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ON AN nTH-ORDER INFINITESIMAL GENERATOR AND TIME-DEPENDENT OPERATOR DIFFERENTIAL EQUATION WITH A STRONGLY ALMOST PERIODIC SOLUTION

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In a Banach space, if u is a Stepanov almost periodic solution of a certain nth-order infinitesimal generator and time-dependent operator differential equation with a Stepanov almost periodic forcing function, then $u, u', \ldots, u^{(n-2)}$ are all strongly almost periodic and $u^{(n-1)}$ is weakly almost periodic.

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1. Introduction. Suppose that X is a Banach space, X^* is the dual space of X, and $\mathbb{R}$ is the real line. A continuous function $f : \mathbb{R} \to X$ is said to be strongly (or Bochner) almost periodic if, given $\varepsilon > 0$, there is a positive real number $r = r(\varepsilon)$ such that any interval of the real line of length r contains at least one point τ for which

$$\sup_{t\in\mathbb{R}} \left\| f(t+\tau) - f(t) \right\| \le \varepsilon. \tag{1.1}$$

A function $f : \mathbb{R} \to X$ is weakly almost periodic if the scalar-valued function $\langle x^*, f(t)\rangle = x^* f(t)$ is almost periodic for each $x^* \in X^*$.

A function $f \in L^p_{\text{loc}}(\mathbb{R}; X)$ with $1 \le p < \infty$ is said to be Stepanov-bounded or S^p-bounded on $\mathbb{R}$ if

$$\|f\|_{S^p} = \sup_{t\in\mathbb{R}} \left[\int_t^{t+1} \left\| f(s) \right\|^p ds \right]^{1/p} < \infty. \tag{1.2}$$

A function $f \in L^p_{\text{loc}}(\mathbb{R}; X)$ with $1 \le p < \infty$ is said to be Stepanov almost periodic or S^p-almost periodic if, given $\varepsilon > 0$, there is a positive real number $r = r(\varepsilon)$ such that any interval of the real line of length r contains at least one point τ for which

$$\sup_{t\in\mathbb{R}} \left[\int_t^{t+1} \left\| f(s+\tau) - f(s) \right\|^p ds \right]^{1/p} \le \varepsilon. \tag{1.3}$$

We designate by $L(X;X)$ the set of all bounded linear operators on X into itself. An operator-valued function $T : \mathbb{R} \to L(X;X)$ is called a strongly continuous group if

$$T(t_1 + t_2) = T(t_1)T(t_2) \quad \forall t_1, t_2 \in \mathbb{R}, \tag{1.4}$$

$$T(0) = I = \text{the identity operator on } X, \tag{1.5}$$

$$T(t)x, \quad t \in \mathbb{R} \longrightarrow X, \text{ is continuous for each } x \in X. \tag{1.6}$$

The infinitesimal generator A of a strongly continuous group $T:\mathbb{R}\to L(X;X)$ is a closed linear operator, with its domain $D(A)$ dense in X, defined by

$$Ax=\lim_{t\to 0}\frac{T(t)x-x}{t}\quad \text{for } x\in D(A) \tag{1.7}$$

(see Dunford and Schwartz [4]).

An operator-valued function $T:\mathbb{R}\to L(X;X)$ is said to be strongly (weakly) almost periodic if $T(t)x$, $t\in\mathbb{R}\to X$ is strongly (weakly) almost periodic for each $x\in X$.

Assume that A and B are two densely defined closed linear operators, having their domains and ranges in a Banach space X, and $f:\mathbb{R}\to X$ is a continuous function. Then, a strong solution of the differential equation

$$u^{(n)}(t)=Au^{(n-1)}(t)+Bu(t)+f(t)\quad \text{a.e. on } \mathbb{R} \tag{1.8}$$

is an n times strongly differentiable function $u:\mathbb{R}\to D(B)$ with $u^{(n-1)}(t)\in D(A)$ for all $t\in\mathbb{R}$, and satisfying equation (1.8) a.e. (almost everywhere) on $\mathbb{R}$.

