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RANDOM SUBGRAPHS OF CERTAIN GRAPH POWERS

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We determine the limiting probability that a random subgraph of the Cartesian power K_a^n or of $K_{a,a}^n$ is connected.

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1. Introduction. A finite, simple, undirected graph G has vertex set $V(G)$ and edge set $E(G)$. The order of G is $|V(G)|$ and the size $e(G)$ of G is $|E(G)|$. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S and $G[S, V(G)-S]$ denote the spanning subgraph of G with edges xy where $x \in S$ and $y \in V(G)-S$. For $U \subseteq V(G)$, let $N_G(U) = \{y \in V(G) : \exists xy \in E(G) \text{ with } x \in U\}$ and $\tilde{N}_G(U) = N_G(U) \cup U$. Of course, $N_G(v) = N_G(\{v\})$ and the degree $d_G(v)$ of v in G is $|N_G(v)|$ for $v \in V(G)$. For $S \subseteq V(G)$, let $b_G(S) = |\{xy \in E(G) : x \in S,\ y \in V(G)-S\}|$ and $b_G(s) = \min\{b_G(S) : S \subseteq V(G),\ |S| = s\}$ $(0 \le s \le |V(G)|)$.

The *Cartesian product* $G\square H$ of graphs G and H is the graph with vertex set $V(G)\times V(H)$ where vertices (g_1,h_1) and (g_2,h_2) are adjacent if and only if $g_1 = g_2$ and $h_1h_2 \in E(H)$, or, $h_1 = h_2$ and $g_1g_2 \in E(G)$. For a graph G, define $G^1 = G$ and $G^n = G^{n-1}\square G$ for $n \ge 2$. We use the following recent isoperimetric result of Tillich [6]. Here K_a denotes the complete graph of order a and $K_{a,a}$ denotes the complete bipartite graph with parts of order a.

Lemma 1.1 (see [6]). *For $G = K_a^n$ with $a \ge 2$ and $n \ge 1$,*

$$b_G(s) \ge (a-1)s(n-\log_a s) \quad \text{for } 1 \le s \le a^n \tag{1.1}$$

and, for $G = K_{a,a}^n$ with $a \ge 1$ and $n \ge 1$,

$$b_G(s) \ge as(n-\log_{2a} s) \quad \text{for } 1 \le s \le (2a)^n. \tag{1.2}$$

Let G be a graph of order n and size N. The probability space $\mathcal{G}(G,p)$ consists of all spanning subgraphs H of G where edges of G are chosen for H independently with probability $0 \le p = p(n) \le 1$, so that, $\Pr(H) = p^{e(H)}q^{N-e(H)}$ where $q = q(n) = 1-p(n)$. (We denote the random graphs in $\mathcal{G}(G,p)$ generally by G_p.)

In this paper, we determine the limiting probability that G_p is connected for $G = K_a^n$ and $K_{a,a}^n$. Specifically, we show that

$$\lim_{n\to\infty} \Pr\left(G_p \in \mathcal{G}(K_a^n, p) \text{ is connected}\right) = e^{-\lambda} \tag{1.3}$$

for fixed $a \geq 2$ where $p = p(n) = 1 - q(n)$ with $q(n) = [(\lambda(n))^{1/n}/a]^{1/(a-1)}$ and $\lim_{n\to\infty} \lambda(n) = \lambda \in (0,\infty)$. In addition, we show that

$$\lim_{n\to\infty} \Pr\left(G_p \in \mathcal{G}(K^n_{a,a}, p) \text{ is connected}\right) = e^{-\lambda} \tag{1.4}$$

for fixed $a \geq 1$ where $p = p(n) = 1 - q(n)$ with $q(n) = [(\lambda(n))^{1/n}/2a]^{1/a}$ and $\lim_{n\to\infty} \lambda(n) = \lambda \in (0,\infty)$. Our first result includes those of Burtin [3], Erdös and Spencer [5], and Bollobás [1] as a special case ($a = 2$). Our approach is similar to [1].

The rth factorial moment of a random variable (r.v.) X is denoted by $E_r(X)$. We write $X_n \xrightarrow{d} X$ when the sequence X_n of r.v.s converges in distribution to the r.v. X. Also, we write P_λ for a r.v. having Poisson distribution with mean λ.

