of B. By hypothesis, $B\otimes_A B[j] \cong M_2(B)$ which is an Azumaya B-algebra, so $B[j]$ is an Azumaya A-algebra ([3], Corollary 1.10, P. 45). Therefore B is Galois over A by using the same argument as given in [7].

In Theorem 3.5, the hypothesis that $B\otimes_A B[j] \cong M_2(B)$ can be replaced by that $B\otimes_A B[j]$ is an Azumaya B-algebra with the same proof.

4. SPECIAL SEPARABLE QUATERNION ALGEBRAS.

Theorem 3.5 tells us that $B[j]$ is an Azumaya A-algebra such that $B\otimes_A B[j] \cong M_2(B)$ when B is Galois over A. In this section, we are going to discuss generalized quaternion algebras $B[j]$ in which 2 is a unit in A when B is projective and separable over A. With a similar argument as given in Lemma 3.1, we have

LEMMA 4.1. The element $a_{11}(1\otimes 1)+a_{12}(1\otimes j)+a_{21}(j\otimes 1)+a_{22}(j\otimes j)$ in $A[j]\otimes_A A[j]$ is a separable idempotent for $A[j]$ if and only if (1) $a_{22} = -a_{11}$ such that $2a_{11} = 1$, and (2) $a_{21} = a_{12}$ such that $2a_{12} = 0$.

THEOREM 4.2. The A-algebra $A[j]$ is separable if and only if 2 is a unit in A.

PROOF. The necessity is clear by Lemma 4.1; the sufficiency is immediate because $(1/2)(1\otimes 1-j\otimes j)$ is a separable idempotent.

Now we give a characterization of $B[j]$ in which 2 is a unit when B is projective and separable over A.

THEOREM 4.3. Let B be separable and projective over A. Then, $B[j]$ is a separable extension over B and projective over $A[j]$ as a bimodule if and only if 2 is a unit in A.

PROOF. Let 2 be a unit in A and let c be (1/2). Then Tr(c) = 1/2+1/2 = 1, and hence B[j] is separable over B by Theorem 3.2. By hypothesis, B is projective over A, so B[j] is left projective over A (for B[j] is left projective over B). Hence B[j] is left projective over A[j] ([3], Proposition 2.3, P. 48). We next claim that B[j] is also right projective over A[j]. In fact, α: $B\otimes_A A[j] \to B[j]$ defined by $\alpha(b\otimes 1+b'\otimes j) = b\otimes b'j$ for all b and b' in B is an isomorphism as right A[j]-modules. But B is projective over A, so $B\otimes_A A[j]$ is right projective over A[j]. This proves that B[j] is right projective over A[j]. Thus $B[j]\otimes_A(B[j])^o$ is projective as A[j]-A[j]-module. Since B[j] is a direct summand of $B[j]\otimes_A(B[j])^o$ as a $B[j]\otimes_A(B[j])^o$-module (for B[j] is separable over A), B[j] is projective as a A[j]-A[j]-module.