Our first result is as follows (see Zaidman [7] for a first-order infinitesimal generator differential equation).

Theorem 1.1. *In a Banach space X, suppose that $f:\mathbb{R}\to X$ is an S^1-almost periodic continuous function, A is the infinitesimal generator of a weakly almost periodic strongly continuous group $T:\mathbb{R}\to L(X;X)$, $B:\mathbb{R}\to L(X;X)$ is a strongly almost periodic operator-valued function, and $u:\mathbb{R}\to X$ is a strong solution of the differential equation*

$$u^{(n)}(t)=Au^{(n-1)}(t)+B(t)u(t)+f(t)\quad \text{a.e. on } \mathbb{R}. \tag{1.9}$$

If u is S^1-almost periodic from $\mathbb{R}$ to X and $u^{(n-1)}$ is S^1-bounded on $\mathbb{R}$, then $u,u',\dots,u^{(n-2)}$ are all strongly almost periodic from $\mathbb{R}$ to X, $T(-t)u^{(n-1)}(t)$ is weakly almost periodic from $\mathbb{R}$ to X, and $u^{(n-1)}$ is bounded on $\mathbb{R}$.

Remark 1.2. See Cooke [3] and Zaidman [8] for some nth- and first-order abstract differential equations with strongly almost periodic solutions.

2. Lemmas

Lemma 2.1. *The $(n-1)$th derivative of any solution of (1.9) admits the representation*

$$u^{(n-1)}(t)=T(t)u^{(n-1)}(0)+\int_0^t T(t-s)[B(s)u(s)+f(s)]ds\quad \text{on } \mathbb{R}. \tag{2.1}$$

Proof. For an arbitrary but fixed $t\in\mathbb{R}$, we have

$$\begin{aligned}\frac{d}{ds}\left[T(t-s)u^{(n-1)}(s)\right]&=T(t-s)\left[u^{(n)}(s)-Au^{(n-1)}(s)\right]\\&=T(t-s)\left[B(s)u(s)+f(s)\right]\quad \text{a.e. on } \mathbb{R}, \text{ by (1.9)}.\end{aligned} \tag{2.2}$$

Hence,

$$\int_0^t\frac{d}{ds}\left[T(t-s)u^{(n-1)}(s)\right]ds=\int_0^t T(t-s)[B(s)u(s)+f(s)]ds, \tag{2.3}$$

which gives the desired representation by (1.5). □

Lemma 2.2. *In a Banach space X, if $g:\mathbb{R}\to X$ is a strongly almost periodic function and if $G:\mathbb{R}\to L(X;X)$ is a strongly (weakly) almost periodic operator-valued function, then $G(t)g(t)$, $t\in\mathbb{R}\to X$, is a strongly (weakly) almost periodic function.*

Proof. See Rao [6, Theorem 1] for weak almost periodicity. □

Lemma 2.3. *In a Banach space X, if $g:\mathbb{R}\to X$ is an S^1-almost periodic continuous function and if $G:\mathbb{R}\to L(X;X)$ is a weakly almost periodic operator-valued function, then $x^*G(t)g(t)$, $t\in\mathbb{R}\to$ scalars, is an S^1-almost periodic continuous function for each $x^*\in X^*$.*

Proof. By our assumption, for an arbitrary but fixed $x^*\in X^*$, the scalar-valued function $x^*G(t)x$ is almost periodic, and hence is bounded on $\mathbb{R}$, for each $x\in X$. So, by the uniform-boundedness principle,

$$\sup_{t\in\mathbb{R}}\left|\left|x^*G(t)\right|\right| = M<\infty. \tag{2.4}$$

The function $x^*G(t)g(t)$ is continuous on $\mathbb{R}$ (see the proof of Theorem 1 of Rao [6]).

