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ON THE INTITIAL VALUE PROBLEM FOR A PARTIAL DIFFERENTIAL EQUATION WITH OPERATOR COEFFICIENTS

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ABSTRACT. In the present work it is studied the initial value problem for an equation of the form

$$L \frac{\partial^k u}{\partial t^k} = \sum_{j=1}^{k} L_j \frac{\partial^{k-j} u}{\partial t^{k-j}} ,$$

where L is an elliptic partial differential operator and $(L_j : j = 1, \ldots , k)$ is a family of partial differential operators with bounded operator coefficients in a suitable function space. It is found a suitable formula for solution. The correct formulation of the Cauchy problem for this equation is also studied.

KEY WORDS AND PHRASES. *Partial Differential Equations, Elliptic Operators, Cauchy Problems and General Solutions.*

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1. INTRODUCTION.

Consider the equation

$$\sum_{|q|=2m} a_q(t)\, D^q D_t^k u = \sum_{j=1}^{k} \sum_{|q|=2m} A_{q,j}(t) D^q D_t^{k-j} u, \tag{1.1}$$

where $q = (q_1, \ldots, q_n)$ is an n-tuple of nonnegative integers, and

$$D^q = \frac{\partial^{|q|}}{\partial x_1^{q_1} \ldots \partial x_n^{q_n}},$$

in which $|q| = q_1 + \ldots + q_n$; $D_t = \frac{\partial}{\partial t}$, and m, k are positive integers.
It is assumed in equation (1.1) that the following conditions are satisfied;

(a) The coefficients $(a_q(t),\ |q| = 2m)$ are continuous functions of t in [0,1].

(b) For each $t \in [0,1]$, $\sum_{|q|=2m} a_q(t)\, D^q$ is an elliptic operator.

(c) The coefficients $(A_{q,j}(t),\ |q| = 2m,\ j = 1,\ldots,k)$ for each $t \in [0,1]$ are linear bounded operators from $L_2(E_n)$ into itself, where $L_2(E_n)$ is the set of all square integrable functions on the n-dimensional Euclidean space E_n.

(d) The operators $(A_{q,j}(t),\ |q| = 2m,\ j = 1,\ldots,k)$, are strongly continuous in $t \in [0,1]$.

In section 2, we shall find a solution $u(x,t)$ of equation (1.1) in a suitable function space so that $t \in (0,1)$, $x \in E_n$, and the solution $u(x,t)$ satisfies the following initial conditions

$$D_t^j u(x,t)\Big|_{t=0} = f_j(x),\ j = 0,1,2,\ldots,k-1. \tag{1.2}$$

The uniqueness of the solution of the problem (1.1), (1.2) is also proved. Under suitable conditions ([3], [4]) we establish the correct formulation of the Cauchy problem (1.1) and (1.2).

2. A GENERAL FORMULA FOR THE SOLUTION

Let $W^m(E_n)$ be the space of all functions $f \in L_2(E_n)$ such that the distributional derivatives $D^q f$ with $|q| \leq m$ all belong to $L_2(E_n)$, [8].

We shall say that u is a solution of equation (1.1) in the space $W^{2m}(E_n)$, if for every $t \in (0,1)$ the derivatives $D_t^j u$, $j = 0,1,2,\ldots, k$ exist and are members of $W^{2m}(E_n)$ and if u satisfies equation (1.1).

We are now able to prove the following theorem.

THEOREM 1. If $f_j \in W^{2m}(E_n)$, $j = 0,1,\ldots, k-1$ and if $4m > n$, then there exists a unique solution u of the initial value problem (1.1), (1.2) in the space $W^{2m}(E_n)$.

PROOF. As in [6] the differential operators (D^q, $|q| = 2m$) can be transformed into

$$D^q f = R^q \nabla^{2m} f, \; f \in W^{2m}(E_n) , \tag{2.1}$$

where $\nabla^2 = D_1^2 + \ldots + D_n^2$, $R^q = R_1^{q_1} \ldots R_n^{q_n}$, R_j is the Riesz-transform defined by

$$R_j f = - i \, \Pi^{-\frac{(n+1)}{2}} \, \Gamma \left(\frac{n+1}{2}\right) \int_{E_n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y) \, dy ,$$

Γ is the gamma function, $i = \sqrt{-1}$ and $|x|^2 = x_1^2 + \ldots + x_n^2$, (see [1]).