Let $[n] = \{1,\dots,n\}$ when n is a positive integer. For a real number x and a positive integer n, let $(x)_0 = 1$ and $(x)_n = (x)\cdots(x-n+1)$. The cardinality of a set S is denoted by $|S|$. The greatest (least) integer at most (least) the real number x is denoted by $\lfloor x \rfloor$ ($\lceil x \rceil$). We write $\leq$ for an inequality that holds absolutely for the parameters considered and $\overset{*}{\leq}$ for an inequality that holds for the parameters considered and all sufficiently large n. We refer the reader to Bollobás [2] for random graph theory and to Durrett [4] for probability.

2. Results.

We use the following result from [1].

Lemma 2.1 (see [1]). *If G is a simple graph having order $n \geq 1$, maximum degree $\Delta(G) \leq \Delta$, average degree $d = d(G) = 2e(G)/n$, and $\Delta + 1 < u < n - \Delta - 1$, then there exists a u-set $U \subseteq V(G)$ with*

$$|\tilde{N}_G(U)| \geq n\frac{d}{\Delta}\left\{1 - \exp\left(-\frac{u(\Delta+1)}{n}\right)\right\}. \tag{2.1}$$

Assume $n \geq 2\Delta + 4$, since the result is vacuously true otherwise, and $\Delta > 0$ (the right-hand side is defined to be 0 for $\Delta = 0$).

We first consider $G = K^n_a$ with $V(G) = [a]^n$ for fixed $a \geq 2$ and for $n \geq 3$. Note that $V(G)$ is totally ordered lexicographically which naturally extends to u-subsets of $V(G)$. In Lemma 2.2 and Theorem 2.5, $\lambda(n) > 0$ for all n.

Lemma 2.2. *For fixed $a \geq 2$, $q = q(n) = [(\lambda(n))^{1/n}/a]^{1/(a-1)}$ where $\lim_{n\to\infty} \lambda(n) = \lambda \in (0,\infty)$, and $p = p(n) = 1 - q(n)$, we have*

$$\lim_{n\to\infty} \Pr\left(G_p \in \mathcal{G}(K^n_a, p) \textit{ has no isolated vertices}\right) = e^{-\lambda}. \tag{2.2}$$

Proof. Recall that $G = K^n_a$. Let $X_n(G_p)$ denote the number of isolated vertices in G_p. Fix $r \in \mathbb{P}$ and let $\mathcal{A}_r$ denote the set of r-tuples of V with distinct coordinates; $\mathcal{B}_r = \{(v_1,\dots,v_r) \in \mathcal{A}_r : e(G[\{v_1,\dots,v_r\}]) \neq 0\}$ and $\mathcal{C}_r = \mathcal{A}_r - \mathcal{B}_r = \{(v_1,\dots,v_r) \in \mathcal{A}_r : e(G[\{v_1,\dots,v_r\}]) = 0\}$. Then $|\mathcal{B}_r| \leq (a^n)_{r-1} ran \leq a^{n(r-1)} ran$ and $|\mathcal{C}_r| = (a^n)_r - |\mathcal{B}_r| \overset{*}{\geq} a^{nr} e^{-r^2/a^n} - a^{n(r-1)} ran$. Observe that the number of edges in G incident with $\{v_1,\dots,v_r\}$ is at least $(a-1)r(n-r)$ for all $(v_1,\dots,v_r) \in \mathcal{A}_r$.

First,

$$\begin{aligned}0 \le \sum_{(v_1,\dots,v_r)\in\mathcal{B}_r} \Pr\left(d_{G_p}(v_1)=\cdots=d_{G_p}(v_r)=0\right) &\le |\mathcal{B}_r|\,q^{(a-1)r(n-r)}\\ &\le a^{n(r-1)}ran\,\frac{(\lambda(n))^{r-r^2/n}}{a^{r(n-r)}} = \frac{(\lambda(n))^{r-r^2/n}ran}{a^{n-r^2}}.\end{aligned} \tag{2.3}$$