Consider the functions on $\mathbb{R}$

$$g_\delta(t)=\frac{1}{\delta}\int_0^\delta g(t+s)\,ds \quad \text{for } \delta>0. \tag{2.5}$$

Since g is S^1-almost periodic from $\mathbb{R}$ to X, it follows that g_δ is strongly almost periodic from $\mathbb{R}$ to X for each fixed $\delta>0$. Further, as shown for scalar-valued functions in Besicovitch [2, pages 80–81], we can prove that $g_\delta\to g$ as $\delta\to 0+$ in the S^1-sense, that is,

$$\sup_{t\in\mathbb{R}}\int_t^{t+1}\left|\left|g(s)-g_\delta(s)\right|\right|ds\longrightarrow 0 \quad \text{as } \delta\longrightarrow 0+. \tag{2.6}$$

Furthermore, we have

$$x^*G(s)g(s)=x^*G(s)\left[g(s)-g_\delta(s)\right]+x^*G(s)g_\delta(s) \quad \text{on } \mathbb{R}, \tag{2.7}$$

and, by (2.4) and (2.6),

$$\begin{aligned}\sup_{t\in\mathbb{R}}&\int_t^{t+1}\left|x^*G(s)\left[g(s)-g_\delta(s)\right]\right|ds\\ &\le M\sup_{t\in\mathbb{R}}\int_t^{t+1}\left|\left|g(s)-g_\delta(s)\right|\right|ds\longrightarrow 0 \quad \text{as } \delta\longrightarrow 0+.\end{aligned} \tag{2.8}$$

By Lemma 2.2, the functions $x^*G(s)g_\delta(s)$ are almost periodic from $\mathbb{R}$ to the scalars. Therefore, from (2.7) and (2.8), it follows that $x^*G(s)g(s)$ is S^1-almost periodic from $\mathbb{R}$ to the scalars. □

Lemma 2.4. *In a Banach space X, if $g:\mathbb{R}\to X$ is an S^1-almost periodic continuous function and if $G:\mathbb{R}\to L(X;X)$ is a strongly almost periodic operator-valued function, then $G(t)g(t)$, $t\in\mathbb{R}\to X$, is an S^1-almost periodic continuous function.*

The proof of this lemma is analogous to that of Lemma 2.3.

Lemma 2.5. *In a reflexive Banach space X, let $h:\mathbb{R}\to X$ be an S^1-almost periodic continuous function and*

$$H(t)=\int_0^t h(s)ds \quad \text{on } \mathbb{R}. \tag{2.9}$$

If H is S^1-bounded on $\mathbb{R}$, then it is strongly almost periodic from $\mathbb{R}$ to X.

Proof. See Rao [5, Notes (ii)]. □

Lemma 2.6. *For an operator-valued function $G:\mathbb{R}\to L(X;X)$, assume that $G^*(t)$ is the adjoint (conjugate) of the operator $G(t)$. If $G^*:\mathbb{R}\to L(X^*;X^*)$ is strongly almost periodic and if $g:\mathbb{R}\to X$ is weakly almost periodic, then $G(t)g(t)$, $t\in\mathbb{R}\to X$, is weakly almost periodic (X a Banach space).*

Proof. See Rao [6, Remarks (iii)]. □

3. Proof of Theorem 1.1. From (2.1), we obtain

$$T(-t)u^{(n-1)}(t)=u^{(n-1)}(0)+\int_0^t T(-s)[B(s)u(s)+f(s)]ds \quad \text{on } \mathbb{R}. \tag{3.1}$$

So, for an arbitrary but fixed $x^*\in X^*$, we have

$$x^*T(-t)u^{(n-1)}(t)=x^*u^{(n-1)}(0)+\int_0^t x^*T(-s)[B(s)u(s)+f(s)]ds \quad \text{on } \mathbb{R}. \tag{3.2}$$

By Lemma 2.4, $B(s)u(s)$, $s\in\mathbb{R}\to X$ is an S^1-almost periodic continuous function. Hence, $[B(s)u(s)+f(s)]$, $s\in\mathbb{R}\to X$, is an S^1-almost periodic continuous function.