Using (2.1) we see that equation (1.1) is formally equivalent to,

$$\sum_{|q| = 2m} a_q(t) \, R^q \, \nabla^{2m} \, D_t^k \, u = \sum_{j=1}^{k} \sum_{|q|=2m} A_{q,j}(t) R^q \nabla^{2m} D_t^{k-j} \, u, \tag{2.2}$$

Using the notations

$$\nabla^{2m} u = v \; , \; \nabla^{2m} f_j = g_j,$$

$$\sum_{|q| = 2m} a_q(t) \, R^q = H_o(t), \quad \sum_{|q| = 2m} A_{q,j}(t) \, R^q = H_j(t),$$

We obtain from (2.2) in a formal way the equation

$$H_o(t)\ D_t^k v = \sum_{j=1}^{k} H_j(t)\ D_t^{k-j} v. \tag{2.3}$$

Since the operator $\sum_{|q| = 2m} a_q(t)\ D^q$ is elliptic, it follows that the operator $H_o\ (t)$ has a unique bounded inverse $H_o^{-1}(t)$ from $L_2\ (E_n)$ into itself, for each $t \in [0,1]$. Applying $H_o^{-1}(t)$ to both sides of (2.3) we get

$$D_t^k\ v = \sum_{j=1}^{k} H_o^{-1}(t)\ H_j(t)\ D_t^{k-j}\ v. \tag{2.4}$$

Since the operators R_j , $j = 1,\ldots, n$ are bounded in $L_2(E_n)$, it can be easily proved that $H_j(t)$, $j = 1,\ldots,k$ are bounded operators in $L_2(E_n)$ for each $t \in [0,1]$. It is convenient to introduce the following notations in order to complete the proof by considering the problem in a Banach space to be defined below.

Let A(t) denote the square matrix,

$$A(t) = \begin{bmatrix} H_1^*(t) & H_2^*(t) & \cdots & H_{k-1}^*(t) & H_k^*(t) \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix},$$

where $H_j^*(t) = H_o^{-1}(t)\ H_j(t)$, , $j = 1,2,\ldots, k$ and I denotes the identity operator.

Equation (2.4) can be written in the form

$$\frac{d\ V(t)}{dt} = A(t)\ \ V(t), \tag{2.5}$$

where V is the column matrix

$$V = \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_k \end{bmatrix}$$

and $v_j = D_t^{k-j} \nabla^{2m} u.$

The column vector V satisfies formally the initial condition

$$V(0) = \begin{bmatrix} g_{k-1} \\ g_{k-2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ g_o \end{bmatrix} = G . \qquad (2.6)$$

Let B denote the space of column vectors V, with the norm

$$||V|| = \sum_{j=1}^{k} ||v_j|| L_2 (E_n) ,$$

where $||f||^2_{L_2(E_n)} = \int_{E_n} f^2(x)\, dx .$

It is clear that B is a Banach space and A(t) is a linear bounded operator from B into itself for each $t \in [0,1]$. According to the conditions imposed on the coefficients $a_q(t)$, $A_{q,j}(t)$, it can be seen that A(t) is strictly continuous on [0,1].

Since $g_j \in L_2(E_n)$, j = 0,1 , ... , k-1, we find that the column vector G is an element of the space B. The abstract Cauchy problem (2.5), (2.6) can be solved by applying the above argument [7]. In other words, there exists for each $t \in (0,1)$ a unique operator Q(t) bounded in the Banach space B such that the formula

$$V (t) = Q(t)\, G, \qquad (2.7)$$

represents the unique solution of the problem (2.5), (2.6) in the space B. The operator Q(t) can be represented in the matrix form

$$Q(t) = \begin{bmatrix} Q_{11}(t) & Q_{12}(t) \dots & Q_{1k}(t) \\ Q_{21}(t) & Q_{22}(t) \dots & Q_{2k}(t) \\ \vdots & \vdots & \vdots \\ Q_{k1}(t) & Q_{k2}(t) \dots & Q_{kk}(t) \end{bmatrix}, \tag{2.8}$$

where ($Q_{rs}(t)$, $r = 1,\dots,k$, $s = 1,\dots, k$) are bounded operatros in the space $L_2(E_n)$ for each t in [0,1].

Using (2.7) and (2.8) one gets

$$v_r(x,t) = D_t^{k-r} \nabla^{2m} u(x,t)$$

$$= \sum_{s=1}^{k} Q_{rs}(t)\, g_{k-s} = \sum_{s=1}^{k} Q_{rs}(t)\, \nabla^{2m} f_{k-s} \tag{2.9}$$

From (2.9) we get immediately

$$u(x,t) = (\nabla^{2m})^{-1} \sum_{s=1}^{k} Q_{ks}(t)\, \nabla^{2m} f_{k-s}, \tag{2.10}$$

where $(\nabla^{2m})^{-1}$ is a closed operator, defined on $L_2(E_n)$ and representing the inverse of ∇^{2m}.

We prove now that the formula (2.10) which we have obtained in a formal way is in fact the required solution of the problem (1.1), (1.2) in the space $W^{2m}(E_n)$.