Next,

$$\begin{aligned}\sum_{(v_1,\dots,v_r)\in\mathcal{C}_r} \Pr\left(d_{G_p}(v_1)=\cdots=d_{G_p}(v_r)=0\right) &= |\mathcal{C}_r|\,q^{(a-1)nr}\\ &\overset{*}{\ge} \left[a^{nr}e^{-r^2/a^n}-a^{n(r-1)}ran\right]\frac{\lambda^r(n)}{a^{nr}}\\ &= \lambda^r(n)e^{-r^2/a^n}-\frac{\lambda^r(n)ran}{a^n}\end{aligned} \tag{2.4}$$

while,

$$\sum_{(v_1,\dots,v_r)\in\mathcal{C}_r} \Pr\left(d_{G_p}(v_1)=\cdots=d_{G_p}(v_r)=0\right) \le a^{nr}q^{(a-1)nr}=\lambda^r(n). \tag{2.5}$$

Hence,

$$\lambda^r(n)e^{-r^2/a^n}-\frac{\lambda^r(n)ran}{a^n} \overset{*}{\le} E_r(X_n) \le \lambda^r(n)+\frac{(\lambda(n))^{r-r^2/n}ran}{a^{n-r^2}} \tag{2.6}$$

so that,

$$\lim_{n\to\infty} E_r(X_n)=\lambda^r \tag{2.7}$$

and $X_n \xrightarrow{d} P_\lambda$ (see [4]). □

Lemma 2.3. *For fixed $a \ge 2$, $q=q(n)=[(\ln n)^{1/n}/a]^{1/(a-1)}$, and $p=p(n)=1-q(n)$, we have*

$$\Pr\left(G_p\in\mathcal{G}(K_a^n,p)\ \textit{has a component of order } s \textit{ with } 2\le s\le a^n/2\right)=o(1)\quad \textit{as } n\longrightarrow\infty. \tag{2.8}$$

Proof. Recall that $G=K_a^n$. Let $\mathcal{A}_s=\{S\subseteq V(G):|S|=s\}$ $(1\le s\le a^n)$. We consider four cases.

Case 1 $(2\le s\le s_1=\lfloor a^{n/2}/n\rfloor)$. We have

$$\begin{aligned}\left|\{S\in\mathcal{A}_s : G[S]\text{ is connected}\}\right| &\le a^n\cdot(a-1)n\cdot 2(a-1)n\cdots(s-1)(a-1)n\\ &\le a^{n+s}n^{s-1}s^s,\end{aligned} \tag{2.9}$$

so that (Lemma 1.1)

$$\begin{aligned}\sum_{S\in\mathcal{A}_s}\Pr\left(G_p[S]\text{ is a component}\right) &\le a^{n+s}n^{s-1}s^sq^{b_G(s)}\\ &= a^{n+s}n^{s-1}s^s\left[\frac{(\ln n)^{1/n}}{a}\right]^{s(n-\log_a s)}\\ &= \frac{1}{n}\left[\frac{ans^2\ln n}{a^{n(1-1/s)}}\right]^s.\end{aligned} \tag{2.10}$$

By examining the derivative $f(s)\ln(ce^2s^2/a^n)$ with respect to s of $f(s) = c^s s^{2s}/a^{n(s-1)}$ with $c = an\ln n$, we see that $f(s)$ is decreasing for $s \in [2, a^{n/2}/ec^{1/2}]$. Here $f(s) \overset{*}{\le} f(2) = 16a^2n^2\ln^2 n/a^n$. Hence,

$$\sum_{s=2}^{s_1}\sum_{S\in\mathcal{A}_s}\Pr(G_p[S] \text{ is a component}) \overset{*}{\le} \sum_{s=2}^{s_1}\frac{16a^2n\ln^2 n}{a^n} = o(1) \quad \text{as } n \longrightarrow \infty. \tag{2.11}$$

Case 2 ($s_1+1 \le s \le s_3 = \lfloor a^n/2\rfloor$). Let $\mathcal{B}_s = \{S \in \mathcal{A}_s : b_G(S) \ge (a-1)s(n-\log_a(s/n))\}$ and $\mathcal{C}_s = \mathcal{A}_s - \mathcal{B}_s = \{S \in \mathcal{A}_s : b_G(S) < (a-1)s(n-\log_a(s/n))\}$.