Obviously, $T(-s)$, $s\in\mathbb{R}\to L(X;X)$, is a weakly almost periodic strongly continuous group. Therefore, by Lemma 2.3, $x^*T(-s)[B(s)u(s)+f(s)]$, $s\in\mathbb{R}\to$ scalars, is an S^1-almost periodic continuous function. By (2.4) and our assumption on $u^{(n-1)}$, $x^*T(-t)u^{(n-1)}(t)$ is S^1-bounded on $\mathbb{R}$. Consequently, by Lemma 2.5, $x^*T(-t)u^{(n-1)}(t)$ is almost periodic from $\mathbb{R}$ to the scalars. That is, $T(-t)u^{(n-1)}(t)$ is weakly almost periodic from $\mathbb{R}$ to X and so is bounded on $\mathbb{R}$.

From (2.4), again by the uniform-boundedness principle,

$$\sup_{t\in\mathbb{R}}||T(t)||<\infty. \tag{3.3}$$

Therefore, $u^{(n-1)}(t)=T(t)[T(-t)u^{(n-1)}(t)]$ is bounded on $\mathbb{R}$.

Consider a sequence $\{\varphi_\kappa(t)\}_{\kappa=1}^\infty$ of infinitely differentiable nonnegative functions on $\mathbb{R}$ such that

$$\varphi_\kappa(t)=0 \quad \text{for } |t|\geq\frac{1}{\kappa}, \qquad \int_{-1/\kappa}^{1/\kappa}\varphi_\kappa(t)dt=1. \tag{3.4}$$

The convolution of u and φ_κ is defined by

$$(u*\varphi_\kappa)(t)=\int_{\mathbb{R}}u(t-s)\varphi_\kappa(s)ds=\int_{\mathbb{R}}u(s)\varphi_\kappa(t-s)ds \quad \text{on } \mathbb{R}. \tag{3.5}$$

Since u is S^1-almost periodic from $\mathbb{R}$ to X, $u * \varphi_\kappa$ is strongly almost periodic from $\mathbb{R}$ to X and hence is bounded on $\mathbb{R}$.

We note that

$$\sup_{t\in\mathbb{R}} \|(u^{(n-1)} * \varphi_\kappa)(t)\| \le \sup_{t\in\mathbb{R}} \|u^{(n-1)}(t)\|, \tag{3.6}$$

and, for $m = 1,2,\ldots,n-1$ and $\kappa = 1,2,\ldots,$

$$(u * \varphi_\kappa)^{(m)}(t) = (u^{(m)} * \varphi_\kappa)(t) \quad \text{on } \mathbb{R}. \tag{3.7}$$

Therefore, $y = u * \varphi_\kappa$ is a bounded solution of the differential equation

$$y^{(n-1)}(t) = (u * \varphi_\kappa)^{(n-1)}(t) \quad \text{on } \mathbb{R}. \tag{3.8}$$

Hence, by Cooke [3, Lemma 2], $u' * \varphi_\kappa, u'' * \varphi_\kappa, \ldots, u^{(n-1)} * \varphi_\kappa$ are all bounded on $\mathbb{R}$. Consequently, $u' * \varphi_\kappa, u'' * \varphi_\kappa, \ldots, u^{(n-2)} * \varphi_\kappa$ are all uniformly continuous on $\mathbb{R}$. So, by Amerio and Prouse [1, Theorem 6, page 6], we conclude successively that $u' * \varphi_\kappa, u'' * \varphi_\kappa, \ldots, u^{(n-2)} * \varphi_\kappa$ are all strongly almost periodic from $\mathbb{R}$ to X.