Conversely, to show that 2 is a unit in A, it suffices to show that A[j] is a separable A-algebra by Theorem 4.2. Since B[j] is a separable extension over B, Tr(c) = 1 for some c in B by Theorem 3.2. Hence Tr: $B \to A \to 0$ is exact. We claim that Tr induces an exact sequence: $B[j] \to A[j] \to 0$ as A[j]-A[j]-modules. We define β: $B[j] \to A[j] \to 0$ by $\beta(b+b'j) = Tr(b)+Tr(b')j$. Clearly, β is an additive group homomorphism. Moreover, for a,a' in A, (b+b'j)(a+a'j) = (ba-b'a')+(ba'+b'a)j, so $\beta((b+b'j)(a+a'j)) = Tr(ba-b'a')+Tr(ba'+b'a)j$ = (aTr(b)-a'Tr(b'))+(a'Tr(b)+aTr(b'))j. Also, $\beta(b+b'j)(a+a'j) = (Tr(b)+Tr(b')j)(a+a'j) = \beta((b+b'j)(a+a'j))$. Thus β is a right A[j]-homomorphism. Similarly, by noting that $Tr = 1+\sigma$ and that $(Tr)\sigma = Tr = \sigma(Tr)$, it is straightforward to verify that β is a left A[j]-homomorphism. But then A[j] is A[j]-A[j] projective such that β is onto (for Tr(c) = 1 in A[j]). This implies that the exact sequence β: $B[j] \to A[j] \to 0$ splits as $A[j]$-$A[j]$-modules. Thus $A[j]$ is an $A[j]$-direct summand of $B[j]$. Now by hypothesis, $B[j]$ is $A[j]$-projective, so $B[j] \otimes_A (B[j])^o$ is $A[j] \otimes_A A[j]$-projective, where $(B[j])^o$ is the opposite algebra of $B[j]$. By hypothesis again, $B[j]$ is separable over A, so $B[j]$ is projective over $A[j] \otimes_A A[j]$. Therefore, the $A[j]$-direct summand $A[j]$ of $B[j]$ is also projective over $A[j] \otimes_A A[j]$. This proves that $A[j]$ is separable over A, and so 2 is a unit in A by Theorem 4.2.

5. EXAMPLES.

This section includes several examples to illustrate our results.

(1) Let Z be the ring of integers, and ZxZ (= B) the ring of direct product of Z under the componentwise operations. Define σ: $ZxZ \to ZxZ$ by $\sigma(a,a') = (a',a)$ for a,a' in Z. Then σ is an automorphism group of order 2 and $\{(a,a) / a \text{ in } Z\}$ (= A) is the subring of ZxZ of the fixed elements under σ. Imbed Z in ZxZ by $a \to (a,a)$. Then we have

(a) ZxZ is a free A-module with a basis $\{(1,0),(0,1)\}$.

(b) ZxZ is separable over Z.

(c) $(ZxZ)[j]$ is a separable extension over ZxZ because $Tr((1,0)) = (1,0)+(0,1) = (1,1)$ by Theorem 3.2.

(d) $Z[j]$ is not separable over Z because 2 is not a unit in Z by Theorem 4.2.

(e) $(ZxZ)[j]$ is not projective over $Z[j]$ because 2 is not a unit in Z by Theorem 4.3.

(2) Let $Z_{(3)}$ be the local ring of Z at the prime ideal (3). Replace Z by $Z_{(3)}$ in Example (1). Then we have

(a) 2 is a unit in $Z_{(3)}$.

(b) All properties (a),(b) and (c) in Example (1) hold.

(c) $(Z_{(3)} x Z_{(3)})[j]$ is projective over $Z_{(3)}[j]$ by Theorem 4.3.

(3) ZxZ and $Z_{(3)} x Z_{(3)}$ in Example (1) and Example (2) are Galois over Z and $Z_{(3)}$ respectively by using Proposition 1.2 on P. 64 in [3], Since Tr((3,-2)) = (3,-2)+(-2,3) = (1,1) which is not in any maximal ideal of ZxZ or $Z_{(3)} x Z_{(3)}$. Thus $(ZxZ) \otimes_Z (ZxZ)[j] \cong M_2(ZxZ)$ and $(Z_{(3)} x Z_{(3)}) \otimes_{Z_{(3)}} (Z_{(3)} x Z_{(3)})[j] \cong M_2(Z_{(3)} x Z_{(3)})$ by Theorem 3.4.

(4) Let i be the usual imaginary unit. Then Z[i] is not separable over Z. Z[i] has an automorphism group $\{\sigma : \sigma(a+bi) = a-bi$ for a,b in $Z\}$ such that $\sigma^2 = 1$ and Z is the fixed ring of σ. Also, (a) (Z[i])[j] is not separable over Z[i], and (b) Z[i] is not Galois over Z.

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