First,

$$\begin{aligned}\sum_{S\in\mathcal{B}_s}\Pr(G_p[S] \text{ is a component}) &\le \binom{a^n}{s}q^{(a-1)s(n-\log_a(s/n))}\\ &\le \left(\frac{ea^n}{s}\right)^s\left[\frac{(\ln n)^{1/n}}{a}\right]^{s(n-\log_a(s/n))}\\ &= \left[\frac{e(\ln n)^{1-(1/n)\log_a(s/n)}}{n}\right]^s\\ &\overset{*}{\le}\left(\frac{e\ln n}{n}\right)^s.\end{aligned} \tag{2.12}$$

Hence,

$$\sum_{s=s_1+1}^{s_3}\sum_{S\in\mathcal{B}_s}\Pr(G_p[S] \text{ is a component}) \overset{*}{\le} \sum_{s=s_1+1}^{s_3}\left(\frac{e\ln n}{n}\right)^s = o(1) \quad \text{as } n \longrightarrow \infty. \tag{2.13}$$

Next, for $S \in \mathcal{C}_s$, let $H = G[S]$. Then

$$(a-1)sn = \sum_{v\in S} d_G(v) = 2e(H) + b_G(S) < 2e(H) + (a-1)s\left(n - \log_a\frac{s}{n}\right), \tag{2.14}$$

so that

$$2e(H) \ge (a-1)s\log_a\frac{s}{n} \tag{2.15}$$

and the average degree d in H satisfies

$$d > (a-1)\log_a\frac{s}{n}. \tag{2.16}$$

Case 3 ($s_1+1 \le s \le s_2 = \lfloor a^n/\ln^2 n\rfloor$). Let $u = \lfloor s/n\rfloor$, so that $(a-1)n+1 \overset{*}{<} u \overset{*}{<} s-(a-1)n-1$, and by Lemma 2.1, for sufficiently large n, there exists $U \subseteq S$, $|U| = u$, and

$$\begin{aligned}|\tilde{N}_H(U)| &\overset{*}{\ge} \frac{s}{n}\log_a\frac{s}{n}\left\{1-\exp\left(-\frac{u[(a-1)n+1]}{s}\right)\right\}\\ &\ge \frac{\delta s}{n}\log_a\frac{s}{n} \quad \text{with } \delta = 1-e^{-1} = 0.631\ldots.\end{aligned} \tag{2.17}$$

Let $t = \lceil(\delta s/n)\log_a(s/n)\rceil$, so that $u \overset{*}{<} t \overset{*}{<} s$, and let $w = s-t = s(1-x)-\tau$ with $x = (\delta/n)\log_a(s/n)$ and $0 \le \tau < 1$. Observe that $\delta/4 \overset{*}{\le} x \overset{*}{\le} \delta$ here. For sufficiently large n, take the smallest such u-set $U = \{d_1,\dots,d_u\}$ in S ($\subseteq V(G)$) which is totally ordered; take the (uniquely determined) first $t-u$ vertices of $(N_G(d_1)\cap(S-U))\cup\cdots\cup(N_G(d_u)\cap(S-U))$ ($\subseteq V(G)$); and add the remaining w vertices W of S. Then

$$S \longmapsto (\{d_1,\dots,d_u\};N_G(d_1)\cap(S-U),\dots,N_G(d_u)\cap(S-U);W) \tag{2.18}$$

is an injection. Hence,

$$\begin{aligned} |\mathscr{C}_s| &\overset{*}{\le} \binom{a^n}{u} 2^{(a-1)nu}\binom{a^n}{w} \\ &\le \left(\frac{ea^n}{u}\right)^u 2^{(a-1)nu}\left(\frac{ea^n}{w}\right)^w \\ &\le \left(\frac{ena^n}{s}\right)^{s/n} 2^{(a-1)s}\left(\frac{ea^n}{s(1-x)}\right)^{s(1-x)}. \end{aligned} \tag{2.19}$$