Since $(\nabla^{2m})^{-1}$ is a closed operator from L_2 (E_n) onto W^{2m} (E_n), it follows immediately from (2.10) that

$$u \in W^{2m} (E_n),$$

for each $t \in [0,1]$.

Now the differential operator $\frac{d}{dt}$ in equation (2.5) denotes the abstract derivative with respect to t in the space $L_2(E_n)$, i.e. if $f_t \in L_2(E_n)$ for each $t \in (0,1)$, then $\frac{d}{dt} f_t$ is defined by

$$\frac{d}{dt} f_t = f_t^*, \text{ where}$$

$$\lim_{t \to 0} \left\| \frac{\Delta f_t}{\Delta t} - f_t^* \right\|_{L_2(E_n)} = 0,$$

and $\Delta f_t = f_{t+\Delta t} - f_t$.

Since $\frac{d}{dt} (\nabla^{2m})^{-1} f_t = (\nabla^{2m})^{-1} \frac{d}{dt} f_t$, $f_t \in L_2(E_n)$,

it follows from (2.9) and (2.10) that

$$\begin{aligned} \frac{d^{k-r}}{dt^{k-r}} u &= (\nabla^{2m})^{-1} \sum_{s=1}^{k} \frac{d^{k-r}}{dt^{k-r}} Q_{ks}(t) \nabla^{2m} f_{k-s} \\ &= (\nabla^{2m})^{-1} \sum_{s=1}^{k} Q_{rs}(t) \nabla^{2m} f_{k-s}. \end{aligned} \tag{2.11}$$

The last formula proves that

$$\frac{d^{k-r}}{dt^{k-r}} u \in W^{2m} (E_n),$$

$r = 1,2, \ldots, k$, and $t \in (0,1)$.

Using (2.4) and (2.11) one gets,

$$\frac{d^k u}{dt^k} \in W^{2m} (E_n),$$

for each $t \in (0,1)$.

In [5] we have proved that if u, $\frac{du}{dt} \in W^{2m} (E_n)$ and $\frac{d}{dt} D^q u \in L_2 (E_n)$, $|q| = 2m$, $4m > n$, then the partial derivative $D_t u$ exists in the usual sense and that it is identical to the corresponding abstract derivative. Since these conditions are satisfied by u in (2.10), therefore the same conclusion applies. In a similar manner we can deduce also that the partial derivatives $D_t^j u$, $j = 1,2,\ldots, k$ exist in the usual sense for each $t \in [0,1]$, $x \in E_n$ and that they are identical to the corresponding abstract derivatives.

Since $Q(0)\,G = G$, we have

$$Q_{rs}(0) = \begin{cases} I & , r = s \\ 0 & , r \neq s, \end{cases}$$

therefore

$$\nabla^{2m} u(x,0) = \sum_{s=1}^{k} Q_{ks}(0)\, \nabla^{2m} f_{k-s} = \nabla^{2m} f_o(x)$$

The last formula leads to $u(x,0) = f_o(x)$. In a similar manner we can prove that

$$D_t^j u(x,0) = f_j(x) \ , \quad j = 0,1,\ldots., k,$$

which complete the proof. (Compare [2]).

THEOREM 2. If the coefficients $(A_{q,j}(t)\ , |q| = 2m,\ j = 1,2,\ldots,k)$ commute with D_r, $r = 1,2,\ldots, n$, then the solution of the problem (1.1), and (1.2) is given by the formula

$$u = \sum_{s=1}^{k} Q_{ks}(t)\, f_{k-s} \tag{2.12}$$

PROOF. For any $f \in W^{2m}(E_n)$, we have $R^q \nabla^{2m} f = \nabla^{2m} R^q f$ (2.13)

Since the operators $(A_{q,j}(t)\ |q| = 2m,\ j = 1,\ldots, k)$ commute with D_r, $r = 1,2,\ldots, n$, it follows that the operators $(A_{q,j}(t)\ |q| = 2m,\ j = 1,\ldots, k)$ commute with $(R^q,\ |q| = 2m)$ and according to (2.2) and (2.13) we can write

$$\nabla^{2m} [H_o(t)\, D_t^k u - \sum_{j=1}^{k} H_j(t)\, D_t^{k-j} u] = 0$$

The last equation leads immediately to

$$D_t^k u = H_o^{-1} \sum_{j=1}^{k} H_j(t)\, D_t^{k-j} u.$$

Applying similar steps to theorem (1), we obtain the required result.

COROLLARY. If the operators $(A_{q,j}(t),\ |q| = 2m,\ j=1,..,k)$ commute with

D_r, r = 1,2,..., n, then the Cauchy problem (1.1), (1.2) is correctly formulated.

PROOF. The proof of this important fact can be deduced immediately by using formula (2.12), (compare [3], [4]).

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