Since $u^{(n-1)}$ is bounded on $\mathbb{R}$, $u^{(n-2)}$ is uniformly continuous on $\mathbb{R}$. Hence, $(u^{(n-2)} * \varphi_\kappa)(t) \to u^{(n-2)}(t)$ as $\kappa \to \infty$, uniformly on $\mathbb{R}$. Therefore, $u^{(n-2)}$ is strongly almost periodic from $\mathbb{R}$ to X and so is bounded on $\mathbb{R}$. We thus conclude successively that $u^{(n-2)}, \ldots, u', u$ are all strongly almost periodic from $\mathbb{R}$ to X, which completes the proof of the theorem.

4. A consequence of Theorem 1.1.

We demonstrate the following result.

Theorem 4.1. *In a Banach space X, assume that A is the infinitesimal generator of a strongly continuous group $T : \mathbb{R} \to L(X;X)$, with the group of adjoint operators $T^* : \mathbb{R} \to L(X^*;X^*)$ being strongly almost periodic, and f, B, and u are defined as in Theorem 1.1. If u is S^1-almost periodic from $\mathbb{R}$ to X and $u^{(n-1)}$ is S^1-bounded on $\mathbb{R}$, then $u, u', \ldots, u^{(n-2)}$ are all strongly almost periodic and $u^{(n-1)}$ is weakly almost periodic from $\mathbb{R}$ to X.*

Proof. By our assumption, for an arbitrary but fixed $x^* \in X^*$, $T^*(t)x^*$, $t \in \mathbb{R} \to X^*$, is strongly almost periodic, and so, $x^*T(t)x$, $t \in \mathbb{R} \to$ scalars, is almost periodic for each $x \in X(x^*T(t) = T^*(t)x^*)$. Consequently, it follows that $T : \mathbb{R} \to L(X;X)$ is a weakly almost periodic group. Hence, by Theorem 1.1, $u, u', \ldots, u^{(n-2)}$ are all strongly almost periodic, and $T(-t)u^{(n-1)}(t)$ is weakly almost periodic from $\mathbb{R}$ to X. So, by Lemma 2.6, $u^{(n-1)}(t) = T(t)[T(-t)u^{(n-1)}(t)]$ is weakly almost periodic from $\mathbb{R}$ to X, which completes the proof of the theorem. □

Remark 4.2. Theorem 4.1 remains valid if $f : \mathbb{R} \to X$ is a weakly almost periodic continuous function.

Proof. By Lemma 2.6, $T(-s)f(s)$, $s \in \mathbb{R} \to X$, is weakly almost periodic. □

5. Note. Now, the proof of the following result is obvious.

Theorem 5.1. *In a reflexive Banach space X, suppose that A is the infinitesimal generator of a strongly almost periodic group $T : \mathbb{R} \to L(X;X)$ and f, B, and u are defined as in Theorem 1.1. If u is S^1-almost periodic from $\mathbb{R}$ to X and $u^{(n-1)}$ is S^1-bounded on $\mathbb{R}$, then $u, u', \dots, u^{(n-1)}$ are all strongly almost periodic from $\mathbb{R}$ to X.*

Remark 5.2. For $n = 1$, Theorem 5.1 holds in a Banach space X.

Proof. For $n = 1$, (3.1) becomes

$$T(-t)u(t) = u(0) + \int_0^t T(-s)[B(s)u(s) + f(s)]ds \quad \text{on } \mathbb{R}. \tag{5.1}$$

Using Lemma 2.4 twice, we can show that $T(-s)[B(s)u(s)+f(s)]$ is S^1-almost periodic from $\mathbb{R}$ to X. So, by Amerio and Prouse [1, Theorem 8, page 79], $T(-t)u(t)$ is uniformly continuous on $\mathbb{R}$. Further, By Lemma 2.4, $T(-t)u(t)$ is S^1-almost periodic from $\mathbb{R}$ to X. Consequently, by Amerio and Prouse [1, Theorem 7, page 78], $T(-t)u(t)$ is strongly almost periodic from $\mathbb{R}$ to X. So, by Lemma 2.2, $u(t) = T(t)[T(-t)u(t)]$ is strongly almost periodic from $\mathbb{R}$ to X. □

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