Then (where $x-1/n \overset{*}{>} 0$, Lemma 1.1)

$$\begin{aligned} &\sum_{S\in\mathscr{C}_s} \Pr(G_p[S] \text{ is a component}) \\ &\quad\le |\mathscr{C}_s| q^{b_G(s)} \\ &\quad\overset{*}{\le} \left(\frac{ena^n}{s}\right)^{s/n} 2^{(a-1)s}\left(\frac{ea^n}{s(1-x)}\right)^{s(1-x)}\left[\frac{(\ln n)^{1/n}}{a}\right]^{s(n-\log_a s)} \\ &\quad= \left[(en)^{1/n}2^{a-1}\left(\frac{e}{1-x}\right)^{1-x}\left(\frac{s}{a^n}\right)^{x-1/n}(\ln n)^{1-(1/n)\log_a s}\right]^s \\ &\quad\overset{*}{\le} \left[(en)^{1/n}2^{a-1}\left(\frac{e}{1-x}\right)^{1-x}(\ln n)^{1+(2/n)-2x-(1/n)\log_a s}\right]^s. \end{aligned} \tag{2.20}$$

Here

$$2x+\frac{1}{n}\log_a s - 1 - \frac{2}{n} \ge \delta - \frac{1}{2} - \frac{4}{n}\log_a n - \frac{2}{n} \overset{*}{\ge} \frac{1}{10}, \tag{2.21}$$

so that

$$\sum_{S\in\mathscr{C}_s} \Pr(G_p[S] \text{ is a component}) \overset{*}{\le} \left[(en)^{1/n}2^{a-1}\left(\frac{e}{1-x}\right)^{1-x}(\ln n)^{-0.1}\right]^s. \tag{2.22}$$

Hence,

$$\begin{aligned} &\sum_{s=s_1+1}^{s_2}\sum_{S\in\mathscr{C}_s} \Pr(G_p[S] \text{ is a component}) \\ &\qquad\overset{*}{\le} \sum_{s=s_1+1}^{s_2}\left[(en)^{1/n}2^{a-1}\left(\frac{e}{1-x}\right)^{1-x}(\ln n)^{-0.1}\right]^s \\ &\qquad= o(1) \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{2.23}$$

Case 4 ($s_2+1 \le s \le s_3$). For $S \in \mathscr{C}_s$ and $H = G[S]$, let $T = \{v \in S : d_H(v) \ge (a-1)n - \log_a^2 n\}$, $t = |T|$ and $H_1 = H[T] = G[T]$. Then

$$\begin{aligned} 2e(H_1) &= 2e(H) - 2e(H[S-T,T]) - 2e(H[S-T]) \\ &> (a-1)s\log_a \frac{s}{n} - 2(a-1)n(s-t) \\ &= (a-1)s\Big[\log_a \frac{s}{n} - \frac{2n}{s}(s-t)\Big]. \end{aligned} \tag{2.24}$$

Here

$$\log_a \frac{s}{n} \overset{*}{\ge} n - 2\log_a n, \tag{2.25}$$

so that

$$\begin{aligned} s(a-1)n - (s-t)\log_a^2 n &\ge \sum_{v\in T} d_H(v) + \sum_{v \in S-T} d_H(v) > (a-1)s\log_a \frac{s}{n} \\ &\overset{*}{\ge} (a-1)s(n - 2\log_a n), \end{aligned} \tag{2.26}$$

hence,

$$t \overset{*}{\ge} s\Big(1 - \frac{2(a-1)}{\log_a n}\Big). \tag{2.27}$$

We take the first t vertices of T for H_1 where $t = s(1-\epsilon)$ with $s\epsilon = \lfloor 2(a-1)s/\log_a n \rfloor$ so that $0 < (a-1)/\log_a n \overset{*}{<} \epsilon \overset{*}{<} 2(a-1)/\log_a n \overset{*}{<} 1/5$. Then

$$2e(H_1) \overset{*}{>} (a-1)s[(1-2\epsilon)n - 2\log_a n] \tag{2.28}$$

and the average degree d_1 in H_1 satisfies

$$d_1 \overset{*}{>} (a-1)\Big[n - \frac{\epsilon}{1-\epsilon}n - \frac{2}{1-\epsilon}\log_a n\Big] \overset{*}{\ge} (a-1)(1-3\epsilon)n. \tag{2.29}$$

Let $u = \lceil a^n/\ln^6 n \rceil$, so that $(a-1)n+1 \overset{*}{<} u \overset{*}{<} t-(a-1)n-1$, and by Lemma 2.1, for all sufficiently large n, there exists $U \subseteq T$, $|U| = u$, and

$$\begin{aligned} |\tilde{N}_H(U)| \ge |\tilde{N}_{H_1}(U)| &\overset{*}{\ge} s(1-\epsilon)(1-3\epsilon)\Big\{1 - \exp\Big(-\frac{u[(a-1)n+1]}{t}\Big)\Big\} \\ &\overset{*}{\ge} s(1-\epsilon)^2(1-3\epsilon) \ge s(1-4\epsilon). \end{aligned} \tag{2.30}$$

Let $t = s - \lfloor 4\epsilon s \rfloor$, so that $u \overset{*}{<} t \overset{*}{<} s$, and $w = \lfloor 4\epsilon s \rfloor$. For sufficiently large n, take the smallest such u-set $U = \{d_1,\dots,d_u\}$ in S ($\subseteq V(G)$); take the (uniquely determined) first $t-u$ vertices of $(N_G(d_1)\cap(S-U)) \cup \cdots \cup (N_G(d_u)\cap(S-U))$ ($\subseteq V(G)$); and add the remaining w vertices W of S. Then

$$S \longmapsto (\{d_1,\dots,d_u\}; N_G(d_1)-S,\dots,N_G(d_u)-S; W) \tag{2.31}$$

is an injection with $|N_G(d_i)-S| \le \lfloor \log_a^2 n \rfloor$ $(1 \le i \le u)$. Hence, with $y = \lfloor \log_a^2 n \rfloor$,

$$\begin{aligned} |\mathscr{C}_s| &\overset{*}{\le} \binom{a^n}{u} \sum_{(k_1,\dots,k_u)\in\{0,\dots,y\}^u} \prod_{i=1}^{u} \binom{(a-1)n}{k_i} \binom{a^n}{w} \\ &\overset{*}{\le} \binom{a^n}{u} (y+1)^u \binom{(a-1)n}{y+1}^u \binom{a^n}{w}, \end{aligned} \tag{2.32}$$

since

$$\binom{(a-1)n}{k} \overset{*}{\le} \binom{(a-1)n}{y+1}, \quad \forall k \in \{0,\dots,y\}. \tag{2.33}$$

Then

$$\begin{aligned} |\mathscr{C}_s| &\overset{*}{\le} \left(\frac{ea^n}{u}\right)^u (y+1)^u \left(\frac{ean}{y+1}\right)^{u(y+1)} \left(\frac{ea^n}{w}\right)^w \\ &\le (e^2 an \ln^6 n)^u \left(\frac{ean}{\log_a^2 n}\right)^{uy} \left(\frac{ea^n}{4\epsilon s}\right)^{4\epsilon s}. \end{aligned} \tag{2.34}$$

Hence, (Lemma 1.1)

$$\begin{aligned} &\sum_{S\in\mathscr{C}_s} \Pr(G_p[S] \text{ is a component}) \\ &\quad\le |\mathscr{C}_s| q^{b_G(s)} \\ &\quad\overset{*}{\le} (e^2 an \ln^6 n)^u \left(\frac{ean}{\log_a^2 n}\right)^{uy} \left(\frac{ea^n}{4\epsilon s}\right)^{4\epsilon s} \left[\frac{(\ln n)^{1/n}}{a}\right]^{s(n-\log_a s)} \\ &\quad= \left[(e^2 an \ln^6 n)^{u/s} (ean \ln^2 a)^{uy/s} \left(\frac{e}{4\epsilon}\right)^{4\epsilon} \left(\frac{s}{a^n}\right)^{1-4\epsilon} (\ln n)^{1-(1/n)\log_a s - 2uy/s}\right]^s. \end{aligned} \tag{2.35}$$

Here

$$\begin{aligned} 1 \le ean \ln^2 a \overset{*}{\le} e^2 an \ln^6 n, \qquad 0 < \frac{u}{s} \overset{*}{\le} \frac{uy}{s} \overset{*}{\le} \frac{5}{\ln^2 n}, \\ 1 - \frac{1}{n}\log_a s - \frac{2uy}{s} \overset{*}{\le} \frac{2}{n}\log_a \ln n - \frac{4}{\ln^4 n} \overset{*}{\le} 0, \end{aligned} \tag{2.36}$$

so that

$$\begin{aligned} \sum_{S\in\mathscr{C}_s} \Pr(G_p[S] \text{ is a component}) &\overset{*}{\le} \left[(e^3 a^2 n^2 \ln^2 a \ln^6 n)^{5/\ln^2 n} \left(\frac{e}{4\epsilon}\right)^{4\epsilon} 2^{4\epsilon - 1}\right]^s \\ &\overset{*}{\le} \left(\frac{2}{3}\right)^s, \end{aligned} \tag{2.37}$$

since $(e^3 a^2 n^2 \ln^2 a \ln^6 n)^{5/\ln^2 n} \to 1$, $(e/4\epsilon)^{4\epsilon} \to 1$ and $\epsilon \to 0$ as $n \to \infty$. Hence,

$$\sum_{s=s_2+1}^{s_3} \sum_{S\in\mathscr{C}_s} \Pr(G_p[S] \text{ is a component}) \overset{*}{\le} \sum_{s=s_2+1}^{s_3} \left(\frac{2}{3}\right)^s = o(1) \quad \text{as } n \longrightarrow \infty. \tag{2.38}$$

□

Remark 2.4. For all $a \geq 2$ and $n \geq 2$, $b_G(s) \geq 2$ when $2 \leq s \leq a^n/2$. Hence, $0 < \tilde{q}(n) \leq q(n)$ implies $(\tilde{q}(n))^{b_G(s)} \leq (q(n))^{b_G(s)}$ when $2 \leq s \leq a^n/2$. Then (2.10), (2.12), (2.20), and (2.35) hold for $G_{\tilde{p}(n)}$ where $\tilde{p}(n) = 1-\tilde{q}(n)$ (the exponent in (2.12) is larger than $b_G(s)$). Hence, Lemma 2.3 holds for $G_{\tilde{p}(n)}$ as well. The inequalities in the proof of Lemma 2.3 hold for all sufficiently large n which can be determined from nineteen appropriate inequalities there.

Theorem 2.5. *For fixed $a \geq 2$, $q = q(n) = [(\lambda(n))^{1/n}/a]^{1/(a-1)}$ where $\lim_{n\to\infty}\lambda(n) = \lambda \in (0,\infty)$, and $p = p(n) = 1-q(n)$, we have*

$$\lim_{n\to\infty} \Pr\left(G_p \in \mathcal{G}(K_a^n, p) \text{ is connected}\right) = e^{-\lambda}. \tag{2.39}$$

Proof. We have

$$\begin{aligned} 0 &\leq \Pr\left(G_p \text{ is disconnected}\right) - \Pr\left(G_p \text{ has isolated vertices}\right) \\ &\leq \Pr\left(G_p \text{ has a component of order } s \text{ with } 2 \leq s \leq a^{n/2}\right) = o(1) \quad \text{as } n \longrightarrow \infty, \end{aligned} \tag{2.40}$$

by Remark 2.4. Hence, Lemma 2.2 gives

$$\lim_{n\to\infty} \Pr\left(G_p \text{ is disconnected}\right) = \lim_{n\to\infty} \Pr\left(G_p \text{ has isolated vertices}\right) = 1-e^{-\lambda}. \tag{2.41}$$

□

We state the result for $G = K_{a,a}^n$ since its proof is similar to the proof of Theorem 2.5.

Theorem 2.6. *For fixed $a \geq 1$, $q = q(n) = [(\lambda(n))^{1/n}/2a]^{1/a}$ where $\lim_{n\to\infty}\lambda(n) = \lambda \in (0,\infty)$, and $p = p(n) = 1-q(n)$, we have*

$$\lim_{n\to\infty} \Pr\left(G_p \in \mathcal{G}(K_{a,a}^n) \text{ is connected}\right) = e^{-\lambda}. \tag{2.42}$$

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